VŠB – Technical University of Ostrava
Faculty of Electrical Engineering and Computer Science

Ph.D. Thesis

Magic Labelings of Graphs

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Acknowledgments

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I also wish to thank my wife, Tereza, for her support and cooperation. Several of the topics, especially from Sections 3 and 4, were discussed during our regular graph theory seminar. It is often difficult to attribute a particular idea to one person since the ideas came spontaneously during brainstorming discussions.

On the other hand, the presentation and formulations are my own and thus I take full responsibility for all errors.

To my parents

Abstract

A graph labeling is a mapping from the set of edges, vertices, or both to a set of labels. Usually the labels are positive integers. Magic labelings were introduced more than forty years ago by Sedláček. In this thesis we focus on three magic-type labelings.

A vertex magic total labeling λ assigns distinct consecutive integers starting at 1 to both vertices and edges of a given graph G so that the sum (weight)

$$w_\lambda(x) = \lambda(x) + \sum_{y \in N(x)} \lambda(xy)$$

is constant for all vertices in the graph. Vertex antimagic total labelings require the weights to form an arithmetic progression. Supermagic labelings assign labels only to edges of G and again we require the weight

$$w_\lambda(x) = \sum_{y \in N(x)} \lambda(xy)$$

to be equal for every vertex in G.

After a summary of known results we give a general result on (s, d)-vertex antimagic total labelings of cycles.

The notion of magic labelings is generalized to use labels from $\mathbf{Z}$ and generalized labelings are used to find vertex magic total labelings of products of regular graphs $G \square H$. Often to construct one magic-type labeling we use the same or a different magic-type labeling of the graphs G and H of the product.

Often a "pattern" plays an important role in constructing a vertex magic total labeling of regular graphs. The "patterns" which proved to be useful are based on Kotzig arrays. The second part of the thesis explores constructions based on Kotzig arrays of various magic-type labelings of copies and products of regular graphs.

If we find a decomposition of a graph G into certain magic-type factors F_i, we can combine the known labelings of F_i to obtain a magic-type labeling of G. Special cases of such general theorems are used to find vertex magic total labelings and supermagic labelings of compositions of graphs.

A conjecture by MacDougall et al. says that any regular graph other than K_2 or $2K_3$ is vertex magic total. All results in this thesis support the conjecture and we give constructions of vertex magic total labelings for certain general classes of regular graphs.

Contents

1 Introduction and known results

1.1 Graph labelings

In this thesis we use standard terminology and notation which is common in graph theory. We give definitions of basic terms. For further reference we suggest any graph theory book, e.g. [West].

Let $G(V, E)$ be a graph. We assume G is a finite simple graph (without multiple edges and loops). A labeling of G is a mapping from the set of vertices, edges, or both vertices and edges to a set of labels. In most applications labels are positive (or nonnegative) integers, though in general real numbers could be used.

Graph labelings have proved to be useful in a number of applications. The various labelings are obtained based on requirements put on the labelings. Among the most common labelings are graceful and harmonious labelings. Graceful labelings are for instance used in decompositions of complete graphs K_{2n+1} into subgraphs with n edges. In graph related problems based on error correcting codes, harmonious labelings are applied.

Magic graph labelings are a natural extension of the well known magic squares and magic rectangles. The origins of the notion of magic squares can be traced back more than 4000 years. The first magic square known to be recorded was a 3 by 3 magic square called *Lo Shu* around 2200 BC. According to the legend about the Chinese Emperor Yu, from the book Yih King, the diagram was found on the shell of a divine turtle (see Figure 1.1).

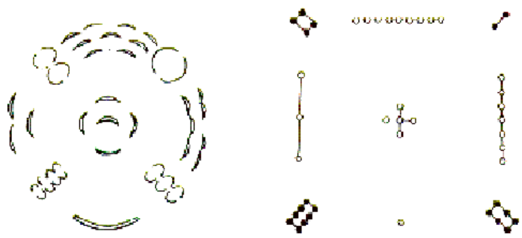


Figure 1.1: *Lo Shu magic square, ancient and newer view.*

The first record of a magic square in Europe is found on a famous engraving from 1514 by the German artist and scientist Albrecht Dürer (see Figure 1.2).

Extending the "magic" idea to graphs we want the sum of labels related to an edge or a vertex to be constant over the graph. The first magic-type labeling was introduced by Sedláček in 1963 (see [Sedl]). He labeled edges of a graph with real numbers and required the sum of labels of all edges incident to a vertex to be constant. Stewart (see [Stew2]) calls a labeling *supermagic* if the labels are integers 1, 2, . . . , $|E|$ and are all different.

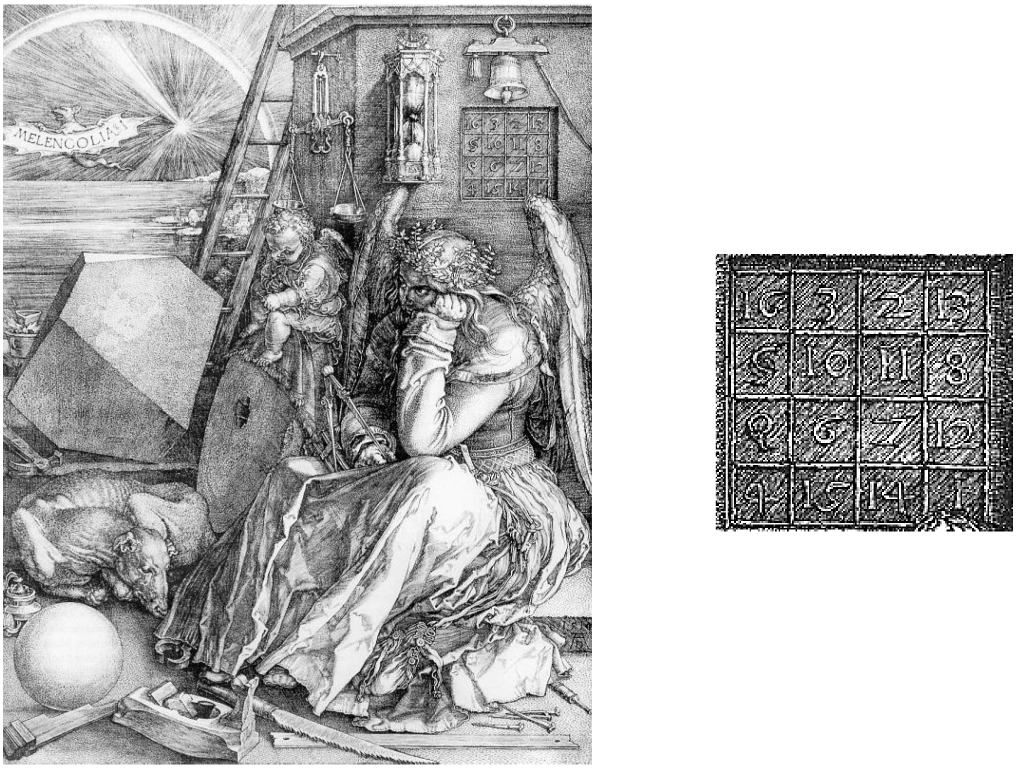


Figure 1.2: *Dürer's "Melancholia" with a magic square in the upper right corner.*

Another labeling was introduced in 1970 by Kotzig and Rosa. They called a labeling *edge magic* if both the edges and the vertices of a graph are labeled by distinct integers 1, 2, ..., $|E| + |V|$ so that the sum of an edge label and the labels of both end vertices does not depend on the edge.

Many other labelings of graphs have been studied since then, including labelings of faces of planar graphs. A unified terminology was proposed by Wallis in [Wal1]. We follow his suggestions in this thesis. If we label just edges, we call the labeling an *edge-labeling.* By labeling just vertices we get a *vertex-labeling.* We call a labeling a *total-labeling* if we label both edges and vertices. If the sum of labels of an edge and both end vertices does not depend on the edge, we call the labeling an *edge-magic type labeling.* If the sum of labels of a vertex and all incident edges is constant, we call the labeling a *vertex-magic type labeling.*

Magic-type labelings are useful when a check sum is required or a look up table has to be avoided. A simple graph can represent a network with nodes and links with addresses (labels) assigned to both links and nodes. If the addresses form an edge-magic labeling, knowing addresses of two vertices is enough to find the address of their link without a look up table simply by subtracting the addresses from the magic constant. Another application of magic-type labelings is in radar pulse codes where an optimal transmitted pulse-train can be obtained.

There is a large number of articles published on magic-type graph labelings and also one monograph [Wal1]. The best source for information is the dynamic survey by Joseph Gallian [Gal].

1.2 Vertex magic total labelings

The concept of vertex magic total graphs is rather new. It was introduced in 1999 by MacDougall, Miller, Slamin, and Wallis in [DMSW]. As the name suggests, we label both vertices and edges of a given graph and evaluate the sum of labels for each vertex.

By $N(x)$ we denote the set of all vertices adjacent to a given vertex x. We call the vertices the *neighbours* of x and the set $N(x)$ the *neighborhood* of x.

Definition 1.1 *Let $G(V, E)$ be a graph with vertex set V and edge set E. We denote $v = |V|$ and $e = |E|$. A one-to-one mapping $\lambda : V \cup E \to \{1, 2, \ldots, v + e\}$ is called a* vertex magic total labeling (VMT) of G *if there exists a constant k such that for every vertex x of G*

$$\lambda(x) + \sum_{y \in N(x)} \lambda(xy) = k. \tag{1}$$

The sum (1) is called the weight of vertex x in the labeling λ *and denoted by $w_\lambda(x)$. The constant k is the* magic constant *for λ.*

Examples of a VMT labeling are in Figure 1.3 and in Figure 1.4. We call a graph which has a VMT labeling a VMT graph. We use the notation $v = |V|$ and $e = |E|$ through the rest of the thesis.

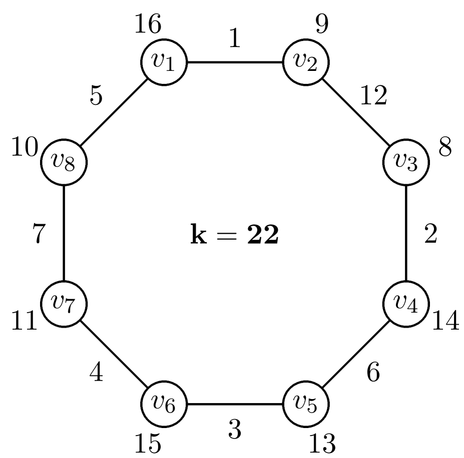


Figure 1.3: *Vertex magic total labeling of C_8 with the magic constant $k = 22$.*

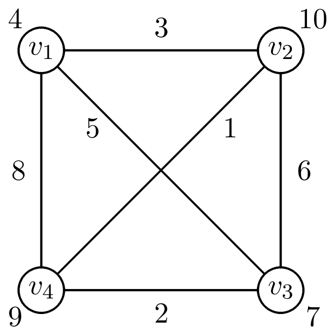


Figure 1.4: *Vertex magic total labeling of K_4 with the magic constant $k = 20$.*

1.3 Summary of known results on VMT labelings

An interesting observation is that for a given graph we can find VMT labelings with different magic constants using the same set of labels (see Figure 1.5). Every vertex label is counted once and every edge label is counted twice, once for each end vertex. Thus, the higher the labels on edges are, the higher the magic constant can be. The set of all possible magic constants is the *spectrum* of the magic labeling problem. A natural question arises: what is the magic spectrum of a given graph.

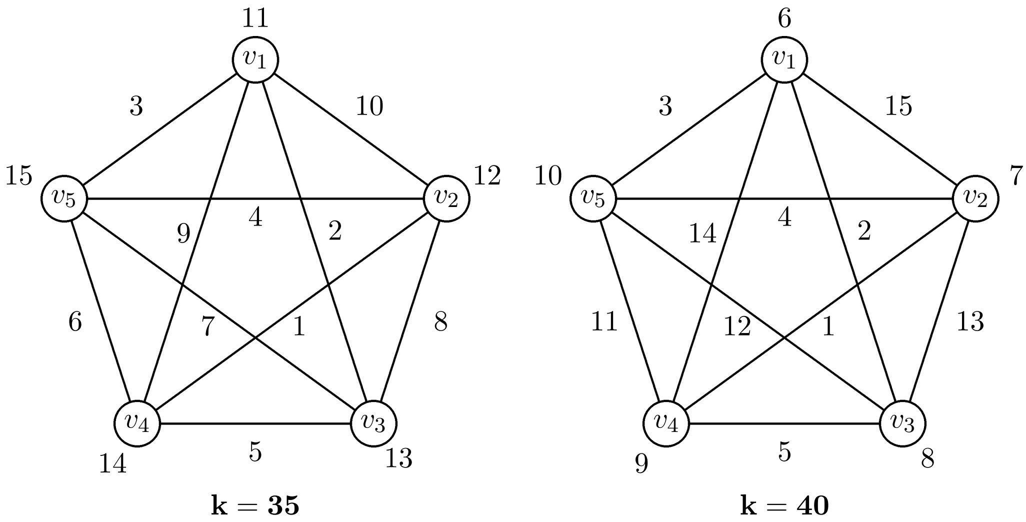


Figure 1.5: *Compare VMT labelings of K_5 with magic constants $k = 35$ and $k = 40$.*

For a regular graph we can find bounds for the magic constant. If we find a VMT labeling in which the edges are labeled with the first e integers, we get the lowest possible magic constant. On the other hand, if we find a VMT labeling in which the vertices are labeled with the first v integers, we obtain the highest magic constant.

$$\frac{\binom{v+e+1}{2}+\binom{e+1}{2}}{v} \leq k \leq \frac{2\binom{v+e+1}{2}-\binom{v+1}{2}}{v}.$$

The computation can be found in [Wal1, DMSW, MsP].

Given a VMT labeling λ of $G(V,E)$, we can create a new labeling by "swapping" the labels. The labeling λ' defined by

$$\begin{aligned} \lambda'(x) &= v+e+1-\lambda(x) \quad && \forall x \in V \\ \lambda'(xy) &= v+e+1-\lambda(xy) \quad && \forall xy \in E \end{aligned}$$

is called the *dual labeling to* λ. Among other results was in [DMSW] shown

Theorem 1.2 *Let λ be a VMT labeling of G. The dual labeling to λ is a VMT labeling if and only if G is a regular graph.*

The dual labeling can have a different magic constant. If G is an r-regular graph with a VMT labeling λ, the magic constant of the dual labeling λ' is

$$k' = (r+1)(v+e+1)-k,$$

where k is the magic constant of λ. The proof is in [DMSW]. Dual labelings are easy to obtain. In a count we do not consider dual labelings as different vertex magic total labelings.

One more general result for regular graphs is shown in [DMSW] and also in [Wal1].

Theorem 1.3 *Let G be an r-regular graph having a VMT labeling λ in which the label 1 is assigned to some edge xy. The graph $G' = G - xy$ has also a VMT labeling.*

The proof is straightforward. In [Wal1] it is shown that the labeling λ' of $G'(V,E)$

$$\begin{aligned} \lambda'(x) &= \lambda(x)-1 \quad && \forall x \in V \\ \lambda'(xy) &= \lambda(xy)-1 \quad && \forall xy \in E \setminus \{xy\} \end{aligned}$$

is a VMT labeling of the graph $G-xy$. The magic constant is $k' = k-(r+1)$. Moreover, in [Wal1, DMSW] the following result is obtained.

Theorem 1.4 *If $G(V,E)$ is an r-regular graph and xy such an edge in E that $G-xy$ has a VMT labeling, then the labeling is derived from a VMT labeling of G by the process described in the proof of Theorem 1.3.*

For $r = 2$ immediately follows:

Corollary 1.5 *Every VMT labeling of P_n is derived from a VMT labeling of C_n.*

There are no sufficient conditions known for a graph to have a VMT labeling, but an unpublished conjecture by MacDougall says that any regular graph other than K_2 or $2K_3$ is vertex magic total (see [McQ]). More concerning the conjecture can be found in Chapter 8.

Graph	*Labeling*	*Notes*
C_n	VMT	[DMSW]
P_n	VMT	$n > 2$ [DMSW]
$K_{m,m}$	VMT	$m > 1$ [DMSW][LM]
$K_{m,m} - e$	VMT	$m > 2$ [DMSW]
$K_{m,n}$	VMT	iff $\lvert m - n\rvert \leq 1$ [DMSW]
	not VMT	if $n > m + 1$
K_n	VMT	for n odd [DMSW]
		for $n \equiv 0 \pmod 4$, $n > 2$ [LM]
		for $n \equiv 2 \pmod 4$, $n > 2$ [DMSW2]
Petersen $P(n, k)$	VMT	for all n and k [BMS]
prisms $C_n \square P_2$	VMT	for all n [SlM]
W_n	VMT	iff $n \leq 11$ [DMSW]
F_n	VMT	iff $n \leq 10$ [DMSW]
friendship graphs	VMT	iff # of triangles ≤ 3 [DMSW]
$G + H$	VMT	$\lvert V(G)\rvert = \lvert V(H)\rvert$
		and $G \cup H$ is VMT [Wal1]
unions of stars	VMT	[Wal1]
tree with n internal vertices		
and more than $2n$ leaves	not VMT	[Wal1]
nG	VMT	n odd, G regular of even
		degree, VMT [Wal2]
nG	VMT	G is regular of odd
		degree, VMT, but not K_1 [Wal2]
$C_n \square C_{2m+1}$	VMT	[FKK1]
$K_5 \square C_{2n+1}$	VMT	[FKK2]
$G \square C_{2n}$	VMT	G $2r + 1$-regular VMT [Kov1]
$G \square K_5$	VMT	G $2r + 1$-regular VMT [Kov1]
$G \square H$	VMT	G r-regular VMT, H $2s$-regular SPM,
		r odd or 4 even and $\lvert H\rvert$ odd [Kov2]
$G \square H$	VMT	G r-regular VMT, H $2s$-regular SPM,
		r odd or r even and $\lvert H\rvert$ odd [Kov2]
two 2-regular graphs joined	VMT	[McQ]
with a perfect matching		

Table 1.1: *Summary of results on vertex magic total labelings.*

The Table 1.1 appeared in the current edition of the survey [Gal]. Preparation of the summary tables was part of a summer project[1].

The table summarizes results on VMT labelings. From the table it is apparent that the existence of VMT labelings is known for most basic families of graphs such as complete graphs, bipartite graphs, cycles, etc. Notice there are only few general results: one result by Wallis concerning VMT labelings of copies of a given VMT graph (see [Wal2]) and a couple of results on Cartesian products of regular graphs. These results are examined in more detail in Section 4.2, Chapter 6 and Chapter 7.

1.4 Supermagic labeling

One of the oldest magic-type labelings is the supermagic labeling introduced by Stewart (see [Stew2]) in 1967. Stewart calls a graph *semi-magic* if there exists a labeling of the edges of the graph with integers such that for every vertex v of the graph the sum of the labels of all edges incident to v equals to the same constant (see [Stew1]). If moreover all the edge labels are distinct positive integers, he calls the labeling *magic*. A special case of a magic labeling is the supermagic labeling.

Definition 1.6 *Let $G(V, E)$ be a graph with vertex set V and edge set E. We denote $e = |E|$. A one-to-one mapping $\lambda : E \to \{1, 2, \ldots, e\}$ is called a* supermagic (SPM) labeling of G *if there exists a constant h such that for every vertex x of G*

$$\sum_{y \in N(x)} \lambda(xy) = h. \tag{2}$$

The constant h is the magic constant *for λ and the sum* (2) *is called the* weight of vertex x in labeling λ *and denoted by $w_\lambda(x)$. We call a graph to be an* SPM graph *if it has an SPM labeling.*

An example of a supermagic labeling is in Figure 1.6.

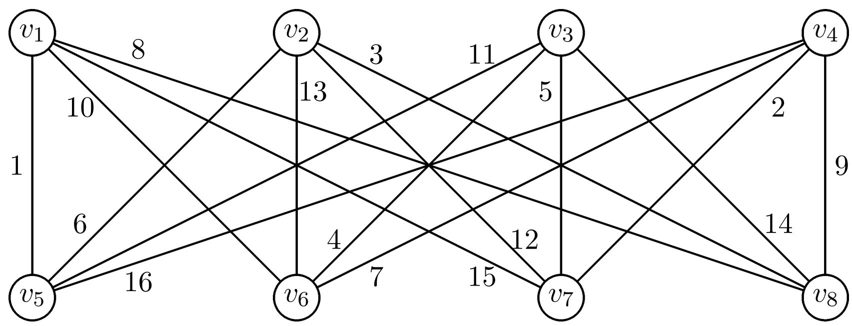


Figure 1.6: *Super magic labeling of $K_{4,4}$ with $h = 34$.*

[1]Funding to include figures, tables and index to survey [Gal] was provided by the University of Minnesota Duluth.

We shall point out that Wallis calls this labeling a *vertex-magic edge-labeling*. In this thesis we use the term *supermagic* introduced by Stewart in [Stew2]. Later, some authors used the term *super* in a different meaning, namely if the set of vertex labels in a total labeling is $\{1, 2, \ldots, |V|\}$. For a detailed overview on labelings we refer to [Gal].

Graph	*Labeling*	*Notes*
K_n	SPM	for $n \geq 5$ iff $n > 5$, $n \not\equiv 0 \pmod 4$ [Stew2]
Möbius ladders M_n	SPM	if $n \geq 3$, n is odd
$C_n \,\square\, P_2$	not SPM	for $n \geq 4$, n even [Sedl]
Composition of C_m and $\overline{K}_n$	SPM	if $m \geq 3$, $n \geq 2$
$K_{\underbrace{n, n, \ldots, n}_{p}}$	SPM	$n \geq 3$, $p > 5$ and $p \not\equiv 0 \pmod 4)$
Composition of r-regular SPM graph and $\overline{K}_n$	SPM	if $n \geq 3$
Composition of K_k and $\overline{K}_n$	SPM	if $k = 3$ or 5, $n = 2$ or n odd
null graph with n vertices		
$mK_{n,n}$	SPM	for $n \geq 2$ iff n is even or both n and m are odd
Q_n	SPM	iff $n = 1$ or $n > 2$ even
$C_m \,\square\, C_n$	SPM	$m = n$ or m, n even [Iv]
$C_m \,\square\, C_n$	SPM ?	for all m and n [Iv]

Table 1.2: *Summary of results on supermagic labelings.*

1.5 Summary of known results on SPM labelings

The classic $n \times n$ magic square corresponds to the supermagic labeling of $K_{n,n}$.

For a given graph G every supermagic labeling has the same magic constant. For an r-regular graph with e edges the magic constant is

$$h = \frac{1}{2} r(1 + e). \tag{3}$$

The dual labeling λ' to an SPM labeling of a regular graph G defined by

$$\lambda'(xy) = e + 1 - \lambda(xy) \quad \forall xy \in E$$

is not of much interest, since it yields the same magic constant.

The Table 1.2 appeared in the recent survey [Gal]. From the table it is apparent that the existence of an SPM labelings is known for most basic families of graphs such as complete graphs, complete multipartite graphs, hypercubes, etc.

Some more general results concerning SPM labelings were published in [Iv]. We prove similar and more general results later in Section 6.2 and in Section 8.2. The question mark after SPM means that the existence of an SPM labeling is conjectured.

1.6 Vertex antimagic total labelings

In general for a magic-type labeling we require the weights of all elements (edges and/or vertices) to be the same. In an antimagic labeling we require the exact opposite: all the weights have to be different. This is not difficult to achieve, thus we put additional restrictions on the labeling. Certainly, most interesting are labelings where the weights form an arithmetic progression. Such labelings were introduced in [BBDMSS] in 2000.

Definition 1.7 *Let $G(V, E)$ be a graph with vertex set V and edge set E. Let $v = |V|$, $e = |E|$. A one-to-one mapping $\lambda : V \cup E \to \{1, 2, \ldots, v+e\}$ is called a* vertex antimagic total labeling of G *if the sums*

$$\lambda(x) + \sum_{y \in N(x)} \lambda(xy) \tag{4}$$

are distinct for all vertices x of G. The sum (4) *is called the* weight of vertex x in labeling λ *and denoted by $w_\lambda(x)$. The labeling λ is called an* (s, d)-vertex antimagic total labeling (VAMT) of G *if the weights* (4) *form an arithmetic progression*

$$s, s + d, \ldots, s + (v - 1)d.$$

Not only are antimagic labelings interesting on their own, but they are a useful tool for constructing vertex magic total (VMT) labelings of products of cycles and other regular graphs, see Chapter 4.

1.7 Summary of known results on VAMT labelings

In [BBDMSS] the followings results are shown:

Theorem 1.8 *The dual labeling to an (s, d)-VAMT labeling of a graph G is an (s', d)-VAMT labeling for some s' if and only if G is regular.*

The lowest weight of the dual labeling is

$$s' = (r + 1)(e + v + 1) - s - (v - 1)d.$$

In [BBDMSS] also a similar result to Theorem 1.3 was shown:

Theorem 1.9 *Let G be an r-regular graph with a total labeling in which some edge xy receives the label 1. Then G has an (s,d)-VAMT labeling if and only if $G - xy$ has an (s',d)-VAMT labeling with $s' = s - (r+1)$.*

This theorem allows us to find a VAMT labeling of the cycle C_n given a VAMT labeling of a path P_n. The other way works only if the label 1 happens to be assigned to some edge. In Chapter 2 we focus on (s,d)-vertex antimagic total labelings of cycles, but using Theorem 1.9 most of the results can be rephrased for paths.

Another set of results in [BBDMSS] is focused on the relationship between various magic-type labelings.

Theorem 1.10 *Every supermagic graph G has an $(s,1)$-VAMT labeling.*

Theorem 1.11 *Let G be a graph with a total labeling whose vertex labels constitute an arithmetic progression with difference d. Then G has a VMT labeling with constant k if and only if G has an $(s,2d)$-VAMT labeling where $s = k + (1-v)d$.*

Theorem 1.12 *Every (s,d)-antimagic graph has an $(s+e+1, d+1)$-VAMT labeling and an $(s+e+v, d-1)$-VAMT labeling,*

(s,d)-antimagic labelings of prisms were studied in [BH]. In [BBDMSS] an open problem was posted: are there other ways how to obtain an (s,d)-VAMT labeling from a VMT labeling besides the construction in [BBDMSS]? One solution of the problem was given in [MsP].

Theorem 1.13 *Let G be a graph on an odd number of vertices $n = 2r+1$ that has a vertex magic total labeling with a magic constant k such that the vertex labels constitute an arithmetic progression with difference d starting with s. Then G has an $(k - \frac{n-1}{2}, d)$-vertex antimagic total labeling.*

Proof. See [MsP]. □

Theorem 1.13 applies to the case that n is odd. The approach presented in Theorem 1.13 cannot be used for n even as follows from the next theorem.

Theorem 1.14 *Let G be a graph on an even number of vertices $n = 2r$. Suppose G has a vertex magic total labeling λ with a magic constant k such that the vertex labels constitute an arithmetic progression with difference d starting with s. Then no permutation of vertex labels given by the labeling λ yields an (s',d)-vertex antimagic total labeling for some s'.*

Proof. See [MsP]. □

The following Table 1.2 appeared in the recent survey [Gal]. From the table it is apparent that the existence of an (s, d)-VAMT labelings is known for some basic families of graphs such as paths, cycles, etc.

Difference	*Labeling*	*Notes*
C_n	(s, d)-VAMT	wide variety of s and d [BBDMSS]
P_n	(s, d)-VAMT	wide variety of s and d [BBDMSS]
$K_{n,n}$	$(s, 1)$-VAMT	$n \geq 3$ [BBDMSS]
A class of quartic graphs R_n	$(s, 1)$-VAMT	[BHL]
generalized Petersen graph $P(n, k)$	(s, d)-VAMT	[MB]
prisms $C_n \square P_2$	(s, d)-VAMT	[BBDMSS] [BH]
antiprisms	(s, d)-VAMT	
W_n	not (s, d)-VAMT	for $n > 20$

Table 1.3: *Summary of results on vertex antimagic total labelings.*

There are several more antimagic-type labelings, such as (s, d)-edge antimagic total labelings or (s, d)-face antimagic labelings. In this thesis we focus only on (s, d)-vertex antimagic total labelings.

2 On (s, d)-vertex antimagic total labelings of cycles

In this chapter we focus on vertex antimagic total labelings of cycles. After a brief summary of results we give a general construction of vertex antimagic total labelings with difference 1 and difference 2.

2.1 (s, d)-VAMT labelings of cycles

Vertex antimagic total labelings of cycles were studied in [BBDMSS] and [MsP].

It is easy to see that using the first $2n$ integers to label vertices and edges of a cycle, one can achieve vertex antimagic total labelings with only a limited set of differences. Since every weight is obtained as the sum of three labels, the lowest possible weight is 6 and the largest possible label is $3n-3$. This observation is a special case of the following theorem.

Theorem 2.1 *Let G be a graph with smallest vertex degree δ and largest vertex degree Δ. Let λ be an (s, d)-vertex antimagic total labeling of G. Then*

(1) $d \leq \frac{(\Delta+1)(2(v+e)-\Delta)-(\delta+1)(\delta+2)}{2(v-1)}$

(2) $\frac{(\delta+1)(\delta+2)}{2} \leq s \leq \frac{1}{2}(\Delta+1)(2(v+e)-\Delta)-(v-1)d.$

The proof of the theorem can be found in [BBDMSS]. However, there was a mistake in the computation of the upper bound of w_{max} and thus the bound on d was not correct. The correct statement and the proof can be also found in [MsP].

Substituting $\delta = \Delta = r$ in Theorem 2.1 we obtain the following corollary.

Corollary 2.2 *Let G be an r-regular graph. If there exists an (s, d)-vertex antimagic total labeling of G, then*

(1) $d \leq \frac{(r+1)(v+e-r-1)}{v-1}$

(2) $\frac{(r+1)(r+2)}{2} \leq s \leq \frac{1}{2}(r+1)(2(v+e)-r)-(v-1)d.$

Moreover, for a cycle is $r = 2$ and evaluating $d \leq \frac{3(2n-3)}{n-1} = 6 - \frac{3}{n-1}$ we get the following corollary.

Corollary 2.3 *For $d \geq 6$, there exists no (s, d)-vertex antimagic total labeling of C_n where the vertex labels and the edge labels are $1, 2, \ldots, 2n$.*

As mentioned in Section 1.6 we can for regular graphs flip the labels: the highest with the lowest, the second highest with the second lowest and so on. The resulting labeling, called the dual labeling, will again have the magic properties. We do not consider dual labelings as separate results, as their existence is apparent.

Graph	*Labeling*	*Notes*
d=1	two $(a+2b+n-1,1)$-VAMT labelings for any n	[BBDMSS], [MsP]
	one $(a+2b+n-1,1)$-VAMT labeling for n odd	[MsP]
d=2	one $(a+2b+2(n-1),2)$-VAMT for any n	[BBDMSS], [MsP]
	one $(a+2b+(n-1)/2,2)$-VAMT for n odd	[BBDMSS], [MsP]
	one $(a+2b-(n+1)/2,2)$-VAMT for n odd	[BBDMSS]
	two $(a+2b+2(n-1),2)$-VAMT for n odd	[MsP]
d=3	one $(a+2b+1,3)$-VAMT for any n	[BBDMSS], [MsP]
	one $(a+2b+n-1,3)$-GVAMT for n odd	[MsP][2]
	one $(a+2b+(n-1)/2,3)$-GVAMT for n odd	[MsP][2]
d=4	one $(a+2b+n-1,4)$-VAMT for n odd	[MsP]
d=5	only small cycles	[MsP]

Table 2.1: *All known different types of (s,d)-vertex antimagic total labelings of cycles.*

The Table 2.1 summarizes all known types of antimagic total labelings of cycles. We consider only cases $d>0$. The constant a is the lowest vertex label and b is the lowest edge label. For $d=0$ we obtain a special type of an (s,d)-VAMT labeling, namely the VMT labeling with the magic constant s.

The notion of VMT and VAMT labelings is generalized in Chapter 3. The primary goal of the generalization is in constructing magic-type labelings of large graphs based on magic labelings of smaller graphs, see Chapter 4.

2.2 Large class of $(s,1)$-VAMT labelings of cycles

Theorem 2.4 *Let n be an integer at least 3. There exist $(n-1)!$ nonisomorphic $(3n+2,1)$-VAMT labelings of C_n where $1,2,\ldots,n$ are the vertex labels and $n+1,n+2,\ldots,2n$ are the edge labels.*

Proof. Let C_n be a graph with vertices $v_1,v_2,\ldots,v_n$, $n\geq 3$, and edges v_iv_{i+1} for $i=1,2,\ldots,n$ where subscripts are taken modulo n if necessary. We use the notation v_0 and v_n interchangeably. Let $a_1,a_2,\ldots,a_n$ be a permutation of integers $1,2,\ldots,n$. Consider the following labeling

$$\begin{aligned}\lambda(v_i) &= a_i\\ \lambda(v_iv_{i+1}) &= 2n+1-a_i\end{aligned}\tag{5}$$

where $i=1,2,\ldots,n$. We prove that λ is a $(3n+2,1)$-vertex antimagic total labeling of C_n for any n (see Figure 2.1).

[2]These two cases consider only *generalized* $(s,d)-VAMT$, see Chapter 3.

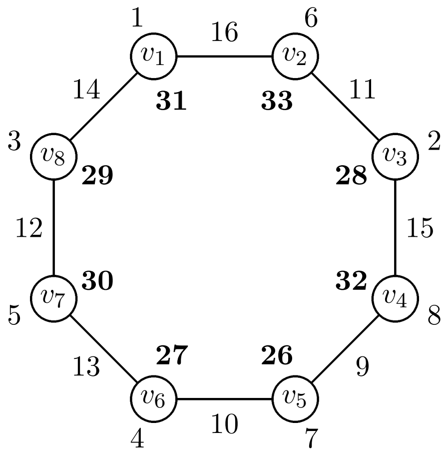


Figure 2.1: *A* $(26, 1)$*-vertex antimagic total labeling of* C_8.

This fact can be verified easily. Evaluating $w_\lambda(v_i)$ for any i we have

$$\begin{aligned} w_\lambda(v_i) &= \lambda(v_i) + \lambda(v_{i-1}v_i) + \lambda(v_i v_{i+1}) \\ &= a_i + 2n + 1 - a_{i-1} + 2n + 1 - a_i \\ &= 4n + 2 - a_{i-1}. \end{aligned}$$

Since the set of labels a_i is $\{1, 2, \ldots, n\}$ is the set of weights $\{3n+2, 3n+3, \ldots, 4n+1\}$. Obviously λ is a bijection to $\{1, 2, \ldots, 2n\}$ and thus λ is a $(3n + 2, 1)$-VAMT labeling. Since there are $\frac{(n-1)!}{2}$ different nonisomorphic permutations of vertex labels, we have $\frac{(n-1)!}{2}$ different nonisomorphic $(3n + 2, 1)$-VAMT labelings of C_n with the vertex labels $1, 2, \ldots, n$ and the edge labels $n + 1, n + 2, \ldots, 2n$.

One can show in a similar way that also the labeling

$$\begin{aligned} \lambda(v_i) &= a_i \\ \lambda(v_i v_{i+1}) &= 2n + 1 - a_{i+1} \end{aligned} \tag{6}$$

where $i = 1, 2, \ldots, n$, yields $\frac{(n-1)!}{2}$ different $(3n+2, 1)$-vertex antimagic total labelings of C_n for any n (see Figure 2.2). Subscripts are taken modulo n if necessary. The labelings given by (5) are different from labelings given by (6), thus we have $(n - 1)!$ different $(3n + 2, 1)$-vertex antimagic total labelings of C_n for any n. □

A natural question arises: how many different (s, d)-vertex antimagic total labelings of cycles there are: This question is in general difficult and there are no results published on this topic. The only counting result was published in [DMSW]. The authors used a computer for an exhaustive search for VMT labelings of K_5 and K_6.

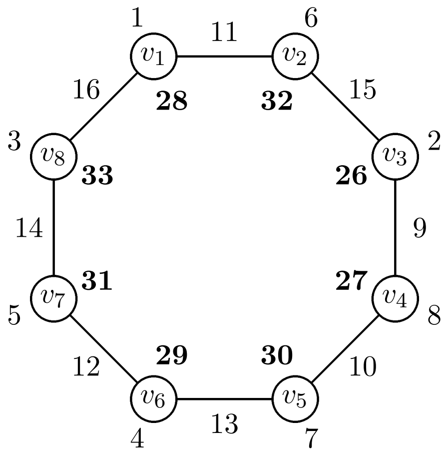
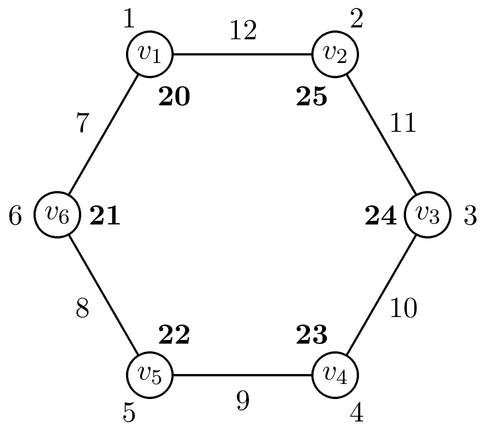


Figure 2.2: *Another* $(26, 1)$*-vertex antimagic total labeling of* C_8*.*

For simplicity we can focus only on the $(3n + 2, 1)$-VAMT labelings of C_n where the set of vertex labels has to be $\{1, 2, \ldots, n\}$. Even in this case the constructions given in Theorem 2.4 do not cover all possibilities. Comparing the two labelings of C_6 given in Figure 2.3 we see the labeling on the left corresponds to the first construction given in Theorem 2.4, but the labeling on the right is a different one. Moreover, notice that in both labelings of C_6 the vertices have the same weights on corresponding vertices, yet we cannot find a second labeling with the same vertex labels and weights as in the labeling in Figure 2.1. This is easy to see, since the vertex v_1 has label 1 and weight 31. To obtain the same weight, we have to label the adjacent edges 14 and 16. By labeling the edge v_1v_2 with the label 16, the remaining labels are forced (this yields the labeling in Figure 2.1). By labeling the edge v_1v_2 with the label 14, we cannot keep the vertex labels and vertex weights, which one can easily verify by trying to do so.



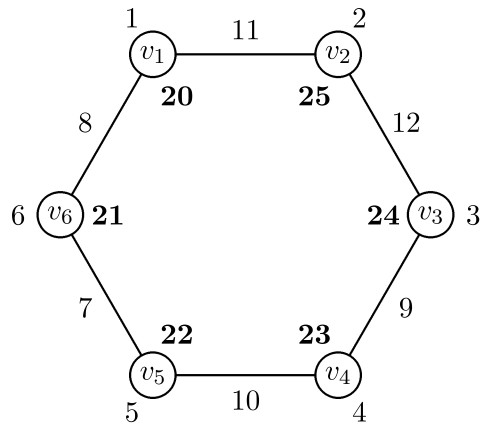


Figure 2.3: *Comparing two* $(20, 1)$*-vertex antimagic total labeling of* C_6*.*

2.3 Large class of $(s, 2)$-VAMT labelings of cycles

Based on the ideas from the proof of Theorem 2.4 we can state a similar result for vertex antimagic total labelings with difference 2.

Theorem 2.5 *Let n be an integer at least 3. There exist $(n-1)!$ nonisomorphic $(2n+3, 2)$-VAMT labelings of C_n where $1, 3, \dots, 2n-1$ are the vertex labels and $2, 4, \dots, 2n$ are the edge labels.*

Proof. Let C_n be a graph with vertices $v_1, v_2, \dots, v_n$, $n \geq 3$, and edges $v_i v_{i+1}$ for $i = 1, 2, \dots, n$ where subscripts are taken modulo n if necessary. We use the notation v_0 and v_n interchangeably. Let $a_1, a_2, \dots, a_n$ be a permutation of the integers $1, 3, 5, \dots, 2n-1$. Consider the following labeling

$$\begin{aligned} \lambda(v_i) &= a_i \\ \lambda(v_i v_{i+1}) &= 2n + 1 - a_i \end{aligned}$$

where $i = 1, 2, \dots, n$. Again it is very easy to prove that λ is a $(2n+3, 2)$-vertex antimagic total labeling of C_n for any n (see Figure 2.4).

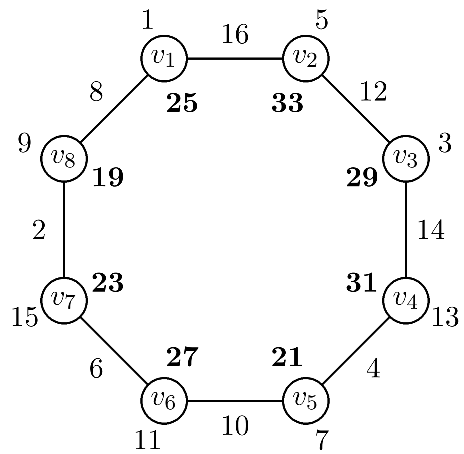


Figure 2.4: *A* $(19, 2)$*-vertex antimagic total labeling of* C_8.

By evaluating $w_\lambda(v_i)$ for every $i = 1, 2, \dots, n$ we have

$$\begin{aligned} w_\lambda(v_i) &= \lambda(v_i) + \lambda(v_{i-1} v_i) + \lambda(v_i v_{i+1}) \\ &= a_i + 2n + 1 - a_{i-1} + 2n + 1 - a_i \\ &= 4n + 2 - a_{i-1}. \end{aligned}$$

Since the set of labels a_i is $\{1, 3, \dots, 2n-1\}$ the set of weights is $\{2n+3, 2n+5, \dots, 4n+1\}$. Obviously λ is a bijection to $\{1, 2, \dots, 2n\}$ and thus λ is a $(2n + 3, 2)$-VAMT labeling for any permutation of vertex labels.

Similarly as in Theorem 2.4 we can consider the second labeling (6) and obtain a total of $(n-1)!$ different nonisomorphic $(2n+3, 2)$-vertex antimagic total labelings of C_n for any n. □

Trying to find the number of all different $(2n+3, 2)$-vertex antimagic total labelings of C_n is even more difficult than the problem of finding all $(3n+2, 1)$-vertex antimagic total labelings of C_n in Section 2.2, since we can have labelings with different sets of vertex and edge labels which yield the same type of antimagic labeling. Notice that in Figure 2.5 we have a different set of vertex labels than in Figure 2.4.

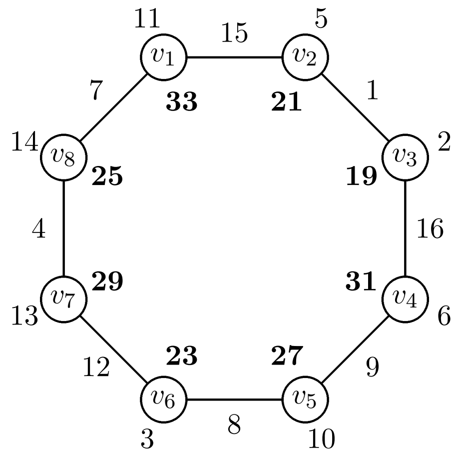


Figure 2.5: *Another* $(19, 2)$*-vertex antimagic total labeling of* C_8*.*

2.4 On $(s, 1)$-VAMT and $(s, 2)$-VAMT labelings of 2-regular graphs

We can extend the ideas from the proofs of Theorems 2.4 and 2.5 to find an $(s, 1)$-VAMT labeling for any 2-regular graph.

Theorem 2.6 *Let G be a 2-regular graph on n vertices. There exists a $(3n+2, 1)$-VAMT labeling of G where $1, 2, \ldots, n$ are the vertex labels and $n+1, n+2, \ldots, 2n$ are the edge labels.*

Proof. We denote the vertices of G by $v_1, v_2, \ldots, v_n$. Because G is a 2-regular graph, G is a collection of cycles. We order the cycles and choose their orientation arbitrarily. By $v_j = \text{out}(v_i)$ we denote the neighbor of v_i on the outgoing edge and by $v_k = \text{in}(v_i)$ we denote the neighbor of v_i on the incoming edge. Every vertex v_i has exactly one "outgoing neighbor" $v_j = \text{out}(v_i)$ and one "incoming neighbor" $v_k = \text{in}(v_i)$, where $j, k \in \{1, 2, \ldots, i-1, i+1, \ldots, n\}$ and $j \neq k$. Consider the labeling

$$\begin{aligned} \lambda(v_i) &= i \\ \lambda(v_i \text{out}(v_i)) &= 2n+1-i \end{aligned}$$

where $i = 1, 2, \ldots, n$ and subscripts are taken modulo n if necessary. We use the notation v_0 and v_n interchangeably. By evaluating the weight $w_\lambda(v_i)$ for every $i = 1, 2, \ldots, n$ we have

$$\begin{aligned} w_\lambda(v_i) &= \lambda(v_i) + \lambda(\mathrm{in}(v_i)v_i) + \lambda(v_i\mathrm{out}(v_i)) \\ &= i + \lambda(v_k v_i) + 2n + 1 - i \\ &= 4n + 2 - k. \end{aligned}$$

where $v_k = \mathrm{in}(v_i)$. Since every vertex v_k for $k \in \{1, 2, \ldots, n\}$ is the "incoming vertex" for some vertex v_i the set of weights is $\{3n+2, 3n+3, \ldots, 4n+1\}$ and thus λ is a $(3n+2, 1)$-vertex antimagic total labeling of G. □

One can show in a similar way that also the labeling

$$\begin{aligned} \lambda(v_i) &= i \\ \lambda(v_i\mathrm{in}(v_i)) &= 2n + 1 - i \end{aligned}$$

where $i = 1, 2, \ldots, n$, yields a $(3n+2, 1)$-VAMT labelings of G for any n.

We can denote the vertices of G by $v_1, v_2, \ldots, v_n$ in any order. For convenience we consider $v_n = v_0$. Counting of different nonisomorphic $(3n+2, 1)$-VAMT labelings of a 2-regular graph G turns into counting the number of isomorphisms of G. Let m be the number of different lengths of cycles in G. By n_s we denote the number of cycles of length l_s in G for $s = 1, 2, \ldots, m$. We can distribute the vertex labels over the vertices of G in $n!$ ways. The number of different nonisomorphic $(3n+2, 1)$-VAMT labelings of G is at least

$$\frac{n!}{\prod_{s=1}^{m} (l_s^{n_s})}.$$

We can state a similar result also for VAMT labelings with difference 2.

Theorem 2.7 *Let G be a 2-regular graph on n vertices. There exists a $(2n+3, 2)$-VAMT labelings of G where $1, 3, \ldots, 2n-1$ are the vertex labels and $2, 4, \ldots, 2n$ are the edge labels.*

Proof. The proof is analogous to the proof of Theorem 2.6. Consider the labeling

$$\begin{aligned} \lambda(v_i) &= 2i - 1 \\ \lambda(v_i\mathrm{out}(v_i)) &= 2n + 2 - 2i \end{aligned}$$

or the labeling

$$\begin{aligned} \lambda(v_i) &= 2i - 1 \\ \lambda(v_i\mathrm{in}(v_i)) &= 2n + 2 - 2i \end{aligned}$$

where $i = 1, 2, \dots, n$ and subscripts are taken modulo n if necessary. By evaluating the weight of every vertex v_i it follows that λ is a $(2n + 3, 2)$-vertex antimagic total labeling of G. □

Example 2.8 Let $G = C_6 + C_7$ be a graph which consists of two disjoint cycles C_6 and C_7. We can construct a $(41, 1)$-VAMT labeling of G using Theorem 2.6, see Figure 2.6.

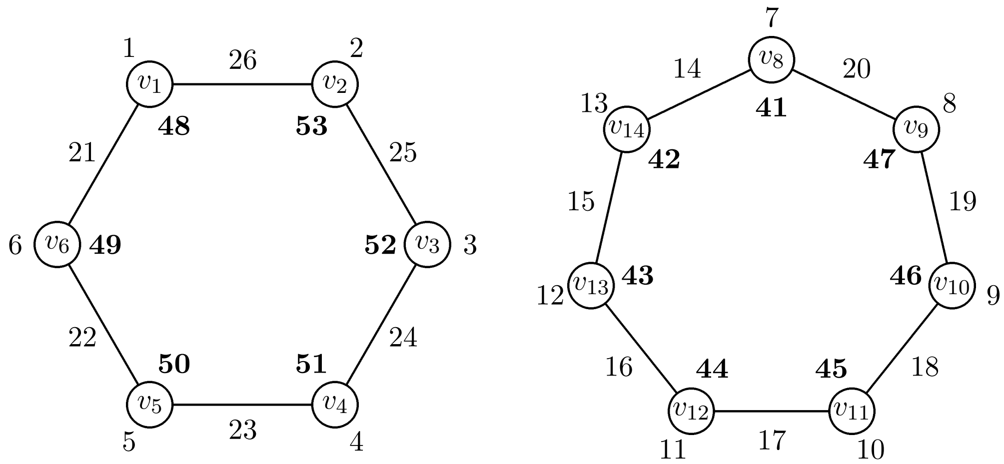


Figure 2.6: *A* $(41, 1)$*-vertex antimagic total labeling of* $C_6 + C_7$*.*

Another example of an $(s, 1)$-vertex antimagic total labeling is given in Example 8.22.

2.5 Remarks

Theorem 2.4 and Theorem 2.5 give a new angle of view on VAMT labelings of cycles. Now we know not only that there are "many" different (s, d)-vertex antimagic total labelings of cycles, but also that the number of different labelings grows at least with the factorial of n.

Another importance of the constructions given in the proofs of the two theorems is that they are used in constructions of other types of magic labelings. Generalized vertex antimagic total labelings are often used as building blocks for vertex magic total labelings. In larger graphs the weights of vertices can be viewed as vertex labels. The result from Theorem 2.4 can be rephrased as:

> Given any labeling of vertices of a cycle C_n using a set of consecutive integers, one can complete the labeling by labeling edges using also consecutive integers in such manner that we obtain an (s, d)-vertex antimagic total labeling of C_n.

According to Theorems 2.6 and 2.7 similar claim is true for any 2-regular graph.

3 Generalized magic-type labelings

The concept of generalized vertex magic total labeling and generalized (s,d)-vertex antimagic total labeling was introduced in [MsP]. A magic-type labeling of a small graph G can be used to find a magic labeling of the same or of a different magic-type of a larger graph. There are two basic approaches.

Based on a certain labeling of G one can obtain a labeling of a graph which consists of copies of G. This approach is described in Section 4.2 and Chapter 6. Obviously in each copy different labels have to be used and none of the copies can be labeled using just the original labels.

Another approach how to obtain magic-type labelings is to decompose the graph G into certain factors and based on a magic-type labeling for each of the factors one can find a magic labeling of G. This method is described in Chapter 4 and Chapter 7.

In either case we need to extend the notion of a vertex magic total labeling. Instead of taking a bijection from the set $V \cup E$ to the set of integers $\{1, 2, \ldots, v+e\}$, we take an injection to the set of all (usually positive) integers.

3.1 Definitions

Definition 3.1 *Let $G(V,E)$ be a graph with vertex set V and edge set E. An injection $\lambda : V \cup E \to \mathbf{Z}$ is called a* generalized vertex magic total labeling (GVMT) of G *if*

$$w_\lambda(x) \quad = \quad \lambda(x) + \sum_{y \in N(x)} \lambda(xy) = k \quad \forall x \in V.$$

The constant k is the magic constant *for λ.*

D. McQuillan introduced in [McQ] another generalization of vertex magic total labelings, called the *magic numbering*, in which he relaxed the injective property. Let n be an integer. A magic numbering λ is any mapping, not just a bijection, from the set of vertices and edges to the set $\{1, 2, \ldots, n\}$ with all vertex weights equal to a constant k. As in the case of generalized vertex magic labelings, magic numberings are used as a tool for constructing vertex magic total labelings of regular graphs.

Similarly we define a generalized supermagic labeling and generalized vertex antimagic total labeling.

Definition 3.2 *Let $G(V,E)$ be a graph with vertex set V and edge set E. An injection $\lambda : E \to \mathbf{Z}$ is called a* generalized supermagic labeling (GSPM) of G *if*

$$w_\lambda(x) \quad = \quad \sum_{y \in N(x)} \lambda(xy) = h \quad \forall x \in V.$$

The constant h is the magic constant *for λ.*

Definition 3.3 *Let $G(V,E)$ be a graph with vertex set V and edge set E. An injection $\lambda : V \cup E \to \mathbf{Z}$ is called a* generalized (s,d)-vertex antimagic total labeling (GVAMT) of G *if the sums*

$$w_\lambda(x) \quad = \quad \lambda(x) + \sum_{y \in N(x)} \lambda(xy) \quad \forall x \in V$$

form an arithmetic progression $s, s+d, \ldots, s+(v-1)d$ for some integers s and d.

The generalized vertex magic total labelings of cycles were studied in [MsP], see Table 2.1.

3.2 Basic transformations

According to the definition VMT and VAMT labelings are special cases of GVMT and GVAMT labelings. It is not difficult to obtain various different generalized vertex magic total labelings from a given VMT or a GVMT labeling.

Theorem 3.4 *Let $G(V,E)$ be an r-regular graph and let λ be a generalized (s,d)-vertex antimagic total labeling of G. Let λ' be a labeling of G defined by*

$$\begin{aligned} \lambda'(x) &= a\lambda(x) + b \quad \textit{for } x \in V \\ \lambda'(xy) &= a\lambda(xy) + c \quad \textit{for } xy \in E \end{aligned} \tag{7}$$

where a, b, and c are any integers such that λ' is injective. Then λ' is a generalized $(b+cr+as, ad)$-vertex antimagic total labeling of G.

Proof. Evaluating the weight we get

$$\begin{aligned} w_{\lambda'}(x) &= \lambda'(x) + \sum_{y \in N(x)} \lambda'(xy) \\ &= a\lambda(x) + b + \sum_{y \in N(x)} (a\lambda(xy) + c) \\ &= b + cr + a\left(\lambda(x) + \sum_{y \in N(x)} \lambda(xy)\right) \\ &= b + cr + aw_\lambda(x). \end{aligned}$$

Since the weights $w_\lambda(x)$ form an arithmetic progression $s, s+d, \ldots, s+(v-1)d$, the weights $w_{\lambda'}(x)$ form an arithmetic progression $b+cr+as, b+cr+as+ad, \ldots, b+cr+as+(v-1)ad$. □

Taking $d = 0$ in Theorem 3.4 we obtain the following corollary.

Corollary 3.5 *Let $G(V,E)$ be an r-regular graph and let λ be a generalized vertex magic total labeling of G with the magic constant k. Let λ' be a labeling of G defined by*

$$\begin{aligned} \lambda'(x) &= a\lambda(x) + b \quad \textit{for } x \in V \\ \lambda'(xy) &= a\lambda(xy) + c \quad \textit{for } xy \in E \end{aligned}$$

where a, b, and c are any integers such that λ' is injective. Then λ' is a generalized vertex magic total labeling of G with the magic constant $k' = b + cr + ak$.

Notice that from the proof of Theorem 3.4 it is apparent that the corollary is true only for regular graphs, otherwise the weight $w_{\lambda'}(x)$ is a function of the degree of x.

A similar proposition is true for supermagic labelings.

Theorem 3.6 *Let $G(V, E)$ be an r-regular graph and let λ be a generalized supermagic labeling of G with the magic constant h. Let λ' be a labeling of G defined by*

$$\lambda'(xy) = a\lambda(xy) + b \quad \text{for } xy \in E$$

where a and b are any integers such that λ' is injective. Then λ' is a generalized supermagic labeling of G with the magic constant $h' = br + ah$.

Proof. Similar to the proof of Theorem 3.4. □

In this thesis we extend those magic-type labelings, which are used in constructions in later chapters. One can generalize also other magic-type labelings.

Not all generalized magic-type labelings are obtained from magic-type labeling using the process from Theorem 3.4, Corollary 3.5, or Theorem 3.6. We compare the VMT labeling in Figure 1.4 and the GVMT in Figure 3.1. Suppose the GVMT labeling in Figure 3.1 is obtained from a VMT labeling using Corollary 3.4, then $a = 1$ since the edge labels are consecutive integers. Moreover $b = 0$, since the label 1 is assigned to the edge v_1v_2. Finally, there is no choice left for c.

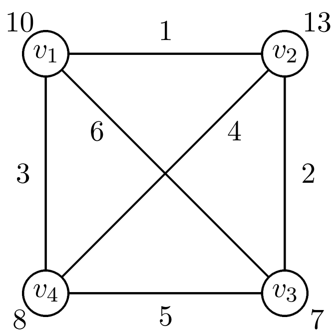


Figure 3.1: *A GVMT of K_4 with the magic constant $k = 20$.*

3.3 Advanced transformations

The use of the theorems from the previous section is limited. The constructions of VMT labelings shown in Chapter 4 and Chapter 6 are more complex. They are not modifying a given VMT labeling of G for the entire graph but of factors of G separately. Hence the following theorem.

Theorem 3.7 *Let $G(V,E)$ be a graph and let λ be a generalized (s,d)-vertex antimagic total labeling of G. Let $H(V,F)$ be an r-regular factor of G. Let λ' be a labeling of G defined by*

$$\begin{aligned} \lambda'(x) &= a\lambda(x)+b \quad \text{for } x \in V \\ \lambda'(xy) &= \begin{cases} a\lambda(xy) & \text{for } xy \in E \setminus F \\ a\lambda(xy)+c & \text{for } xy \in F \end{cases} \end{aligned} \tag{8}$$

where a, b, and c are any integers such that λ' is injective. Then λ' is a generalized $(b+cr+as, ad)$-vertex antimagic total labeling of G.

Proof. Evaluating the weight we get

$$\begin{aligned} w_{\lambda'}(x) &= \lambda'(x) + \sum_{y\in N(x)\cap F} \lambda'(xy) + \sum_{y \in N(x)\cap E\setminus F} \lambda'(xy) \\ &= a\lambda(x)+b+\sum_{y\in N(x)\cap F}(a\lambda(xy)+c) + \sum_{y\in N(x)\cap E\setminus F} a\lambda(xy) \\ &= b+cr+a\left(\lambda(x)+\sum_{y\in N(x)}\lambda(xy)\right) \\ &= b+cr+aw_\lambda(x). \end{aligned}$$

Since the weights $w_\lambda(x)$ form an arithmetic progression $s, s+d, \ldots, s+(v-1)d$, the weights $w_{\lambda'}(x)$ form an arithmetic progression $b+cr+as, b+cr+as+ad, \ldots, b+cr+as+(v-1)ad$. □

By setting $d=0$ we obtain the following.

Corollary 3.8 *Let $G(V,E)$ be a graph and let λ be a generalized vertex magic total labeling of G with the magic constant k. Let $H(V,F)$ be an r-regular factor of G. Let λ' be a labeling of G defined by*

$$\begin{aligned} \lambda'(x) &= a\lambda(x)+b \quad \text{for } x \in V \\ \lambda'(xy) &= \begin{cases} a\lambda(xy) & \text{for } xy \in E \setminus F \\ a\lambda(xy)+c & \text{for } xy \in F \end{cases} \end{aligned} \tag{9}$$

where a, b, and c are any integers such that λ' is injective. Then λ' is a generalized vertex magic total labeling of G with the magic constant $k' = ak+cr+b$.

Again an analogous result for supermagic labelings follows.

Theorem 3.9 *Let $G(V,E)$ be a graph and let λ be an generalized supermagic labeling of G with the magic constant h. Let $H(V,F)$ be an r-regular factor of G. Let λ' be a labeling of G defined by*

$$\lambda'(xy) = \begin{cases} a\lambda(xy) & \text{for } xy \in E \setminus F \\ a\lambda(xy)+b & \text{for } xy \in F \end{cases}$$

where a and b are any integers such that λ' is injective. Then λ' is a generalized supermagic labeling of G with the magic constant $h' = br + ah$.

Proof. Analogous to the proof of Theorem 3.7. □

If a graph has a factorization into regular factors, then by repeating the construction from Theorem 3.7 we obtain the following.

Theorem 3.10 *Let $G(V, E)$ be a graph and let λ be a generalized (s, d)-vertex antimagic total labeling of G. Let $H_1(V, F_1)$ be an r_1-regular factor of G, $H_2(V, F_2)$ be an r_2-regular factor of G, $\dots$, $H_n(V, F_n)$ be an r_n-regular factor of G, where the factors H_1, $\dots$, H_n are edge-disjoint. Let λ' be a labeling of G defined by*

$$\begin{aligned} \lambda'(x) &= a\lambda(x) + b \qquad \text{for } x \in V \\ \lambda'(xy) &= \begin{cases} a\lambda(xy) + c_1 & \text{for } xy \in F_1 \\ a\lambda(xy) + c_2 & \text{for } xy \in F_2 \\ \vdots & \\ a\lambda(xy) + c_n & \text{for } xy \in F_n \\ a\lambda(xy) & \text{otherwise} \end{cases} \end{aligned} \tag{10}$$

where a, b, and c_1, c_2, $\dots$, c_n are any integers such that λ' is injective. Then λ' is a generalized $(b + \sum_{i=1}^{n} r_i c_i + as, ad)$-vertex antimagic total labeling of G.

Proof. By induction on the factors H_1, H_2, $\dots$, H_n.

Let $v = |V|$. We take the generalized VMT labeling λ of G and the r_1-regular factor H_1 and construct a labeling λ_1. According to the proof of Theorem 3.7 (the parameters a and b have the same value in both theorems, $c = c_1$ and $r = r_1$) we see that the weights of vertices in λ_1 constitute an arithmetic progression $b + r_1c_1 + as, b + r_1c_1 + as + ad, \dots, b + r_1c_1 + as + (v-1)ad$. Notice that it may happen that λ_1 is not injective.

For the inductive step ($i = 2, 3, \dots, n$) we take the r_i-regular factor H_i and the labeling λ_{i-1} in which the vertex weights constitute an arithmetic progression $b + \sum_{t=1}^{i-1} c_i r_i + as, b + \sum_{t=1}^{i-1} c_i r_i + as + ad, \dots, b + \sum_{t=1}^{i-1} c_i r_i + as + (v-1)ad$. In the proof of Theorem 3.7 we take $a = 1$, $b = 0$, and $c = c_i$ and we obtain the labeling λ_i in which the weights of vertices constitute an arithmetic progression $b + \sum_{t=1}^{i} c_i r_i + as, b + \sum_{t=1}^{i} c_i r_i + as + ad, \dots, b + \sum_{t=1}^{i} c_i r_i + as + (v-1)ad$.

We have $\lambda' = \lambda_n$. λ_i is not necessarily injective but since the constants a, b, and c_1, c_2, $\dots$, c_n are chosen so that λ' is injective, we have a generalized $(b + \sum_{i=1}^{n} r_i c_i + as, ad)$-vertex antimagic total labeling of G. □

Again the case of generalized VMT labelings is a special case for $d = 0$.

Corollary 3.11 *Let $G(V, E)$ be a graph and let λ be a generalized vertex magic total labeling of G with the magic constant k. Let $H_1(V, F_1)$ be an r_1-regular factor of G,*

$H_2(V, F_2)$ be an r_2-regular factor of G, ..., $H_n(V, F_n)$ be an r_n-regular factor of G, where the factors H_1, ..., H_n are edge-disjoint. Let λ' be a labeling of G defined by

$$\lambda'(x) = a\lambda(x) + b \quad \text{for } x \in V$$

$$\lambda'(xy) = \begin{cases} a\lambda(xy) + c_1 & \text{for } xy \in F_1 \\ a\lambda(xy) + c_2 & \text{for } xy \in F_2 \\ \vdots & \\ a\lambda(xy) + c_n & \text{for } xy \in F_n \\ a\lambda(xy) & \text{otherwise} \end{cases} \tag{11}$$

where a, b, and c_1, c_2, ..., c_n are any integers such that λ' is injective. Then λ' is a generalized vertex magic total labeling of G with the magic constant $k' = ak + \sum_{i=1}^{n} r_i c_i + b$.

We can apply Theorem 3.10 and Corollary 3.11 also in subgraphs of a given graph. A similar result can be obtained for supermagic labelings and for other magic-type labelings.

Theorem 3.12 *Let $G(V, E)$ be a graph and let λ be a generalized supermagic labeling of G with the magic constant h. Let $H_1(V, F_1)$ be an r_1-regular factor of G, $H_2(V, F_2)$ be an r_2-regular factor of G, ..., $H_n(V, F_n)$ be an r_n-regular factor of G, where the factors H_1, ..., H_n are edge-disjoint. Let λ' be a labeling of G defined by*

$$\lambda'(xy) = \begin{cases} a\lambda(xy) + b_1 & \text{for } xy \in F_1 \\ a\lambda(xy) + b_2 & \text{for } xy \in F_2 \\ \vdots & \\ a\lambda(xy) + b_n & \text{for } xy \in F_n \\ a\lambda(xy) & \text{otherwise} \end{cases} \tag{12}$$

where a and b_1, b_2, ..., b_n are any integers such that λ' is injective. Then λ' is a generalized supermagic labeling of G with the magic constant $h' = ah + \sum_{i=1}^{n} r_i b_i$.

Proof. Analogous to the proof of Theorem 3.10. □

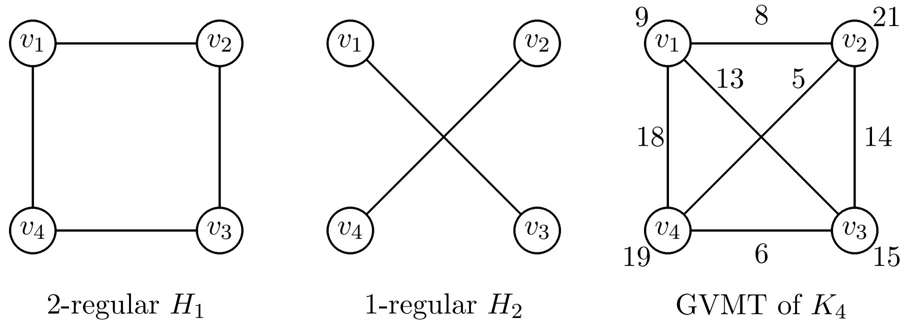


Figure 3.2: *A 2-regular factor H_1, a 1-regular factor H_2 and a generalized vertex magic total labeling of K_4 with the magic constant $k = 48$.*

Example 3.13 We take the VMT labeling of K_4 from Figure 1.4 with the magic constant $k = 20$. We take the factors H_1 and H_2 shown in Figure 3.2. Taking $a = 2$, $b = 1$, $c_1 = 2$, and $c_2 = 3$ in Corollary 3.11 we obtain a GVMT of K_4 with the magic constant $k' = 2 \cdot 20 + (2 \cdot 2 + 1 \cdot 3) + 1 = 48$, see Figure 3.2.

3.4 Remarks

Corollary 3.11 is used to show magic properties of certain labelings of regular graphs in Chapters 4, 6, and 7. Theorem 3.12 is used to construct VMT and SPM labelings of regular graphs in Section 6.2.

Notice that graphs which are not regular but have a magic-type labeling and some regular factors satisfy the conditions of the theorems above, see Figure 3.3. In Corollary 3.8 we picked $a = 2$, $b = 1$, $c = 4$ and the only 1-factor of P_4 with edges v_1v_2 and v_3v_4. From a VMT labeling of P_4 with magic constant $k = 9$ we obtained a GVMT labeling with magic constant $k = 23$.

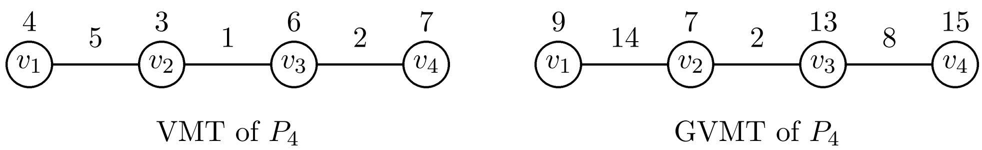


Figure 3.3: *VMT of P_4 with the magic constant $k = 9$ and GVMT of P_4 with the magic constant $k = 23$.*

4 Cartesian products of graphs

4.1 Definition of the Cartesian product of graphs

The Cartesian product of graphs is one of the most common operations on graphs.

Definition 4.1 *Let G and H be graphs with the vertex sets $V(G) = U$ and $V(H) = V$. The* Cartesian product *of graphs G and H is the graph $G \square H$ with the vertex set $V(G \square H) = U \times V$. Two vertices (u, v) and (u', v') are adjacent in $G \square H$ if $u = u'$ and $vv' \in E(H)$ or if $v = v'$ and $uu' \in E(G)$.*

The Cartesian product is called sometimes also the *square product.* The following example shows why.

Example 4.2 The product of P_2 and P_3 is the graph $P_2 \square P_3$ shown in Figure 4.1.

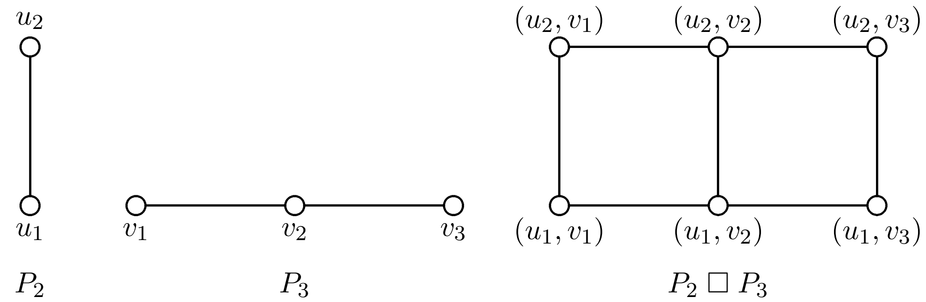


Figure 4.1: *Cartesian product $P_2 \square P_3$.*

In general the product of two paths is a "grid," the product of a cycle and a path is a prism and the product of two cycles is a "torus." Notice, that we can decompose the product $G \square H$ into two factors: one factor consisting of $|H|$ copies of G, the second factor consisting of $|G|$ copies of H.

It was shown in [SlM] that $P_2 \square C_n$ has a VMT labeling. It was shown in [MsP] that $C_{2m+1} \square C_{2n+1}$ has a VMT labeling. This result was extended in [MsT] by finding a VMT labeling for $C_m \square C_{2n}$. In [MsP] were given VMT labelings also for $K_5 \square C_n$ and for $K_5 \square C_{2n}$ (with a different magic constant). In the following sections VMT labelings of products of certain regular graphs are given.

We also deal with vertex magic total labelings of Cartesian products of graphs in Chapter 7.

4.2 Copies of VMT graphs

By nG we denote the graph which consists of n copies of G. For a graph $G(V,E)$ for every $x \in V$ we denote

$$N_E(x) = \{f \in E \mid f \text{ is incident with } x\}\,.$$

Lemma 4.3 *Let n, r be positive integers. Let G be a $(2r+1)$-regular VMT graph which can be factorized into two factors: an $(r+1)$-regular factor and an r-regular factor. Then the graph nG is also a VMT graph.*

Proof. Let $G(V,E)$ be a $(2r+1)$-regular VMT graph with vertices $x_1, x_2, \ldots, x_v$ and edges $f_1, f_2, \ldots, f_e$ where $v = |V|$ and $e = |E|$. We denote the $(r+1)$-regular factor by H_1 and the r-regular factor by H_2. Let λ be a VMT labeling of G with the magic constant k_λ.

Since λ is a VMT labeling of G is $\forall x \in V$

$$\lambda(x) + \sum_{y \in N(x)} \lambda(xy) \;=\; k_\lambda.$$

We construct n copies G_i of the graph G for $i = 1, 2, \ldots, n$. In G_i we denote the copies of vertices by $x_{i,1}, x_{i,2}, \ldots, x_{i,v}$ and edges by $f_{i,1}, f_{i,2}, \ldots, f_{i,e}$ and

$$E_i = \{f_{i,1}, f_{i,2}, \ldots, f_{i,e}\}.$$

Consider the following labeling λ' of nG

$$\begin{aligned} \lambda'(x_{i,j}) &= n(\lambda(x_j) - 1) + i && \text{for } x_{i,j} \in V(nG) \\ \lambda'(f_{i,k}) &= \begin{cases} n(\lambda(e_k) - 1) + (n+1) - i & \text{if } e_k \in H_1 \\ n(\lambda(e_k) - 1) + i & \text{if } e_k \in H_2 \end{cases} \end{aligned} \tag{13}$$

where $i = 1, 2, \ldots n$, $j = 1, 2, \ldots, v$, and $k = 1, 2, \ldots, e$.

First we show that λ' is a bijection. Let x be an element (a vertex or an edge) in G and let a be the label of x assigned by the labeling λ. According to the definition of λ', we label all the n elements in nG corresponding to the element x with distinct labels from the "a-th" n-tuple: $na - n + 1, na - n + 2, \ldots, na$. Since in λ we used all distinct labels $1, 2, \ldots, v + e$, in λ' we used all labels $1, 2, \ldots, n(v + e)$, each of them exactly once.

We use Corollary 3.11 to show that λ' is a vertex magic total labeling of nG with the magic constant $nk_\lambda + (1 - n)(r + 1)$ in every copy of G. We take $a = n$, $b = i - n$, $r_1 = r+1$, $c_1 = 1-i$ (the $(r+1)$-regular factor H_1), $r_2 = r$, and $c_2 = i-n$ (the r-regular factor H_2) and we get

$$\begin{aligned} k' &= nk_\lambda + (r + 1)(1 - i) + r(i - n) - n + i \\ &= nk_\lambda + (r + 1)(1 - n). \end{aligned}$$

Thus, the labeling given by (13) is a vertex magic total labeling of nG with the magic constant $k' = nk_\lambda + (1-n)(r+1)$. □

Note 4.4 There is a similar result due to Wallis (see Lemma 6.1) published in [Wal2]. The result by Wallis is more general because it puts no restriction on the existence of the factors H_1 and H_2 and less restriction on the degree r of G. We mention Lemma 4.3 not only because the construction from the proof yields a different magic constant, but also it helps to understand the construction in Theorem 4.6. A generalization of Lemma 4.3 is in Proof 2 of Lemma 6.1.

Example 4.5 We take the VMT labeling of K_4 with the magic constant $k = 20$ from Figure 1.4 in Section 1.2. K_4 is 3-regular and can be decomposed into a 2-regular factor H_1 (cycle $x_1x_2x_3x_4$) and a 1-regular factor H_2 (two paths x_1x_3 and x_2x_4); see Figure 4.2.

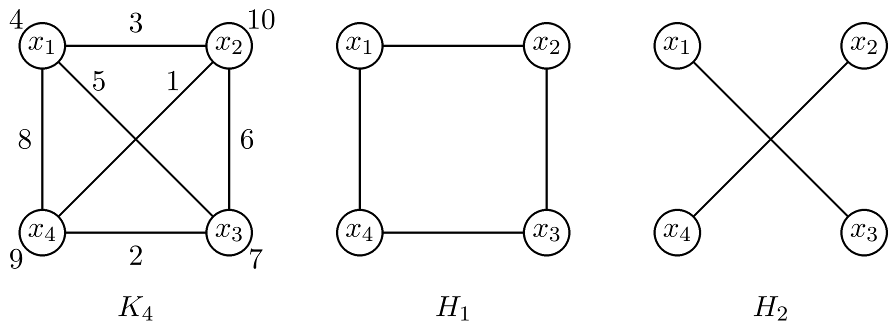


Figure 4.2: *Vertex magic total labeling of K_4 with the magic constant $k = 20$, a 2-regular factor H_1 and a 1-regular factor H_2.*

Using the construction from Lemma 4.3 we get a VMT labeling of $4K_4$. The magic constant is $k_{\lambda'} = 4 \cdot 20 - 3 \cdot 2 = 74$, see Figure 4.3.

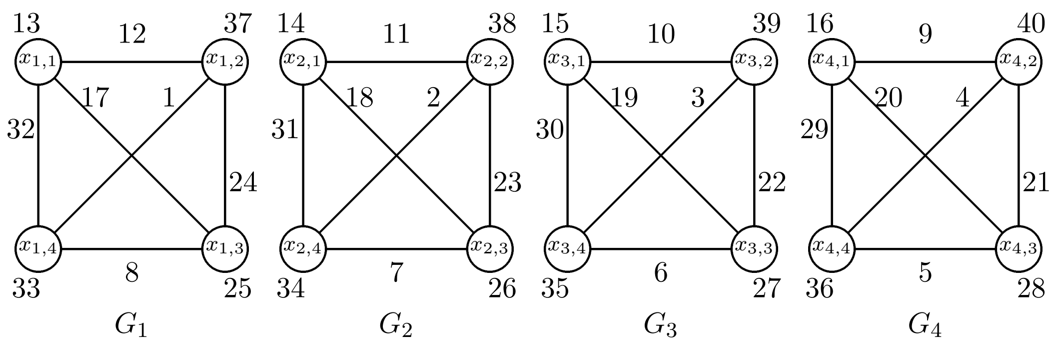


Figure 4.3: *Vertex magic total labeling of $4K_4$ with the magic constant $k = 74$.*

4.3 Products of $2r+1$-regular graphs and even cycles

Theorem 4.6 *Let n, r be positive integers, let $n \geq 3$ be even. Let G be a $(2r+1)$-regular VMT graph which can be factorized into two factors: an $(r+1)$-regular factor and an r-regular factor. Then there exists a vertex magic total labeling of $G \,\square\, C_n$.*

Proof. We can decompose the product $G \,\square\, C_n$ into two factors: one factor H_1 consisting of n copies of G, the second factor H_2 consisting of $|G|$ copies of C_n. We use the same notation as in Lemma 4.3. We denote the edges of copies C_n by $x_{i,j}x_{i+1,j}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, v$ where subscript i is taken modulo n if necessary. For convenience we consider $x_{n,j} = x_{0,j}$.

Consider the following labeling λ' of $G \,\square\, C_n$

$$\begin{aligned}
\lambda'(x_{i,j}) &= \begin{cases} n(\lambda(x_j)-1)+i & \text{if } i=1 \\ n(\lambda(x_j)-1)+(n+2)-i & \text{if } i=2,3,\ldots,n \end{cases} \\
\lambda'(f_{i,k}) &= \begin{cases} n(\lambda(e_k)-1)+(n+1)-i & \text{if } e_k \in H_1 \\ n(\lambda(e_k)-1)+i & \text{if } e_k \in H_2 \end{cases} \\
\lambda'(x_{i,j}x_{i+1,j}) &= \begin{cases} n(v+e)+n(j-1)+i & \text{if } i \text{ is odd} \\ n(v+e)+n(v-j)+i & \text{if } i \text{ is even} \end{cases}
\end{aligned} \tag{14}$$

where $i = 1, 2, \ldots n$, $j = 1, 2, \ldots, v$, and $k = 1, 2, \ldots, e$. We show that λ' is a vertex magic total labeling of nG with the magic constant $k_{\lambda'} = n(k_\lambda + 3v + 2e - r - 1) + r + 2$.

We use the fact that the sums of labels of just edges of G differ from one copy to the other by 1. The vertices and the edges of C_n are labeled using generalized $(s, 1)$-vertex antimagic total labeling. Altogether we obtain a vertex magic total labeling of $G \,\square\, C_n$, see Figure 4.4.

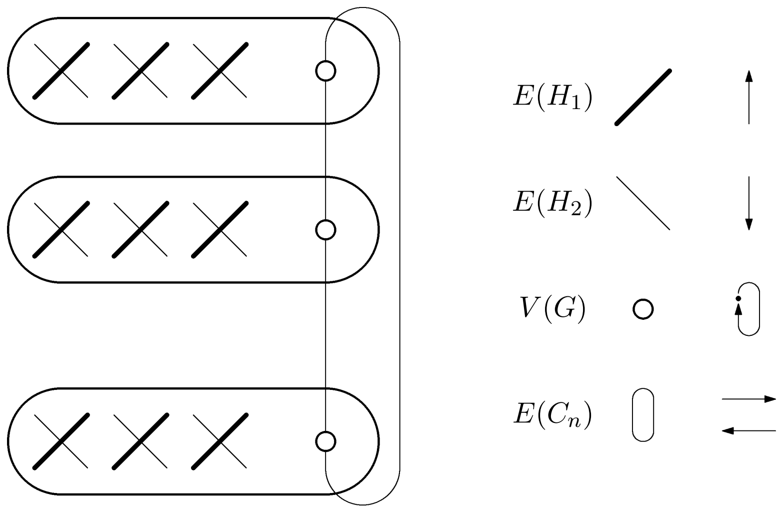


Figure 4.4: *Distribution of labels in $G \,\square\, C_n$.*

The weight of every vertex $x_{i,j}$ for $1 < i \leq n$ odd is

$$\begin{aligned}
w_{\lambda'}(x_{i,j}) &= \lambda(x_{i,j}) + \sum_{f \in N_{E_i}(x_{i,j})} \lambda(f) \\
&= \lambda(x_{i,j}) + \sum_{f \in N_{E_i}(x_{i,j}) \cap E(H_1)} \lambda(f) + \sum_{f \in N_{E_i}(x_{i,j}) \cap E(H_2)} \lambda(f) + \lambda(x_{i-1,j}x_{i,j}) + \lambda(x_{i,j}x_{i+1,j}) \\
&= n \left(\lambda(x_j) + \sum_{f \in N_{E_i}(x_{i,j}) \cap E(H_1)} \lambda(f) + \sum_{f \in N_{E_i}(v_{i,j}) \cap E(H_2)} \lambda(f) \right) + 2 - i + \\
&\quad (r+1)(1-i) + r(i-n) + 2n(v+e) + n(v-j) + (i-1) + n(j-1) + i \\
&= n(k_\lambda + 3v + 2e - r - 1) + r + 2.
\end{aligned}$$

Similarly, for $1 \leq i \leq n$ even we get

$$\begin{aligned}
w_{\lambda'}(x_{i,j}) &= \lambda(x_{i,j}) + \sum_{f \in N_{E_i}(x_{i,j}) \cap E(H_1)} \lambda(f) + \sum_{f \in N_{E_i}(x_{i,j}) \cap E(H_2)} \lambda(f) + \lambda(x_{i-1,j}x_{i,j}) + \lambda(x_{i,j}x_{i+1,j}) \\
&= n \left(\lambda(x_j) + \sum_{f \in N_{E_i}(x_{i,j}) \cap E(H_1)} \lambda(e_k) + \sum_{f \in N_{E_i}(x_{i,j}) \cap E(H_2)} \lambda(e_k) \right) + 2 - i + \\
&\quad (r+1)(1-i) + r(i-n) + 2n(v+e) + n(j-1) + (i-1) + n(v-j) + i \\
&= n(k_\lambda + 3v + 2e - r - 1) + r + 2.
\end{aligned}$$

Finally, if $i = 1$, we get

$$\begin{aligned}
w_{\lambda'}(x_{i,j}) &= \lambda(x_{1,j}) + \sum_{f \in N_{E_1}(x_{1,j}) \cap E(H_1)} \lambda(f) + \sum_{f \in N_{E_1}(x_{1,j}) \cap E(H_2)} \lambda(f) + \lambda(x_{n,j}x_{1,j}) + \lambda(x_{1,j}x_{2,j}) \\
&= n \left(\lambda(x_j) + \sum_{f \in N_{E_1}(x_{1,j}) \cap E(H_1)} \lambda(e_k) + \sum_{f \in N_{E_1}(x_{1,j}) \cap E(H_2)} \lambda(e_k) \right) + \\
&\quad 1 - n + r(1-n) + 2n(v+e) + n(v-j) + n + n(j-1) + 1 \\
&= n(k_\lambda + 3v + 2e - r - 1) + r + 2.
\end{aligned}$$

It is easy to observe that λ' is a bijection to $\{1, 2, \ldots, n(2v+e)\}$. The first $n(v+e)$ labels are used to label vertices and edges of the factor consisting of n copies of G, the next nv labels are used to label the edges of the factor consisting of v even cycles.

Thus the labeling given by (14) is a vertex magic total labeling of $G \,\square\, C_n$ for n even with the magic constant $n(k_\lambda + 3v + 2e - r - 1) + r + 2$. □

Notice that the labels of vertices have changed in comparison to Lemma 4.3. We could have used Corollary 3.11 but the magic constant differs in the factor consisting of n copies of G from one copy of G to the other. Then we would take into count the labels of edges of the factor consisting of copies of C_n. It was easier to evaluate the weight of every vertex.

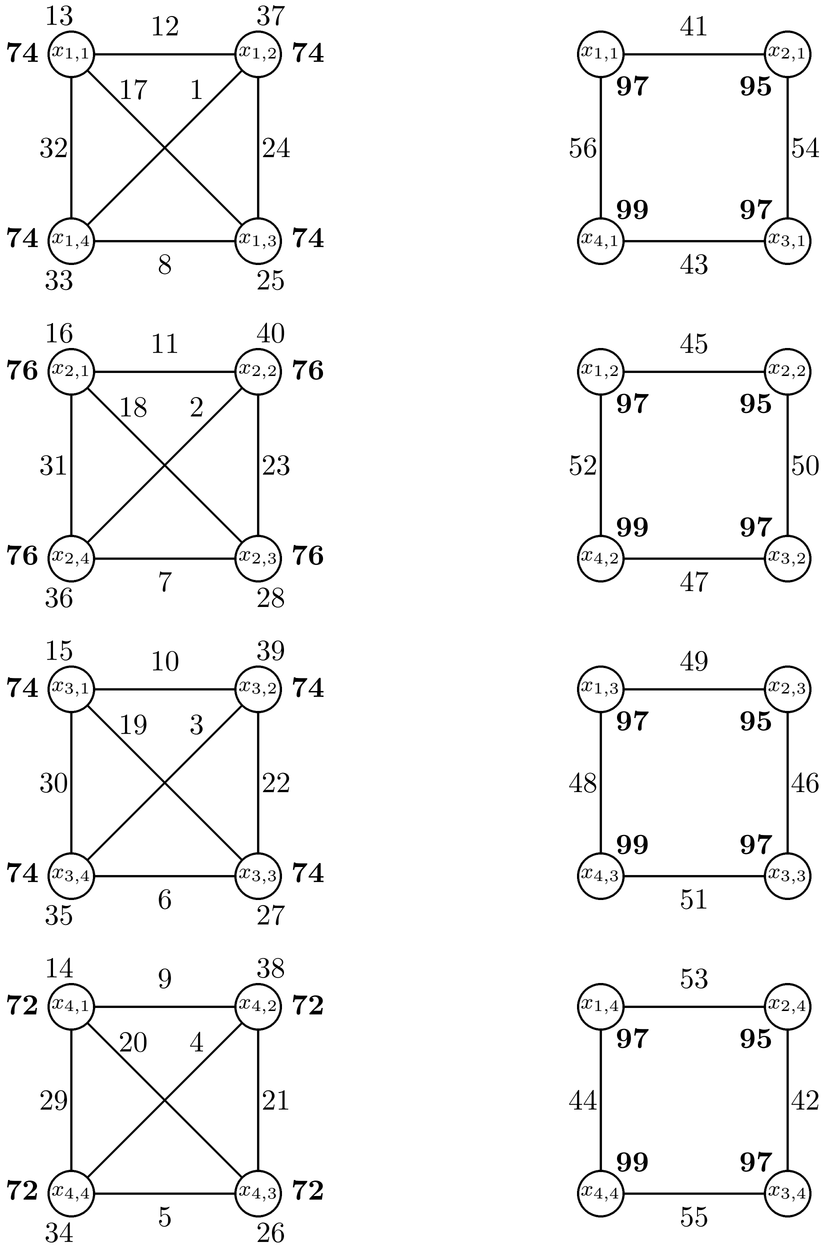


Figure 4.5: *Vertex magic total labeling of $K_4 \square C_4$ with the magic constant $k = 171$.*

Example 4.7 We can find a VMT labeling of $K_4 \square C_4$ using the construction from the proof of Theorem 4.6, see Figure 4.5. We use the VMT labeling and the factors H_1 and H_2 from Figure 4.2.

Evaluating the magic constant for $n = 4$, $v = 4$, $e = 6$, $r = 1$, and $k_\lambda = 20$ we get $k = 4(20 + 3 \cdot 4 + 2 \cdot 6 - 1 - 1) + 1 + 2 = 171$. In Figure 4.5 are the partial sums in the copies of K_4 and of C_4 given in boldface. The partial sums on corresponding vertices have to be added together to obtain the magic constant $k = 171$.

4.4 Products of $2r+1$-regular graphs and K_5

In this section we will take some $(2r+1)$-regular VMT graph G, namely such a graph that can be decomposed into an $(r+1)$-regular and an r-regular factor and give a VMT labeling for the product $G \,\square\, K_5$.

Theorem 4.8 *Let r be a positive integer. Let G be a $(2r+1)$-regular VMT graph which can be factorized into two factors: an $(r+1)$-regular factor and an r-regular factor. Then there exists a vertex magic total labeling of $G \,\square\, K_5$.*

Proof. The proof is very similar to the proof of Theorem 4.6. We use the same notation. We can decompose the product $G \,\square\, K_5$ into two factors: one factor H_1 consisting of five copies of G, the second H_2 factor consisting of $|G|$ copies of K_5. The edges of the copies K_5 we denote by $x_{i,j}x_{k,j}$ for $i,k = 1,2,\ldots,5$, $i \neq k$, and $j = 1,2,\ldots,v$ where subscripts i and k are taken modulo 5 if necessary. For convenience we consider $x_{5,j} = x_{0,j}$.

Consider the labeling

$$\begin{aligned}
\lambda(x_{i,j}) &= 5(\lambda(x_j)-1)+6-i \\
\lambda(f_{i,k}) &= \begin{cases} 5(\lambda(f_k)-1)+6-i & \text{if } f_k \in E(H_1) \\ 5(\lambda(f_k)-1)+i & \text{if } f_k \in E(H_2) \end{cases} \\
\lambda(x_{1,j}x_{2,j}) &= 5(v+e)+10(v-j)+4 \\
\lambda(x_{2,j}x_{3,j}) &= 5(v+e)+10(v-j)+2 \\
\lambda(x_{3,j}x_{4,j}) &= 5(v+e)+10(v-j)+10 \\
\lambda(x_{4,j}x_{5,j}) &= 5(v+e)+10(v-j)+8 \\
\lambda(x_{5,j}x_{1,j}) &= 5(v+e)+10(v-j)+6 \\
\lambda(x_{1,j}x_{3,j}) &= 5(v+e)+10(j-1)+7 \\
\lambda(x_{3,j}x_{5,j}) &= 5(v+e)+10(j-1)+3 \\
\lambda(x_{5,j}x_{2,j}) &= 5(v+e)+10(j-1)+9 \\
\lambda(x_{2,j}x_{4,j}) &= 5(v+e)+10(j-1)+5 \\
\lambda(x_{4,j}x_{1,j}) &= 5(v+e)+10(j-1)+1
\end{aligned} \tag{15}$$

where $i = 1,2,\ldots,5$ and $j = 1,2,\ldots,v$. It is easy to see that the sum of labels of edges *not* contained in the factors H_1 and H_2 is

$$\sum_{\substack{k=1 \\ k\neq i}}^{5} \lambda(x_{i,j}x_{k,j}) = 20(e+v)+20(v-1)+16+2i. \tag{16}$$

The first $5(v+e)$ labels are used to label vertices and edges of the factor consisting of five copies of G, the next $10v$ labels are used to label the edges of the factor consisting of v copies of K_5 (even labels on the edges of the v cycles $x_{1,j}x_{2,j}x_{3,j}x_{4,j}x_{5,j}$ and odd labels on the remaining edges of the cycles $x_{1,j}x_{3,j}x_{5,j}x_{2,j}x_{4,j}$). Every label in λ is used exactly once and the injective property of λ is easy to observe. Thus λ is a bijection to $\{1, 2, \dots, 5(3v + e)\}$.

To show that λ is a VMT labeling of nG we use Corollary 3.11 for each copy of G. We take $a = 5$, $b = 1 - i$. For the two factors H_1 and H_2 of G we take $r_1 = r + 1$, $c_1 = 1 - i$, $r_2 = r$, and $c_2 = i - 5$. Thus for the i-th copy G_i of the graph G we get

$$\begin{aligned}
\lambda(x_{i,j}) + \sum_{f \in N_{E(H_1) \cup E(H_2)}(x_{i,j})} \lambda(f) &= 5k_\lambda + \sum_{l=1}^{2} r_l c_l + 1 - i \\
&= 5k_\lambda + (r+1)(1-i) + r(i-5) + 1 - i \\
&= 5k_\lambda - 4r + 2 - 2i.
\end{aligned}$$

Adding also the sum of labels of edges incident with $x_{i,j}$ which are not in $E(H_1) \cup E(H_2)$ (given by (16)), we get

$$w_{\lambda'}(x_{i,j}) = 5(k_\lambda + 8v + 4e) - 4r - 2.$$

We see that the labeling given by (15) is a vertex magic total labeling of $G \,\square\, K_5$ with the magic constant $5(k_\lambda + 8v + 4e) - 4r - 2$. □

Example 4.9 Using Theorem 4.8 we find a VMT labeling of $K_4 \,\square\, K_5$. We use the VMT labeling and the factors H_1 and H_2 from Figure 4.2.

Evaluating the magic constant for $n = 4$, $v = 4$, $e = 6$, $r = 1$, and $k_\lambda = 20$ we get $k = 5(20 + 8 \cdot 4 + 4 \cdot 6) - 4 \cdot 1 - 2 = 374$. The partial sums on corresponding vertices have to be added together to obtain the magic constant 374, see Figure 4.6.

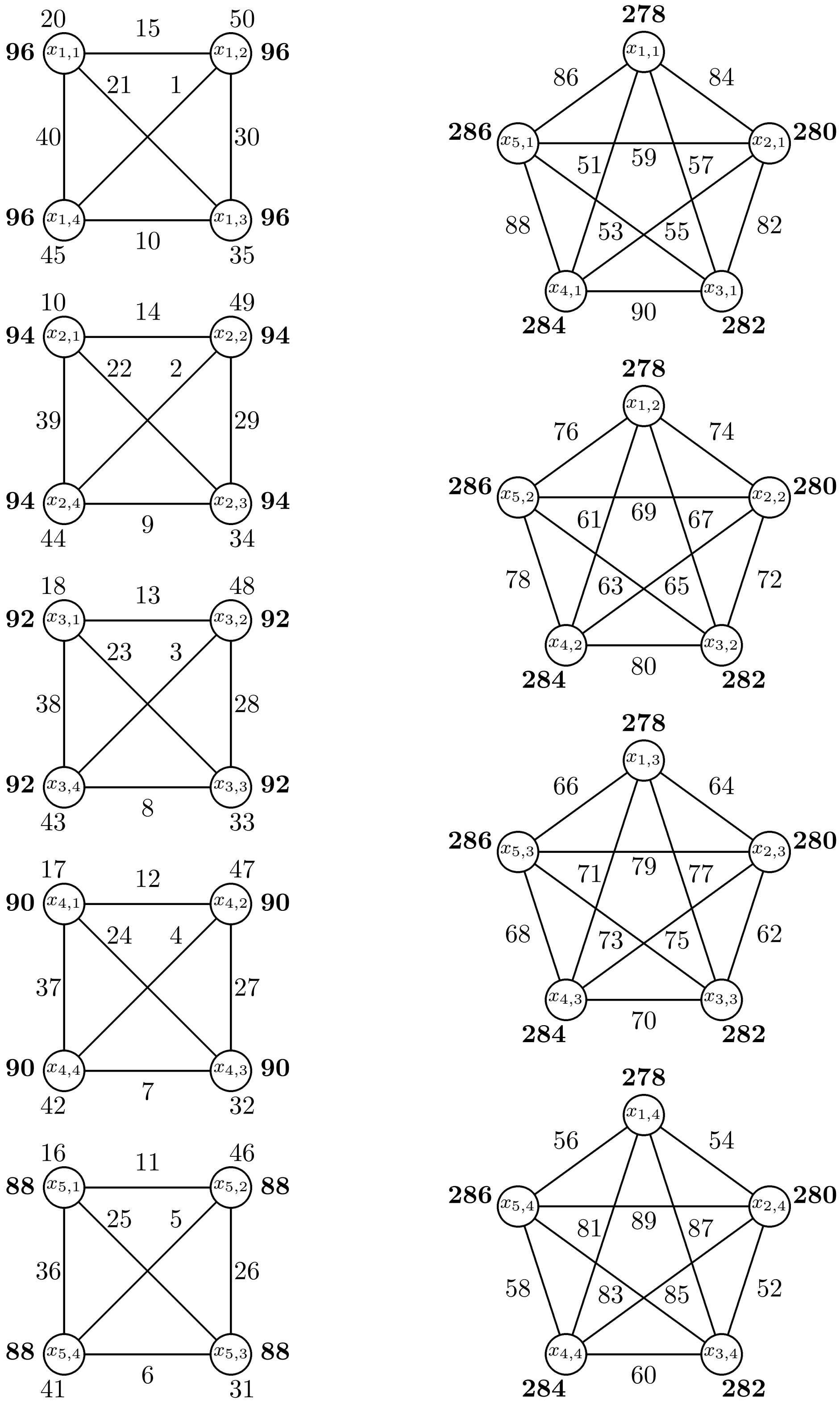


Figure 4.6: *Vertex magic total labeling of $K_4 \square K_5$ with the magic constant $k = 374$.*

The labeling given in the proof of the following theorem is very similar to the labeling given in proof of Theorem 4.8. Yet, it allows to find VMT labelings for different classes of graphs. One such graph is described in Example 4.11.

Theorem 4.10 *Let r be a positive integer. Let G be a $(2r+1)$-regular VMT graph which can be factorized into two factors: an $(r+2)$-regular factor and an $(r-1)$-regular factor. Then there exists a vertex magic total labeling of $G \,\square\, K_5$.*

Proof. The proof is very similar to the proof of Theorem 4.8. We denote the $(r+2)$-regular factor by H_1 and the $(r-1)$-regular factor by H_2. We can use the same notation and the same technique to show that the labeling (17) is a VMT labeling of $G \,\square\, K_5$ with the magic constant $5(k_\lambda + 8v + 4e) - 4r - 2$.

$$\begin{aligned}
\lambda(x_{i,j}) &= 5(\lambda(x_j) - 1) + i \\
\lambda(f_{i,k}) &= \begin{cases} 5(\lambda(f_k) - 1) + 6 - i & \text{if } f_k \in E(H_1) \\ 5(\lambda(f_k) - 1) + i & \text{if } f_k \in E(H_2) \end{cases} \\
\lambda(x_{1,j}x_{2,j}) &= 5(v+e) + 10(v-j) + 4 \\
\lambda(x_{2,j}x_{3,j}) &= 5(v+e) + 10(v-j) + 2 \\
\lambda(x_{3,j}x_{4,j}) &= 5(v+e) + 10(v-j) + 10 \\
\lambda(x_{4,j}x_{5,j}) &= 5(v+e) + 10(v-j) + 8 \\
\lambda(x_{5,j}x_{1,j}) &= 5(v+e) + 10(v-j) + 6 \\
\lambda(x_{1,j}x_{3,j}) &= 5(v+e) + 10(j-1) + 7 \\
\lambda(x_{3,j}x_{5,j}) &= 5(v+e) + 10(j-1) + 3 \\
\lambda(x_{5,j}x_{2,j}) &= 5(v+e) + 10(j-1) + 9 \\
\lambda(x_{2,j}x_{4,j}) &= 5(v+e) + 10(j-1) + 5 \\
\lambda(x_{4,j}x_{1,j}) &= 5(v+e) + 10(j-1) + 1
\end{aligned} \tag{17}$$

where $i = 1, 2, \ldots, 5$ and $j = 1, 2, \ldots, v$. □

Notice that the magic constant of the labeling (17) is equal to the magic constant of the labeling (15), since the sets of vertex labels and edge labels are same.

Example 4.11 To see the difference between Theorem 4.8 and Theorem 4.10 notice that we can give a VMT labeling of $G \,\square\, K_5$ where G is the 3-regular graph shown in Figure 4.7. G satisfies the conditions of Theorem 4.10 but not the conditions of Theorem 4.8, since G has no 1-regular factor and no 2-regular factor. The VMT labeling of G was found using a computer.

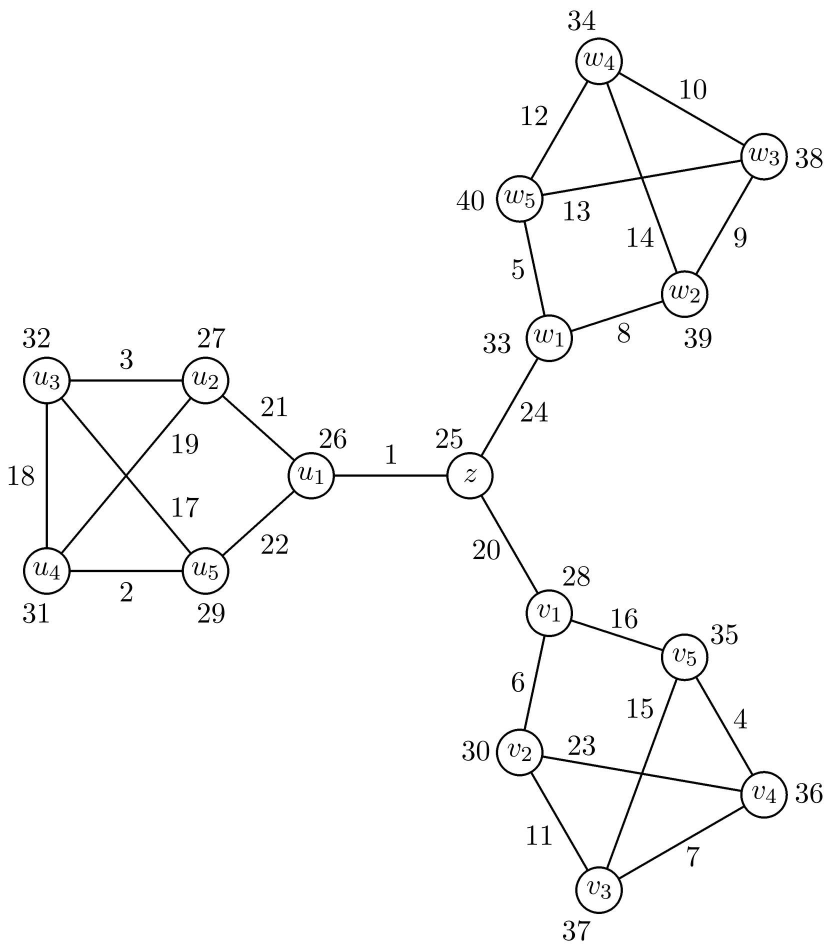


Figure 4.7: *Vertex magic total labeling with the magic constant* $k = 70$ *of a* 3*-regular graph without a* 1*-factorization.*

4.5 Products of certain even-regular VMT graphs

Under certain restriction we take two VMT graphs and find a VMT labeling of their product.

Theorem 4.12 *Let* r, q *be positive integers. Let* G *be a* $2r$*-regular VMT graph which can be factorized into two* r*-regular factors. Let* H *be a* $2q$*-regular VMT graph which can be factorized into two* q*-regular factors, where moreover the vertex labels are consecutive integers. Then there exists a vertex magic total labeling of* $G \,\square\, H$.

Proof. We denote the vertices of $G(U,E)$ by $u_1, u_2, \ldots, u_m$ where $m = |U|$, the two r-regular factors by G_1 and G_2 and the VMT labeling of G by λ_G. We also denote the vertices of $H(V,F)$ by $v_1, v_2, \ldots, v_n$ where $n = |V|$, the two q-regular factors by H_1 and H_2 and the VMT labeling of H by λ_H. We denote the magic constant of λ_G by

k_G and the magic constant of λ_H by k_H. We suppose the vertices are ordered so that $\lambda_H(v_{i+1}) = \lambda_H(v_i) + 1$ for $1 < i \leq n$.

We denote the vertices of $G \,\square\, H$ by $x_{i,j}$, where $x_{i,j} = (u_j, v_i)$, $u_j \in U$, $v_i \in V$, $1 \leq j \leq m$, and $1 \leq i \leq n$.

Consider the labeling λ given by

$$\begin{aligned}
\lambda(x_{i,j}) &= n(\lambda_G(u_j) - 1) + i \\
\lambda(x_{i,j}x_{i,k}) &= \begin{cases} n(\lambda_G(u_ju_k) - 1) + (n+1) - j & \text{if } u_ju_k \in E(G_1) \\ n(\lambda_G(u_ju_k) - 1) + j & \text{if } u_ju_k \in E(G_2) \end{cases} \\
\lambda(x_{i,j}x_{l,j}) &= \begin{cases} n(m + |E|) + |F|(j-1) + \lambda_H(v_iv_l) & \text{if } v_iv_l \in E(H_1) \\ n(m + |E|) + |F|(m-j) + \lambda_H(v_iv_l) & \text{if } v_iv_l \in E(H_2) \end{cases}
\end{aligned} \tag{18}$$

where $i, l = 1, 2, \ldots, n$ and $j, k = 1, 2, \ldots, m$.

We evaluate the weight $w_\lambda(x_{i,j})$ for every $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

$$\begin{aligned}
w_\lambda(x_{i,j}) &= \lambda(x_{i,j}) + \sum_{f \in N_E(x_{i,j}) \cap E(G_1)} \lambda(f) + \sum_{f \in N_E(x_{i,j}) \cap E(G_2)} \lambda(f) + \sum_{f \in N_F(x_{i,j}) \cap E(H_1)} \lambda(f) + \sum_{f \in N_F(x_{i,j}) \cap E(H_2)} \lambda(f) \\
&= n\left(\lambda_G(u_j) + \sum_{f \in N_E(x_{i,j})} \lambda_G(u_ju_k)\right) - n - 2rn + i + r(n+1) + \\
&\quad 2qn(m + |E|) + q|F|(m-1) + \sum_{f \in N_F(x_{i,j})} \lambda_H(v_iv_l).
\end{aligned}$$

Since the vertex labels in λ_H are consecutive integers, we can take an integer c so that $\lambda_H(v_i) = (c-1) + i$ for $1 \leq i \leq n$. Using $i = \lambda_H(v_i) - c + 1$ we get

$$\begin{aligned}
w_\lambda(x_{i,j}) &= nk_G - n - rn + \lambda_H(v_i) - c + 1 + r + \\
&\quad 2qn(m + |E|) + q|F|(m-1) + \sum_{f \in N_F(x_{i,j})} \lambda_H(v_iv_l) \\
&= n\,(k_G - r - 1) + 2qn(m + |E|) + q|F|(m-1) + k_H + r - c + 1.
\end{aligned}$$

Again λ is a bijection. The first $n(m+|E|)$ labels are used to label copies of G, the next $m|F|$ labels are used to label edges of copies of H.

Thus the labeling given by (18) is a vertex magic total labeling of $G \,\square\, H$ with the magic constant $n\,(k_G - r - 1) + 2qn(m + |E|) + q|F|(m-1) + k_H + r - c + 1$, where c is the lowest vertex label. □

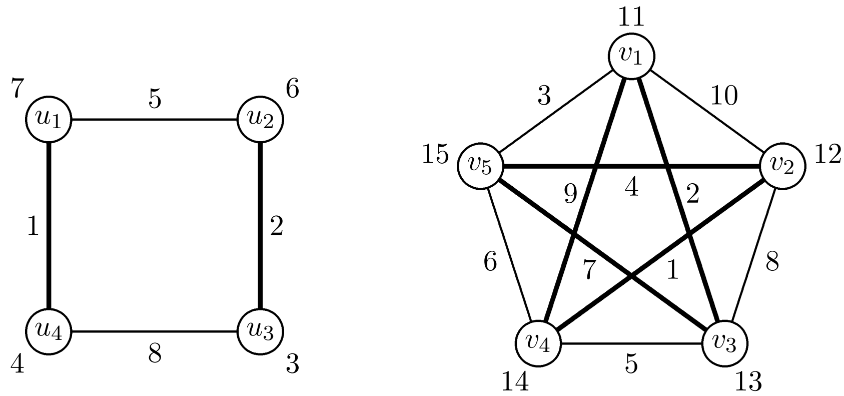
We could have used Corollary 3.11 to evaluate the sums of labels in the factor consisting of copies of G. Taking $a = n$, $b = i - n$, $r_1 = r_2 = r$, $c_1 = 1 - j$, and $c_2 = j - n$ we evaluate the partial sum of vertices in the i-th copy of G and we get

$$nk_G + r(1-n) + i - n \qquad \text{for } i = 1, 2, \ldots, n.$$

On the other hand evaluating the partial sums in the factor consisting of copies of H is somewhat cumbersome, since we are not using the vertex labels of λ_H. Thus, in the proof we evaluated the weight of every vertex.

Example 4.13 Using the construction from the proof of Theorem 4.12 we can find a VMT labeling of $C_4 \,\square\, K_5$. Since the product has twenty vertices of degree six, we give rather the two factors: five copies of C_4 and four copies of K_5. From the names of the vertices it is apparent which pairs of vertices correspond to each other, see Figure 4.8 and Figure 4.9.

Evaluating the magic constant for $m = 4$, $|E| = 4$, $n = 5$, $|F| = 10$, $r = 1$, $q = 2$, $k_G = 13$, $k_H = 35$, and $c = 11$ we get $k = 5(13-1-1)+2\cdot 2\cdot 5(4+4)+2\cdot 10(4-1)+35+1-11+1 = 301$. In Figure 4.8 the edges of the factor G_1 in C_4 and of the factor H_1 in K_5 are drawn in "thick". In Figure 4.9 the partial sums in the copies of C_4 and of K_5 are again given in boldface. Adding together the partial sums on corresponding vertices we obtain the magic constant $k = 301$.



K_4 with $k_G = 13$. K_5 with $k_H = 35$.

Figure 4.8: *Vertex magic total labeling of C_4 with $k_G = 13$ and of K_5 with $k_H = 35$.*

Notice, that the product $G \,\square\, H$ obtained in Theorem 4.12 is a $2(r+q)$-regular VMT graph, which can be decomposed into two $(r+q)$-regular factors. The vertex labels of $G \,\square\, H$ are consecutive integers if VMT labelings of both G and H had vertices labeled by consecutive integers. Thus we can construct VMT labelings of repeated products.

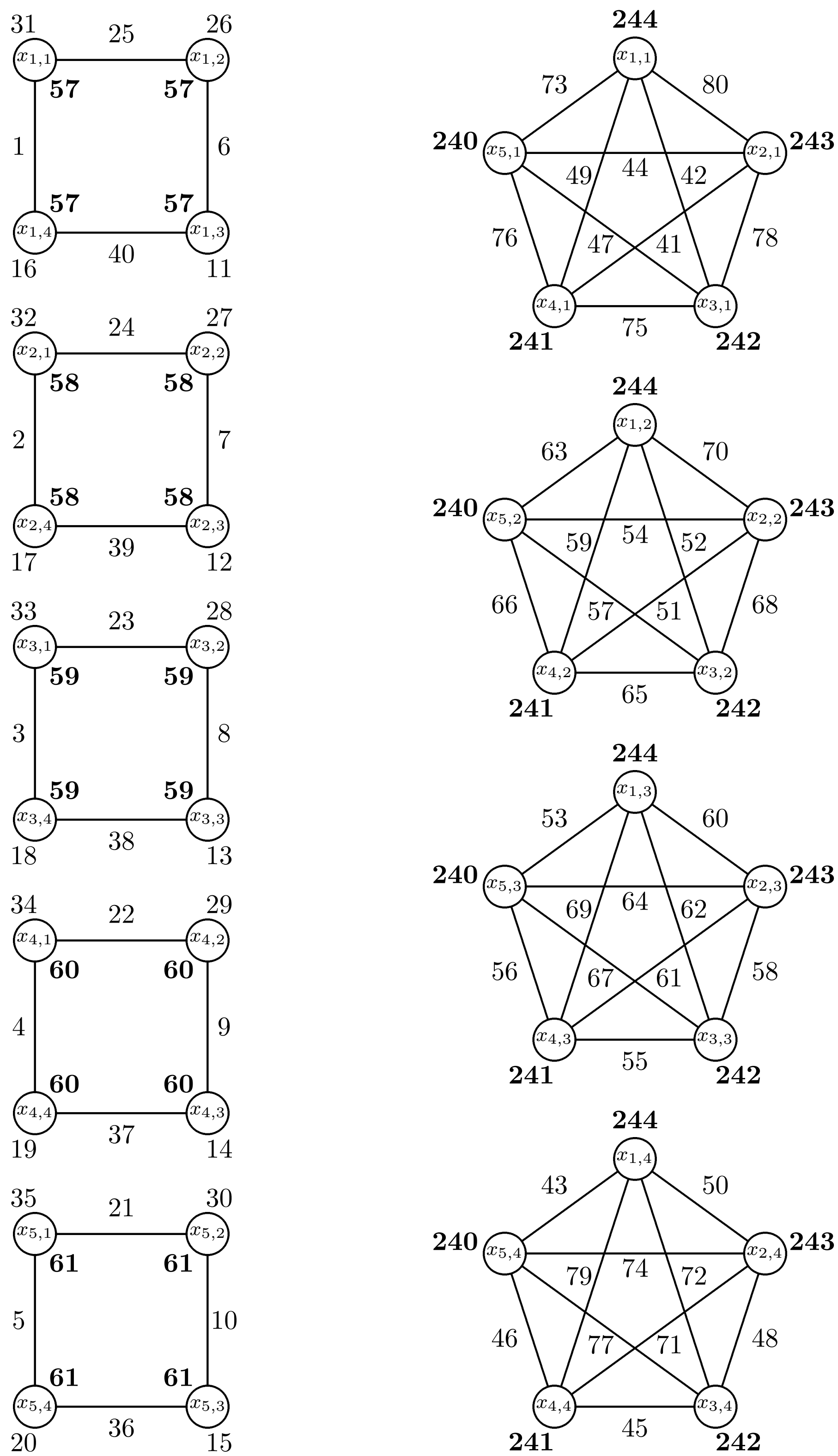


Figure 4.9: *Vertex magic total labeling of* $C_4 \,\square\, K_5$ *with the magic constant* $k = 301$.

4.6 Products of odd cycles and K_5

In the previous sections vertex magic total labelings of Cartesian products of several classes of regular graphs are given. None of them allows to find a VMT labeling of products of graphs with odd cycles. We give a vertex magic total labeling for the small class $C_{2n+1} \square K_5$ in this section.

Theorem 4.14 *Let n be an odd positive integers, $n \geq 3$. There exists a vertex magic total labeling of $C_n \square K_5$.*

Proof. We denote the vertices along the cycle C_n by $x_1, x_2, \ldots, x_n$. We can decompose the product $C_n \square K_5$ into a factor consisting of 5 copies of C_n and a factor consisting of n copies of K_5. We denote the vertices of the copies of C_n by $x_{i,j}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, 5$.

Consider the following labeling

$$\begin{aligned}
\lambda(x_{i,j}) &= 2n(j-1) + 2i - 1 \\
\lambda(x_{i,j}x_{i+1,j}) &= \begin{cases} 2n(5-j) + 1 + i & \text{if } i \text{ is odd} \\ 2n(5-j) + (n+1) + i & \text{if } i \text{ is even} \end{cases} \\
\lambda(x_{i,1}x_{i,2}) &= 12n - i + 1 \\
\lambda(x_{i,2}x_{i,3}) &= 19n - i + 1 \\
\lambda(x_{i,3}x_{i,4}) &= 16n - i + 1 \\
\lambda(x_{i,4}x_{i,5}) &= 20n - i + 1 \\
\lambda(x_{i,5}x_{i,1}) &= 15n - i + 1 \\
\lambda(x_{i,1}x_{i,3}) &= 14n - i + 1 \\
\lambda(x_{i,3}x_{i,5}) &= 13n - i + 1 \\
\lambda(x_{i,5}x_{i,2}) &= 18n - i + 1 \\
\lambda(x_{i,2}x_{i,4}) &= 11n - i + 1 \\
\lambda(x_{i,4}x_{i,1}) &= 17n - i + 1
\end{aligned} \tag{19}$$

where $i = 1, 2, \ldots n$ and $j = 1, 2, \ldots, 5$. The subscript i is taken modulo n if necessary. For convenience we consider $x_{n,j} = x_{0,j}$.

One can check that the sum of edge labels in the i-th K_5 at vertex $x_{i,j}$ is

$$\sum_{\substack{k=1 \\ k \neq j}}^{5} \lambda(x_{i,j}x_{i,k}) = n(56 + 2j) + 4 - 4i.$$

The weight of every vertex $x_{i,j}$ for $1 < i \leq n$ odd is

$$w_\lambda(x_{i,j}) = \lambda(x_{i,j}) + \lambda(x_{i-1,j}x_{i,j}) + \lambda(x_{i,j}x_{i+1,j}) + \sum_{\substack{k=1 \\ k \neq j}}^{5} \lambda(x_{i,j}x_{i,k})$$

$$\begin{aligned} w_\lambda(x_{i,j}) &= 8n - 2nj + 2i + (i-1) + (n+1) + i + n(56+2j) + 4 - 4i \\ &= 75n + 4. \end{aligned}$$

Similarly, for $1 < i \le n$ even is

$$\begin{aligned} w_\lambda(x_{i,j}) &= 8n - 2nj + 2i + i + (n+1) + (i-1) + n(56+2j) + 4 - 4i \\ &= 75n + 4. \end{aligned}$$

Finally, for $i = 1$ we get

$$\begin{aligned} w_\lambda(x_{1,j}) &= 8n - 2nj + 1 + n + 1 + 2 + n(56+2j) + 4 - 4 \\ &= 75n + 4. \end{aligned}$$

To observe that λ is a bijection we notice that the first $10n$ labels are assigned to the vertices (odd labels) and edges (even labels) of the factor containing the five copies of C_n. The next $10n$ labels are used to label the edges of the factor consisting of n copies of K_5. Thus the labeling given by (19) is a vertex magic total labeling of $C_n \,\square\, K_5$ for n odd with the magic constant $k = 75n + 4$. $\square$

Example 4.15 An example of a VMT labeling of $C_5 \,\square\, K_5$ is in Figure 4.10. By evaluating the magic constant we get $k = 75 \cdot 5 + 4 = 379$. The partial sums in the copies of C_5 and of K_5 are given in boldface. Adding the partial sums at corresponding vertices we obtain the magic constant $k = 379$.

4.7 Remarks

According to Table 1.1 we see that the products of cycles are known to have a VMT labeling (see [FKK3]) if at least one of the cycles is odd. Unfortunately none of the theorems in this chapter allows to find a VMT labeling for $C_{2m} \,\square\, C_{2n}$. Theorem 4.12 cannot be of use, because in [MsP] it was shown that an r-regular graph on an even number of vertices cannot have a VMT labeling in which the vertex labels constitute an arithmetic progression with an odd difference. The VMT labeling of $C_{2m} \,\square\, C_{2n}$ remains an open problem. On the other hand VMT labeling of $C_m \,\square\, C_{2n+1}$ is a special case of Theorem 4.12 because all odd cycles have VMT labelings with consecutive vertex labels, see e.g. [Wal1].

The results given in Theorem 4.3 and in Theorem 4.4 were presented on *Conference MIGHTY XXXVII* ([Kov1]) and the result from Theorem 4.14 on *Conference MIGHTY XXXVI* ([FKK1]).

Further constructions of VMT labelings of Cartesian products of regular graphs are given in Chapter 7. Notice that the constructions do *not* include the results from this chapter, because there is no SPM labeling of C_n or K_5 (see [Stew2]).

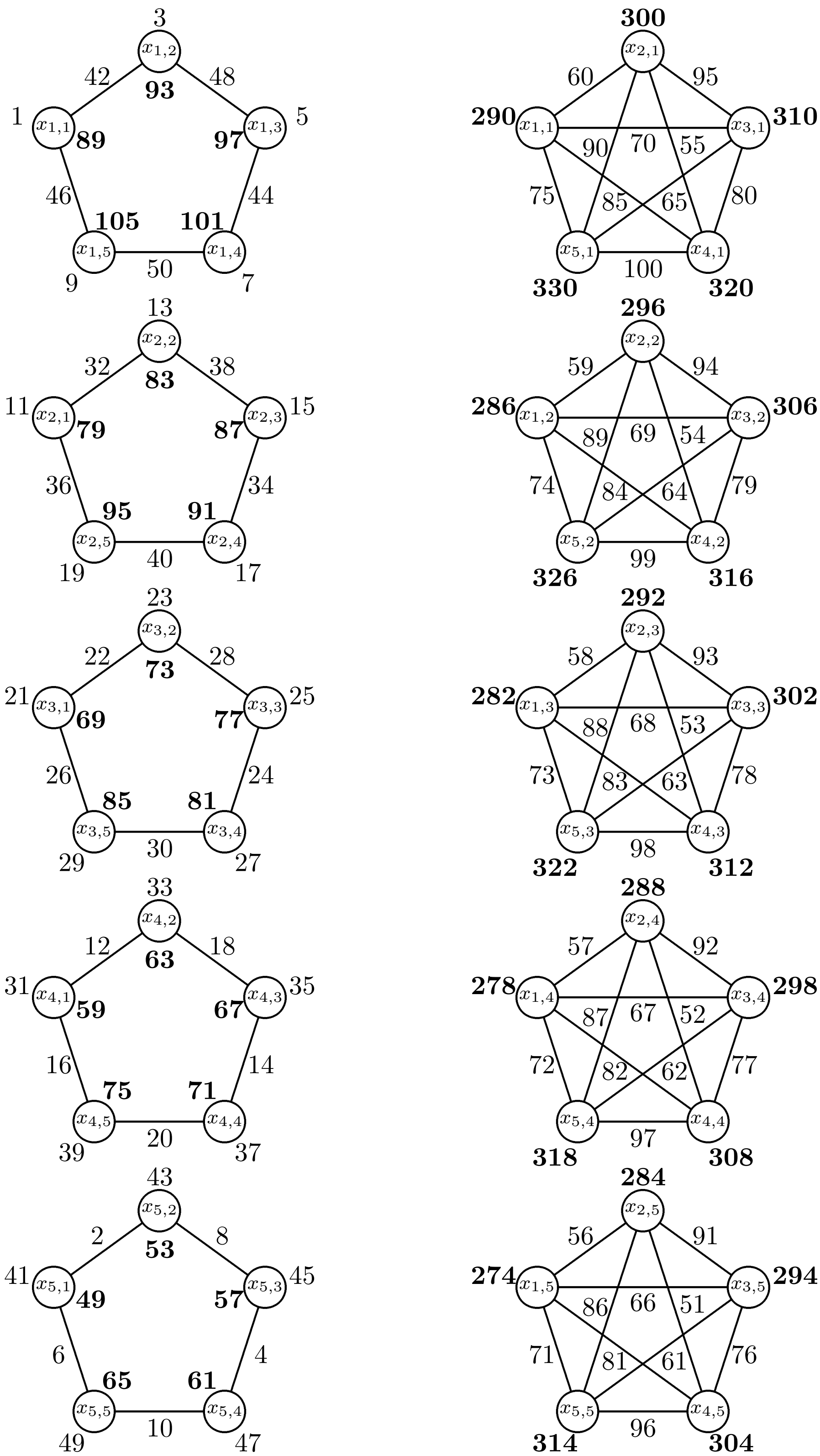


Figure 4.10: *VMT labeling of $C_5 \,\square\, K_5$ with the magic constant $k = 379$.*

5 Kotzig arrays

Kotzig arrays will become very useful in the following sections. Many constructions make use of a "scheme" with a property of having the same sum over every row and/or column.

5.1 Definition

Magic squares, a popular mathematical recreation, can be generalized to magic rectangles. If we keep the property of having the same sum over every row and column but relax the property of having distinct element, we arrive at the topic of Kotzig arrays.

Definition 5.1 *An $m \times n$ matrix such that*

(1) every row contains a permutation of $\{0, 1, \ldots, n-1\}$

(2) every column has the same sum

is called a Kotzig array of size *$m \times n$.*

The name was introduced by Wallis [Wal1] based on an unpublished paper by Kotzig.

5.2 Basic properties

The sum of all entries in a Kotzig array of size $m \times n$ is $\frac{1}{2}mn(n-1)$, thus the sum over each column is $\frac{1}{2}m(n-1)$. Since the sum over a column has to be an integer, a Kotzig array exists only if m is even or if n is odd.

A $2 \times n$ Kotzig array is e.g.

$$\begin{bmatrix} 0 & 1 & 2 & \cdots & n-2 & n-1 \\ n-1 & n-2 & n-3 & \cdots & 1 & 0 \end{bmatrix}. \tag{20}$$

The sum over every column is $n-1$.

A $3 \times n$ Kotzig array exists only for n odd. We denote $n = 2r+1$ for some integer $r \geq 0$. The following example was taken from [Wal1] and is due to Kotzig

$$\begin{bmatrix} 0 & 1 & \cdots & r-1 & r & r+1 & r+2 & \cdots & n-2 & n-1 \\ 2r & 2r-2 & \cdots & 2 & 0 & 2r-1 & 2r-3 & \cdots & 3 & 1 \\ r & r+1 & \cdots & 2r-1 & 2r & 0 & 1 & \cdots & r-2 & r-1 \end{bmatrix}. \tag{21}$$

The sum over every column is $\frac{3}{2}(n-1) = 3r$.

The two examples (20) and (21) are used as building stones, since Kotzig arrays with more rows are obtained by joining copies of (20) and (21). Immediately we get

Lemma 5.2 *Let m, n be positive integers, $m > 1$. There exists a Kotzig array of size $m \times n$ if and only if $m(n-1)$ is even.*

Given a Kotzig array A of size $m \times n$ one can obtain different Kotzig arrays by permuting columns or rows of A.

6 Copies of magic graphs

In Section 4.2 a method was given how to find VMT labelings of copies of certain $(2r+1)$-regular VMT graphs. A more general result by Wallis was mentioned in Table 1.1. In this chapter we repeat the result by Wallis and extend the ideas to a VMT labeling with another magic constant and also to other magic-type labelings.

6.1 Copies of VMT graphs

Wallis published a nice construction based on Kotzig arrays in [Wal2]. He showed that if an r-regular graph has a VMT labeling, then a graph consisting of n copies of G also has a VMT labeling. The only restriction comes from the fact that there is no Kotzig array with an odd number of rows and an even number of columns. If G is of even regularity, we know how to construct a VMT labeling only for an odd number of copies of G. We do not have a general method for constructing a VMT labeling for an even number of copies.

We state the theorem by Wallis and give an alternate proof.

Theorem 6.1 (W. D. Wallis [Wal2]) *Suppose G is a regular graph of degree r other than K_1, which has a vertex magic total labeling.*

(1) If r is even, then nG has a vertex magic total labeling whenever n is an odd positive integer.

(2) If r is odd, then nG has a vertex magic total labeling for every positive integer n.

Proof 1. Given in [Wal2]. □

Let $G(V,E)$ be an r-regular VMT graph. We denote $v = |V|$ and $e = |E|$. We denote the vertices of G by $v_1, v_2, \ldots, v_v$, the edges by $e_1, e_2, \ldots, e_e$, and a Kotzig array of size $(r+1) \times n$ by $A = (a_{j,i})$. By Vizing's Theorem there exists a proper edge coloring of G with $r+1$ colors. In addition we color every vertex x with the $(r+1)$-st color missing among the r colors of r edges incident with x. We obtain a *total* $r+1$ coloring η of G. Wallis showed that given a VMT labeling λ of G with the magic constant k, the labeling

$$\begin{aligned} \lambda'(v_{i,p}) &= \lambda(v_p) + (e+v)a_{\eta(v_p),i} && \text{for } p = 1, 2, \ldots, v \\ \lambda'(e_{i,q}) &= \lambda(e_q) + (e+v)a_{\eta(e_q),i} && \text{for } q = 1, 2, \ldots, e \end{aligned} \tag{22}$$

is a VMT labeling of n copies of G with the magic constant $k' = k+\frac{1}{2}(e+v)(r+1)(n-1)$. We give a second proof with a construction which yields a different magic constant.

Proof 2. Let G be a VMT r-regular graph from Theorem 6.1. The Kotzig array $A = (a_{j,i})$ of size $(r+1)\times n$ always exists under the conditions on r and n of Theorem 6.1.

Take a total coloring $\eta : (V \cup E) \to \{1, 2, \dots, r+1\}$ of G. We will construct n copies G_i of graph G for $i = 1, 2, \dots, n$. In G_i we denote the copies of vertices by $v_{i,1}, v_{i,2}, \dots, v_{i,v}$ and the copies of edges by $e_{i,1}, e_{i,2}, \dots, e_{i,e}$.

Consider the following labeling

$$\begin{aligned} \lambda'(v_{i,p}) &= n(\lambda(v_p) - 1) + a_{\eta(v_p),i} + 1 \qquad \text{for } p = 1, 2, \dots, v \\ \lambda'(e_{i,q}) &= n(\lambda(e_q) - 1) + a_{\eta(e_q),i} + 1 \qquad \text{for } q = 1, 2, \dots, e. \end{aligned} \tag{23}$$

To show that (23) is a VMT labeling we evaluate the weight $w_{\lambda'}(v_{i,p})$ of every vertex $v_{i,p}$ for $i = 1, 2, \dots, n$ and $p = 1, 2, \dots, v$.

$$\begin{aligned} w_{\lambda'}(v_{i,p}) &= \lambda'(v_{i,p}) + \sum_{e_{i,q} \in N_E(v_{i,p})} \lambda'(e_{i,q}) \\ &= n\left(\lambda(v_p) + \sum_{e_q \in N_E(v_p)} \lambda(e_q)\right) - n(r+1) + a_{\eta(v_p),i} + \sum_{e_q \in N_E(v_p)} a_{\eta(e_q),i} + (r+1). \end{aligned}$$

Now using the fact that λ is a VMT labeling with the magic constant k and knowing the sum $\frac{1}{2}(r+1)(n-1)$ over a column of the Kotzig array A, we get

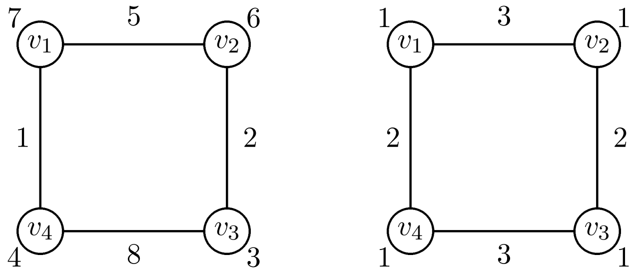
$$\begin{aligned} w_{\lambda'}(v_{i,p}) &= nk - n(r+1) + \frac{1}{2}(r+1)(n-1) + (r+1) \\ &= nk - \frac{1}{2}(r+1)(n-1). \end{aligned}$$

We see that the weight of every vertex in G_i for $i = 1, 2, \dots, n$ equals to the same magic constant $k' = nk + \frac{1}{2}(1-n)(r+1)$.

Now looking at all n vertices $v_{i,p}$ (or n edges $e_{i,q}$, respectively), which correspond to the vertex v_p (or edge e_q) of the original graph G, the labels form the set $\{n\lambda(v_p), n\lambda(v_p)+1, \dots, n\lambda(v_p)+n-1\}$ (or $\{n\lambda(e_q), n\lambda(e_q)+1, \dots, n\lambda(e_q)+n-1\}$). There are $e+v$ such sets and they are disjoint since λ assigns a different label to every vertex (or edge). This means that λ' uses each of the $n(e+v)$ labels exactly once. Thus λ' is a VMT labeling of n copies of G with the magic constant $nk - \frac{1}{2}(r+1)(n-1)$. □

Corollary 3.11 is not relevant, since G does need to have a regular factorization. We could use the coloring η to set up a 1-factorization only in the special case when G has a proper edge r-coloring and η assigns the $(r+1)$-st color only to the vertices of G (compare with Theorem 6.4).

Example 6.2 To compare the VMT labeling of $3C_4$ by Wallis and the labeling from the alternate proof we need a VMT labeling of C_4, a total coloring of C_4 (colors 1, 2, and 3) and a Kotzig array of size 3×3, see Figure 6.1. The labeling of $3C_4$ based on the labeling (22) is in Figure 6.2 and the labeling of $3C_4$ based on the labeling (23) is in Figure 6.3.



$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

C_4 colors 1, 2, and 3 3×3 Kotzig array

Figure 6.1: *VMT labeling of C_4 with the magic constant $k = 13$, a total coloring of C_4, and a 3×3 Kotzig array.*

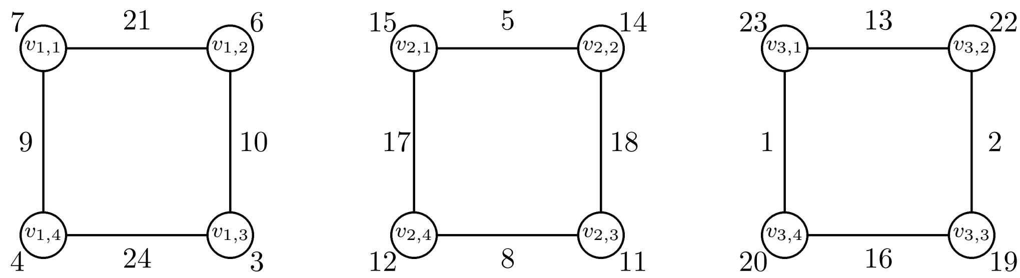


Figure 6.2: *VMT labeling of $3C_4$ with the magic constant $k = 37$.*

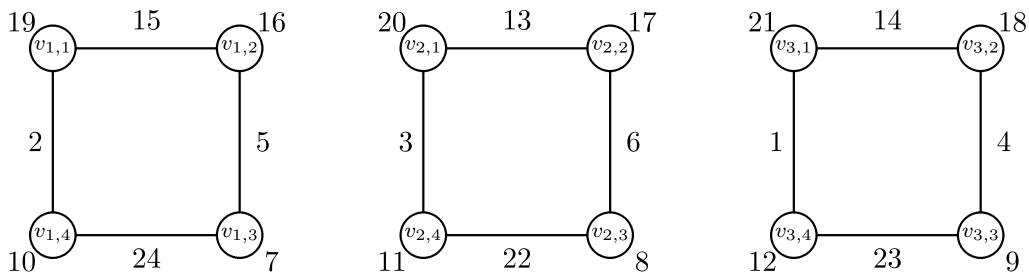


Figure 6.3: *VMT labeling of $3C_4$ with the magic constant $k = 36$.*

Notice that for even regular graphs G Theorem 6.1 guarantees a construction only for an odd number of copies. This does not mean that an even number of copies of G does not have a VMT labeling, we just do not have a general method how to find the labeling. According to the conjecture by MacDougall we expect all copies of regular VMT graphs to have a VMT labeling (with the exception of K_2 and $2K_3$). The Figure 6.4 gives a VMT labeling for $2C_4$ (found by trial and error).

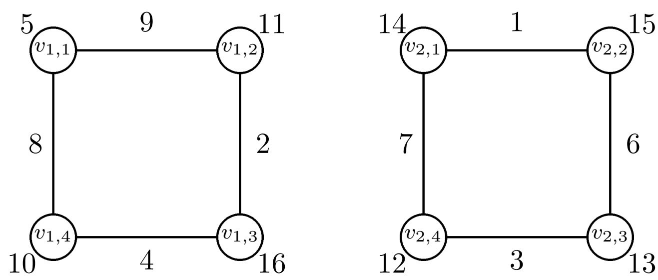


Figure 6.4: *VMT labeling of* $2C_4$ *with the magic constant* $k = 22$.

6.2 Copies of SPM graphs

Unfortunately, for copies of r-regular supermagic graphs we cannot use exactly the same nice approach as Wallis in Theorem 6.1 for r-regular vertex magic total graphs. The construction is based on Vizing's Theorem, which guarantees that for an r-regular graph G there exists always a proper edge coloring with r or $r+1$ colors. We can color every vertex v with the $r+1$-st color which is not used among the r colors of all edges incident with v and thus there exists a proper $r+1$ total coloring (both edges and vertices) of G. Since there are no vertex labels in an SPM graph, we always have to find a proper edge coloring of G with just r colors. Instead of a general statement as in Theorem 6.1, we obtain a result only for *Class 1* r-regular graphs[3].

Note 6.3 The following result was already known, see in [Iv]. We give a different proof based on Kotzig arrays. A single labeling covers all possible cases. We also provide a unified approach for labelings of copies of supermagic graphs as well as for labelings of vertex magic total graphs. Moreover, we show that the construction covers all admissible cases.

Theorem 6.4 *Let r be an integer at least 3. Let G be an r-regular graph with a proper edge r coloring, which has an SPM labeling.*

(1) If r is odd, then nG has a supermagic labeling whenever n is an odd positive integer.

(2) If r is even, then nG has a supermagic labeling for every positive integer n.

Proof. We consider $r \geq 3$ because the only 1-regular graph is K_2 and there can be no 2-regular supermagic graph. The proof continues similarly to the proof of Theorem 6.1.

Let $G(V, E)$ be an r-regular SPM graph which satisfies the conditions above. Take an SPM labeling λ of G with magic constant h and a Kotzig array $A = (a_{j,i})$ of size

[3] Let $\chi'(G)$ be the edge-chromatic number of G. We say a graph G is *Class 1* if $\chi'(G) = \Delta(G)$ and *Class 2* if $\chi'(G) = \Delta(G) + 1$.

$r \times n$. Under the conditions on r and n of the theorem such an array always exists. Take a proper edge r coloring $\eta : E \to \{1, 2, \ldots, r\}$ of G. We denote the vertices of G by $v_1, v_2, \ldots, v_v$ and edges by $e_1, e_2, \ldots, e_e$. We will construct n copies G_i of the graph G for $i = 1, 2, \ldots, n$. In G_i we denote the copies of vertices by $v_{i,1}, v_{i,2}, \ldots, v_{i,v}$ and edges by $e_{i,1}, e_{i,2}, \ldots, e_{i,e}$.

Consider the following labeling

$$\lambda'(e_{i,q}) = n(\lambda(e_q) - 1) + a_{\eta(e_q),i} + 1 \quad \text{for } q = 1, 2, \ldots, e. \tag{24}$$

We observe that all n edges $e_{i,q}$ for $i = 1, 2, \ldots, n$ which correspond to the edge e_q of the original graph G have their labels from the set $\{n\lambda(e_q), n\lambda(e_q)+1, \ldots, n\lambda(e_q)+n-1\}$. There are e such sets and they are disjoint since λ assigns a different label between 1 and e to each edge, so λ' is a bijection from $E(nG)$ to $\{1, 2, \ldots, ne\}$.

We show that (24) is an SPM labeling using Theorem 3.12 for every copy G_i of G. We decompose G to r factors according to the edge coloring η so that factor F_j contains only the edges of color j. Since η is a proper edge coloring, this is always possible. Taking $a = n$, $r_1 = r_2 = \ldots = r_r = 1$, and $b_j = a_{j,i} + 1 - n$ for $j = 1, 2, \ldots, r$ we get

$$h' = nh + \sum_{i=1}^{r} (a_{j,i} + 1 - n).$$

Using the fact that $\frac{1}{2}r(n-1)$ is the sum over a column of the $r \times n$ Kotzig array A, we get

$$\begin{aligned} h' &= nh + \frac{1}{2}r(n-1) + r(1-n) \\ &= nh - \frac{1}{2}r(n-1) \end{aligned}$$

which using (3) simplifies to

$$h' = \frac{1}{2}r(ne+1).$$

Thus λ' is an SPM labeling of nG with the magic constant $h' = \frac{1}{2}r(ne+1)$. □

We can give a second labeling by

$$\lambda''(e_{i,q}) = \lambda(e_q) + ea_{\eta(e_q),i} \quad \text{for } q = 1, 2, \ldots, e. \tag{25}$$

The magic constant $h'' = h + \frac{1}{2}re(n-1)$ also simplifies to $h'' = \frac{1}{2}r(ne+1)$.

The following theorem shows that by Theorem 6.4 we have covered all possible cases of copies of Class 1 regular supermagic graphs.

Theorem 6.5 *Let r be an integer at least 3. Let G be an r-regular graph with a proper edge r coloring, which has an SPM labeling. If r is odd and n is a positive even integer, then there is no supermagic labeling of nG.*

Proof. We denote the number of edges of G by e, the SPM labeling of G by λ and the magic constant of λ by h. According to (3) is $h = \frac{1}{2}r(1+e)$. Since h is an integer and r is odd, then e must be odd. Suppose by way of contradiction that there exists an SPM labeling λ' of nG. The magic constant of λ' is $h' = \frac{1}{2}r(1+ne)$. Because n is even, r is odd and $1+ne$ is odd h' is not an integer. We have a contradiction, because h' is a sum of integers. Thus there is no SPM labeling of nG where r is odd and n is even. □

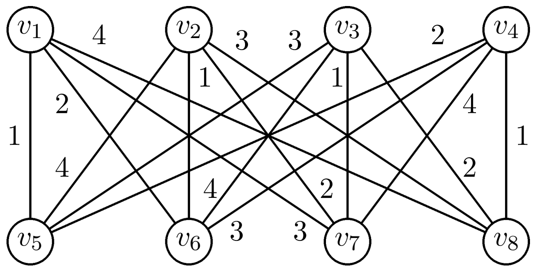


Figure 6.5: *Proper edge coloring of $K_{4,4}$ using colors 1, 2, 3, and 4.*

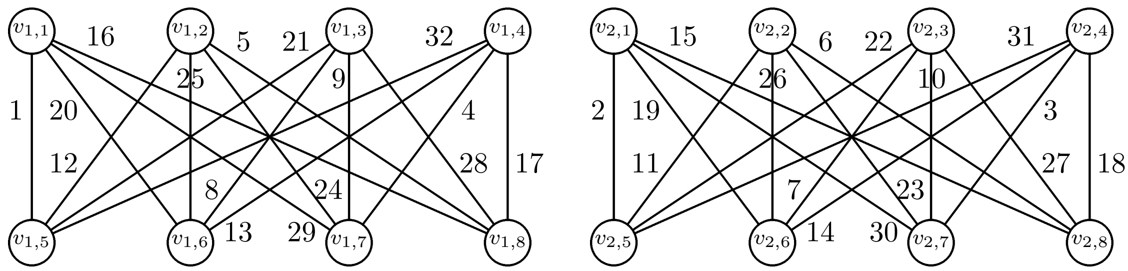


Figure 6.6: *Supermagic coloring of $2K_{4,4}$ with the magic constant $h' = 66$.*

Example 6.6 In Figure 1.6 is an SPM labeling of $K_{4,4}$, in Figure 6.5 is a proper edge coloring of $K_{4,4}$ using colors 1, 2, 3, and 4. A 4×2 Kotzig array is easy to construct.

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

To find an SPM labeling of $2K_{4,4}$ we use the labeling (24) from the proof of Theorem 6.4. The magic constant of $2K_{4,4}$ is $h' = 66$, see Figure 6.6.

We can state a result analogous to Lemma 4.3 also for supermagic graphs.

Lemma 6.7 *Let n, r be positive integers. Let G be a $2r$-regular SPM graph which can be factorized into two r-regular factors. Then the graph nG is also an SPM graph.*

Proof. See Theorem 6.9. □

Since every odd regular VMT graph satisfies the conditions of Theorem 6.1 is Lemma 4.3 a special case. On the other hand if we compare Theorem 6.4 and Lemma 6.7 it is easy to find regular SPM graphs which satisfy conditions of just one of them. Any odd-regular SPM graph G does not satisfy conditions of Lemma 6.7, but we can find an SPM labeling of nG using Theorem 6.4. According to [Stew2] we have an SPM labeling for K_9 (see also Table 1.2). K_9 does not satisfy conditions of Theorem 6.4, because there exists no proper edge 8-coloring of K_9 since there is no 1-factorization of a graph on an odd number of vertices. Yet, we can decompose K_9 into two 4-regular factors (see Figure 6.7) and use Lemma 6.7 to find an SPM labeling for nG for any n.

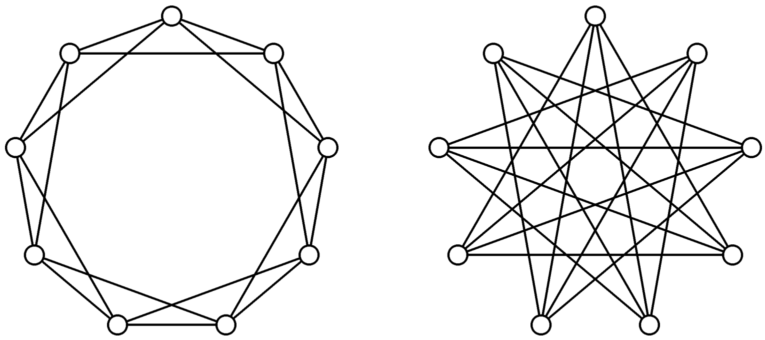


Figure 6.7: *Two* 4*-regular factors of* K_9*.*

Now we prove a more general statement. Both Theorem 6.4 and Lemma 6.7 are special cases of the following theorem.

Note 6.8 The following result was already published in [Iv]. We present a construction based on Kotzig arrays which covers all possible cases by a single labeling.

Theorem 6.9 *Let n, k, and r be positive integers, where $k \geq 2$ and $kr \geq 3$. Let G be a kr-regular SPM graph which can be factorized into k r-regular factors.*

(1) If k is odd, then nG has an SPM labeling whenever n is an odd positive integer.

(2) If k is even, then nG has an SPM labeling for every positive integer n.

Proof. We consider $k \geq 2$ and $kr \geq 3$ because there is no $1 \times n$ Kotzig array and the only 1-regular graph is K_2 and there can be no 2-regular supermagic graph. The proof is a generalization of the proof of Theorem 6.4. We use the same notation. Let $A = (a_{j,i})$ be a Kotzig array of size $k \times n$. Under the conditions on k and n of the theorem such an array always exists. By $F_1, F_2, \ldots, F_k$ we denote the r-regular factors of G.

Consider the following labeling

$$\lambda'(e_{i,q}) \quad = \quad n(\lambda(e_q) - 1) + a_{j,i} + 1 \quad \text{for } q = 1, 2, \ldots, e, \text{ where } e_q \in F_j. \qquad (26)$$

Again λ' is a bijection from $E(nG)$ to $\{1, 2, \ldots, ne\}$ and we show that (26) is an SPM labeling using Theorem 3.12 for every copy G_i of G. While $a = n$, $r_1 = r_2 = \ldots = r_k = r$, and $b_j = a_{j,i} + 1 - n$ for $j = 1, 2, \ldots, k$ we get

$$h' \quad = \quad nh + r \sum_{i=1}^{k} (a_{j,i} + 1 - n).$$

Since $\frac{1}{2}k(n-1)$ is the sum over a column of the $k \times n$ Kotzig array A, is

$$\begin{aligned} h' &= nh + \frac{1}{2}kr(n-1) + r(1-n) \\ &= nh - \frac{1}{2}kr(n-1) \end{aligned}$$

which using (3) simplifies to

$$h' \quad = \quad \frac{1}{2}kr(ne+1).$$

Thus λ' is an SPM labeling of nG with the magic constant $h' = \frac{1}{2}kr(ne+1)$. □

An alternate labeling is given by

$$\lambda''(e_{i,q}) \quad = \quad \lambda(e_q) + ea_{j,i} \quad \text{for } q = 1, 2, \ldots, e, \text{ where } e_q \in F_j.$$

The magic constant $h'' = h + \frac{1}{2}kre(n-1)$ also simplifies to $h'' = \frac{1}{2}kr(ne+1)$.

Theorem 6.4 follows from Theorem 6.9 when $r = 1$ and Lemma 6.7 is a special case for $k = 2$.

For the remaining case in Theorem 6.9 when k is odd and n is even we can exclude the case of r being odd, since then nG does not allow an SPM labeling. We omit the proof because the statement follows from Theorem 6.5.

Theorem 6.10 *Let k and r be positive integers, where $k \geq 2$ and $kr \geq 3$. Let G be a kr-regular SPM graph which can be factorized into k r-regular factors. If kr is odd and n is a positive even integer, then there is no supermagic labeling of nG.*

We could state a result similar to Theorem 6.9 also for VMT graphs. Since any r-regular VMT graph satisfies Theorem 6.1, such result would give a VMT labeling only for classes of r-regular graphs which are already covered by Theorem 6.1. In particular, for r even we cannot find an $r+1 \times n$ Kotzig array for n even and thus the method from Theorem 6.1 fails for an even number of copies. By splitting the r-regular graph G into factors, we have to split the integer $r+1$ into an *odd* number of integers q $(r+1 = k \cdot q)$ and thus we cannot construct labelings of an even number of copies of G either.

The results concerning vertex magic total labelings of copies of regular graphs and copies of supermagic graphs are used is the next chapter to construct vertex magic total labelings of products of VMT and SPM graphs.

7 Cartesian products of graphs revisited

We investigated Cartesian products already in Chapter 4. We gave VMT labelings of products $G \,\Box\, H$ of certain regular graphs based on VMT labelings and VAMT labelings of graphs G and H. In this chapter we give VMT and SPM labelings of Cartesian products $G \,\Box\, H$ combining ideas from Chapter 6. We label products using VMT labelings and/or SPM labelings of copies of graphs G and H based on Kotzig arrays.

7.1 Products of regular VMT and SPM graphs

As mentioned in Chapter 4 on page 27 we can decompose the product $G \,\Box\, H$ into two factors: one factor consisting of $|H|$ copies of G, the second factor consisting of $|G|$ copies of H. Combining the ideas from the proofs of Theorem 6.1 and Theorem 6.4 we obtain the following result.

Theorem 7.1 *Let G be an r-regular VMT graph on m vertices. Let H be an s-regular Class 1 SPM graph on n vertices. Suppose*

(1) r is odd or n is odd,

(2) s is even or m is odd,

then there exists a VMT labeling of $G \,\Box\, H$.

Proof. Let λ_G be a VMT labeling of G with the magic constant k and let λ_H be an SPM labeling of H with the magic constant h. We can decompose $G \,\Box\, H$ into two factors: first factor F_G containing n copies of G and the second factor F_H containing m copies of H. Since r and n satisfy the properties of Theorem 6.1, we can construct a VMT labeling λ'_G of the factor F_G with the magic constant $k_G = nk - \frac{1}{2}(r+1)(n-1)$ using (23). Moreover, s and n satisfy the properties of Theorem 6.4 and we can construct an SPM labeling λ'_H of the factor F_H with the magic constant $h_H = \frac{1}{2}s(m|E(H)|+1)$ using (24).

We add a constant $n(m+|E(G)|)$ to every label in F_H to obtain a generalized supermagic labeling of F_G. The labeling

$$\begin{aligned}
\lambda(x) &= \lambda'_G(x) && \forall x \in V(F_G) \\
\lambda(xy) &= \lambda'_G(xy) && \forall xy \in E(F_G) \\
\lambda(xy) &= \lambda'_H(xy) + n(m+|E(G)|) && \forall xy \in E(F_H)
\end{aligned} \tag{27}$$

is a VMT labeling of $G \,\Box\, H$ with the magic constant

$$k_G + h_H + s(m+|E_G|) = nk + \frac{1}{2}(1-n)(r+1) + \frac{1}{4}s(mns+2) + \frac{1}{2}mns(r+2).$$

The magic property is easy to observe, since both the labelings λ'_G and λ'_H are magic. It is trivial to check that every label between 1 and $mn + n|E(G)| + m|E(H)|$ was used exactly once. □

An alternative labeling can be obtained by adding $m|E(H)|$ to all labels in the labeling λ'_G which results in a VMT labeling of $G \,\square\, H$ with the magic constant

$$k_G + h_H + (r+1)m|E_H| = nk + \frac{1}{2}(r+1)(1-n+mns) + \frac{1}{4}s(mns+2).$$

Moreover, we can use the labeling given by (22) instead of the labeling given by (23) to label F_G and obtain a VMT labeling of $G \,\square\, H$ with the magic constant

$$k + \frac{1}{4}m(r+2)(m+1)(n-1) + \frac{1}{4}s(mns+2) + \frac{1}{2}mns(r+2)$$

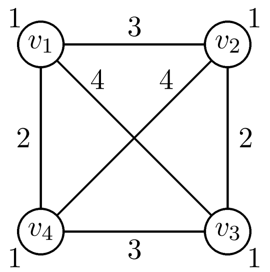
or alternatively

$$k + \frac{1}{4}m(r+2)(m+1)(n-1) + \frac{1}{4}s(mns+2) + \frac{1}{2}mns(r+1).$$

The result from Theorem 7.1 is further extended in Section 8.1.

Example 7.2 We find a VMT labeling of $K_4 \,\square\, K_{4,4}$ using Theorem 7.1. We take the VMT labeling of K_4 from Figure 1.4, the SPM labeling of $K_{4,4}$ from Figure 1.6 and the edge 4-coloring of $K_{4,4}$ from Figure 6.5. A total coloring of K_4, a Kotzig array of size 4×4 and a Kotzig array of size 4×8 are given in Figure 7.1.

Evaluating the magic constant for $m = 4$, $n = 8$, $r = 3$, $s = 4$, and $k = 20$ we get $k = 8 \cdot 20 - \frac{1}{2}7 \cdot 4 + \frac{1}{4}4 \cdot 130 + \frac{1}{2}128 \cdot 5 = 596$. One can verify that adding together the the partial sums (in boldface) on corresponding vertices in Figure 7.2 and Figure 7.3 we obtain the magic constant $k = 596$.



$$A_H = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{pmatrix} \qquad A_G = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

Figure 7.1: *Total coloring of K_4 using colors 1, 2, 3, and 4, a 4×4 Kotzig array and a 4×8 Kotzig array.*

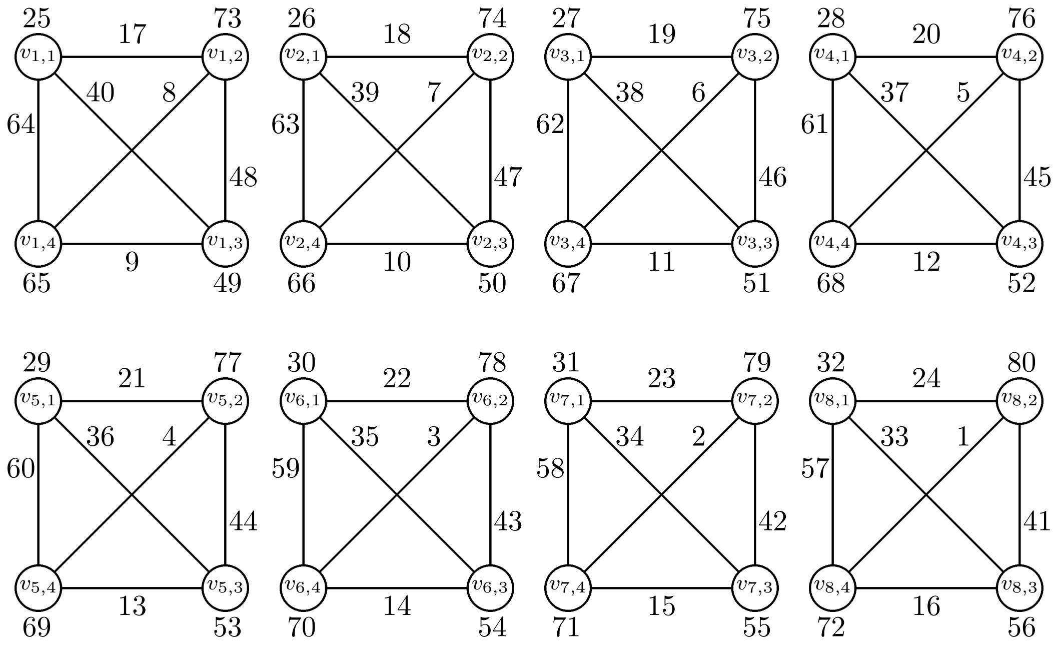


Figure 7.2: *Vertex magic total labeling of* $8K_4$ *with the magic constant* $k = 146$.

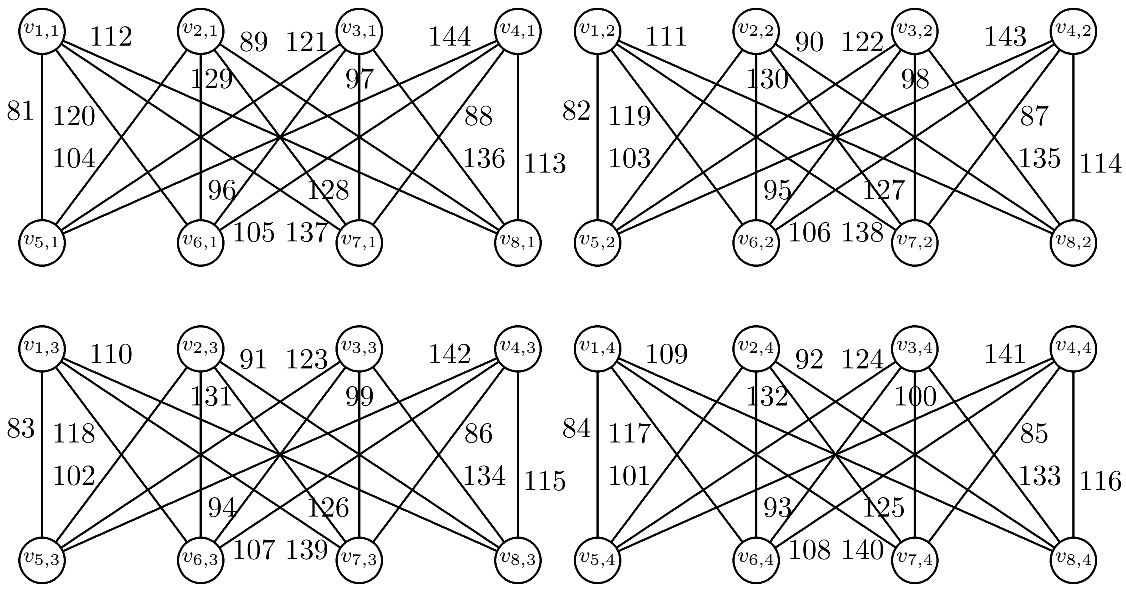


Figure 7.3: *Generalized supermagic labeling of* $4K_{4,4}$ *with the magic constant* $h = 450$.

7.2 Products of regular SPM graphs

The following result was already published in [Iv]. We state this result on SPM labelings of products of SPM graphs as an analogue to Theorem 7.1. The proof is constructive.

Theorem 7.3 *Let G be an r-regular Class 1 SPM graph on m vertices and let H be an s-regular Class 1 SPM graph on n vertices. Suppose*

(1) r is even or n is odd,

(2) s is even or m is odd,

then there exists an SPM labeling of $G \,\square\, H$.

Proof. The proof is similar to the proof of Theorem 7.1. Let λ_G be an SPM labeling of G and let λ_H be an SPM labeling of H. We use the same notation as in the proof of Theorem 7.1. We label edges of F_G by an SPM labeling λ'_G with the magic constant $h_G = \frac{1}{2}r(m|E_G|+1)$ and edges of F_H by an SPM labeling λ'_H with the magic constant $h_H = \frac{1}{2}s(n|E_H|+1)$ using (24).

We add a constant $n|E(G)|$ to every label in F_H. We obtain a GSPM of F_H. The labeling

$$\begin{aligned} \lambda(xy) &= \lambda'_G(xy) && \forall xy \in E(F_G) \\ \lambda(xy) &= \lambda'_H(xy) + n|E(G)| && \forall xy \in E(F_H) \end{aligned} \tag{28}$$

is an SPM labeling of $G \,\square\, H$ with the magic constant

$$\begin{aligned} h_G + h_H + ns|E_G| &= \frac{1}{2}r(m|E_G|+1) + \frac{1}{2}s(n|E_H|+1) + ns|E(G)| \\ &= \frac{1}{4}(mr+ns)^2 + \frac{1}{2}(r+s). \end{aligned}$$

We used the fact that $|E_G| = \frac{1}{2}mr$ and $|E_H| = \frac{1}{2}ns$. The magic property of λ follows from the magic property of the labelings λ'_G and λ'_H. It is trivial to check that every label between 1 and $n|E(G)| + m|E(H)|$ was used exactly once. $\square$

An alternative labeling can be obtained by adding $m|E(H)|$ to all labels in the labeling λ'_G. Moreover, we can use the labeling (25) instead of the labeling (24). It is easy to observe that all such labelings yield the same magic constant since the magic constant of a supermagic labeling of a regular graph depends only on the number of vertices and edges of a given graph.

For a further generalization of Theorem 7.3 see Section 8.2.

Example 7.4 We can use the labeling (28) to find an SPM labeling of the product $K_{4,4} \,\square\, K_{4,4}$. We take an SPM labeling of $K_{4,4}$ (Figure 1.6), a total coloring of $K_{4,4}$ and a Kotzig array of size 4×8 (Figure 7.1). The magic constant of such labeling is $h = \frac{1}{4}(32+32)^2 + \frac{1}{2}(4+4) = 1028$.

7.3 Products of regular VMT graphs

We can use Kotzig arrays to obtain a theorem analogous to Theorem 4.12 with a similar construction as in the proof of Theorem 6.9.

Theorem 7.5 *Let r, s, p, and q be positive integers, where $p, q \geq 2$. Let G be an rp-regular VMT graph on m vertices which can be factorized into r p-regular factors and let H be an sq-regular graph on n vertices which can be factorized into s q-regular factors and which has a VMT labeling with vertices labeled by consecutive integers. Suppose*

(1) r is even or n is odd,

(2) s is even or m is odd,

then there exists a VMT labeling of $G \,\square\, H$.

Proof. We denote the vertices of G by $u_1, u_2, \ldots, u_m$, the VMT labeling of G by λ_G and the p-regular factors by E_1, E_2, $\ldots$, E_r. We also denote the vertices of H by $v_1, v_2, \ldots, v_n$, the VMT labeling of H by λ_H and the q-regular factors by F_1, F_2, $\ldots$, F_s. We denote the magic constant of λ_G by k_G and the magic constant of λ_H by k_H. We suppose the vertices are ordered so that $\lambda_H(v_{j+1}) = \lambda_H(v_j) + 1$ for $1 < j \leq n$, thus if we denote the lowest vertex label by c, we have $\lambda_H(v_j) = (c-1) + j$ for $1 \leq j \leq n$.

We denote the vertices of $G \,\square\, H$ by $x_{i,j}$, where $x_{i,j} = (u_i, v_j)$, $u_i \in U$, $1 \leq i \leq m$, and $v_j \in V$, $1 \leq j \leq n$. We can decompose the product $G \,\square\, H$ into two factors: one factor F_G consisting of n copies of G, the second factor F_H consisting of m copies of H.

Take a Kotzig array $A = (a_{w,i})$ of size $r \times n$ and a Kotzig array $B = (b_{z,j})$ of size $s \times m$. Under the conditions on m, n, r, and s of the theorem such arrays always exist.

Consider the labeling λ given by

$$\begin{aligned}
\lambda(x_{i,j}) &= n(\lambda_G(u_j) - 1) + i && \forall x_{i,j} \in V(G \,\square\, H) \\
\lambda(x_{i,j}x_{i,l}) &= n(\lambda_G(u_j u_l) - 1) + a_{w,i} + 1 && \forall x_{i,j}x_{i,l} \in E(F_G); u_j u_l \in E_w \\
\lambda(x_{i,j}x_{k,j}) &= \frac{1}{2}mn(rp+2) + \frac{1}{2}nsqb_{z,j} + \lambda_H(v_i v_k) && \forall x_{i,j}x_{k,j} \in E(F_H); v_i v_k \in F_z
\end{aligned} \tag{29}$$

where $i, k = 1, 2, \ldots, n$ and $j, l = 1, 2, \ldots, m$.

It is easy to observe that the labels $1, 2, \ldots, \frac{1}{2}mn(rp+2)$ are used to label vertices and edges in F_G. Since λ_G is a bijection and there are n distinct elements in every row of the Kotzig array A, we use each label exactly once. The next $\frac{1}{2}mnsq$ labels are used to label edges of F_H. All m edges corresponding to a certain edge $v_i v_k$ of H obtain labels congruent to $\lambda_H(v_i v_k)$ modulo $\frac{1}{2}nsq$, thus we use each of them exactly once. Altogether λ is a bijection to $\{1, 2, \ldots, \frac{1}{2}mn(rp+sq+2)\}$.

We can use the proof of Corollary 3.11 for every copy of G in the factor F_G. Taking $a = n$, $b = -n + i$, $r = p$ (a p-factorization), and $c_i = a_{w,i} + 1 - n$ we evaluate the partial sum of labels at vertices in the factor F_G. We get

$$nk_G - n + i + \sum_{\substack{l=1\\l\neq j}}^{r}(a_{w,i} - n + 1) = nk_G - n + i + \frac{1}{2}rp(n-1) - rpn + rp$$

using the fact that the sum over a column of an $r \times n$ Kotzig array is $\frac{1}{2}r(n-1)$. This further simplifies to

$$n(k_G - 1) + i - \frac{1}{2}rp(n-1).$$

The partial sum of labels over edges adjacent to vertices in F_H is

$$\sum_{x_{k,j}\in N(E(F_H))} \lambda(x_{i,j}x_{k,j}) = \frac{1}{2}mnsq(rp+2) + \frac{1}{2}nsq\sum_{\substack{k=1\\k\neq i}}^{s} b_{z,j} + \sum_{\substack{k=1\\k\neq i}}^{s} \lambda_H(v_iv_k).$$

Using the fact that the sum over a column of an $s \times m$ Kotzig array is $\frac{1}{2}s(m-1)$ and using $\lambda_H(v_i) = (c-1) + i$, we get

$$\sum_{x_{k,j}\in N(E(F_H))} \lambda(x_{i,j}x_{k,j}) = \frac{1}{2}mnsq(rp+2) + \frac{1}{4}(sq)^2n(m-1) + k_H - c + 1 - i.$$

The weight of every vertex in $G \,\square\, H$ is the sum of both partial sums.

$$\begin{aligned} w_\lambda(x_{i,j}) &= n(k_G-1) + i - \frac{1}{2}rp(n-1) + \frac{1}{2}mnsq(rp+2) + \frac{1}{4}(sq)^2n(m-1) + \\ &\quad k_H - c + 1 - i \\ &= n(k_G-1) + k_H - \frac{1}{2}rp(n-1) + \frac{1}{2}mnsq(rp+2) + \frac{1}{4}(sq)^2n(m-1) - c + 1 \end{aligned}$$

So λ is a vertex magic total labeling of $G \,\square\, H$ with the magic constant $n(k_G-1) + k_H - \frac{1}{2}rp(n-1) + \frac{1}{2}mnsq(rp+2) + \frac{1}{4}(sq)^2n(m-1) - c + 1$. □

Example 7.6 We can find a VMT labeling of $C_4 \,\square\, K_5$ using Theorem 7.5. We take the VMT labeling of C_4 and K_5, the 1-factorization of C_4 and the 2-factorization of K_5 from Figure 6.1 If we take the "thick" factor of C_4 to be E_1, the "thin" factor of C_4 to be E_2, the "thick" factor of K_5 to be F_1 and the "thin" factor of K_5 to be F_2 and if we take the Kotzig array A of size 2×5 and the Kotzig array B of size 4×4 to be

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{pmatrix},$$

we obtain the same VMT labeling of $C_4 \,\square\, K_5$ as given by Theorem 4.12, see Figure 4.9. Taking $m = 4$, $n = 5$, $r = 2$, $p = 1$, $s = 2$, $q = 2$, $c = 11$, $k_G = 13$, and $k_H = 35$ the magic constant is $k = 5(13-1)+35-\frac{1}{2}2\cdot1(5-1)+\frac{1}{2}4\cdot5\cdot2\cdot2(2\cdot1+2)+\frac{1}{4}(2\cdot2)^2 5(4-1)-11+1 = 301$.

Note 7.7 It should be pointed out that if a graph H satisfies the conditions of Theorem 7.5, then n has to be even. It was proved in [MsP, p. 54] that for a regular on an even number of vertices exists no vertex magic total labeling such that the vertex labels constitute an arithmetic progression with an odd difference.

7.4 Remarks

The results given in Chapter 4 in Theorem 4.6, Theorem 4.8 and Theorem 4.10 are *not* special cases of theorems given in this section, because K_5 has no SPM labeling and no 2-regular graph has an SPM labelings.

Theorem 4.12 is a special case of Theorem 7.5 for $p = q = 2$ and for certain Kotzig arrays. In comparison to Theorem 4.12 we allow G be an odd regular graph, so we can find VMT labelings for e.g. $G \square H = K_4 \square K_5$.

An important observation is that if one of the graphs G and H is Class 1, then $G \square H$ for nontrivial graphs is always Class 1 (see [West, p. 284]). This means that we can construct VMT labelings for repeated products such as $((K_4 \square K_{6,6}) \square M_6) \square K_{4,4} \square \cdots$ and that we can construct also SPM labelings for repeated products such as $((Q_4 \square K_{4,4}) \square K_{6,6}) \square M_8 \square \cdots$.

8 Factors and compositions of graphs

As mentioned in Section 1.3 an unpublished conjecture by MacDougall says that any regular graph other than K_2 or $2K_3$ is vertex magic total[4]. Because obviously no graph containing K_2 as a component can have a VMT labeling we rephrase the conjecture:

> Any r-regular graph for $r > 1$, with the exception of $2K_3$, has a vertex magic total labeling.

All results in this thesis support the conjecture. According to results given in the previous chapters, many classes of Cartesian products of regular graphs have a VMT or an SPM labeling. The theorems in this chapter further extend the set of classes of regular graphs which are known to have magic-type labelings. Moreover we give VMT, SPM and VAMT labelings also for certain classes of nonregular graphs.

8.1 On VMT labelings of graphs

Theorem 8.1 *Let G be a graph. If G can be decomposed into a VMT factor F_0 and regular SPM factors $F_1, F_2, \ldots, F_n$, then G has a VMT labeling.*

Proof. Let $G(V, E)$ be a graph satisfying the conditions above. We denote the VMT labeling of F_0 by λ_{F_0} and the magic constant of λ_{F_0} by k_{F_0}. Similarly we denote the SPM labeling of the r_i-regular factor F_i by λ_{F_i} and the magic constant of each λ_{F_i} by h_{F_i} for $i = 1, 2, \ldots, n$. While $v = |V|$ and $e = |E|$ we define

$$\begin{aligned} b_1 &= v + |E(F_0)| \\ b_i &= b_1 + \sum_{j=1}^{i-1} |E(F_j)| = v + |E(F_0)| + \frac{1}{2} v \sum_{j=1}^{i-1} r_j \end{aligned}$$

for $i = 2, 3, \ldots, n$. We show that the labeling λ given by

$$\begin{aligned} \lambda(x) &= \lambda_{F_0}(x) \qquad \text{for } x \in V \\ \lambda(xy) &= \begin{cases} \lambda_{F_0}(xy) + b_1 & \text{for } xy \in E(F_0) \\ \lambda_{F_1}(xy) + b_2 & \text{for } xy \in E(F_1) \\ \vdots & \\ \lambda_{F_n}(xy) + b_n & \text{for } xy \in E(F_n) \end{cases} \end{aligned} \tag{30}$$

is a vertex magic total labeling of G.

It is easy to observe that λ is a bijection to $\{1, 2, \ldots, v + e\}$. The first $v + |E(F_0)|$ labels are used to label the vertices and edges of F_0. The next $|E(F_1)|$ labels are assigned

[4] A note about the conjecture can be found in [McQ, MS]. The author learned from personal communication with J. MacDougall that the conjecture was not published yet.

to the edges of F_1, notice definition of the b_i's. Similarly, different labels are used over the factors F_2, F_3, $\ldots$, F_n. Since for $i = 1, \ldots, n$ each of the labelings λ_{F_i} is an SPM labeling with the magic constant h_{F_i}, from Theorem 3.9 follows that λ is a VMT labeling with the magic constant

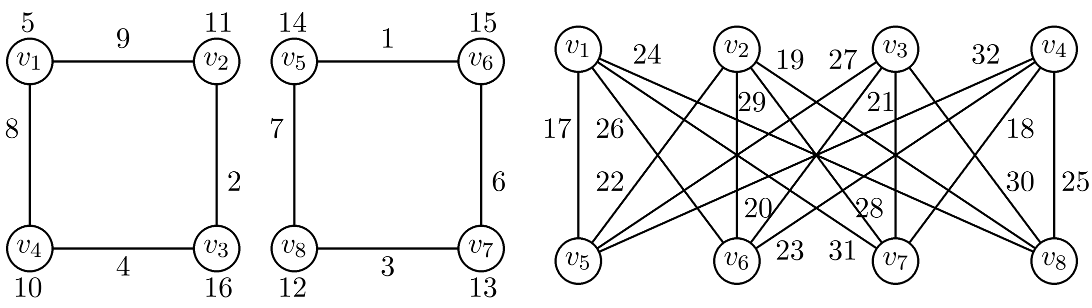
$$k = k_{F_0} + \sum_{i=1}^{n} (h_{F_i} + r_i b_i)$$

which using $h_{F_i} = \frac{1}{2} r_i(1 + |E(F_i)|) = \frac{1}{4} r_i(2 + vr_i)$ for $j = 1, 2, \ldots, n$ and the definition of b_i simplifies to

$$\begin{aligned}
k &= k_{F_0} + \frac{1}{2}\sum_{i=1}^{n} r_i + \frac{1}{4} v \sum_{i=1}^{n} r_i^2 + \sum_{i=1}^{n} r_i \left(v + |E(F_0)| + \frac{1}{2} v \sum_{j=1}^{i-1} r_j \right) \\
&= k_{F_0} + \left(\frac{1}{2} + v + |E(F_0)| \right) \sum_{i=1}^{n} r_i + \frac{1}{4} v \left(\sum_{i=1}^{n} r_i^2 + 2 \sum_{i=1}^{n} \sum_{j=1}^{i-1} r_i r_j \right) \\
&= k_{F_0} + \left(\frac{1}{2} + v + |E(F_0)| \right) \sum_{i=1}^{n} r_i + \frac{1}{4} v \left(\sum_{i=1}^{n} r_i \right)^2 .
\end{aligned}$$

Thus, we see that λ is a vertex magic total labeling of G with the magic constant $k = k_{F_0} + \left(\frac{1}{2} + v + |E(F_0)|\right) \sum_{i=1}^{n} r_i + \frac{1}{4} v \left(\sum_{i=1}^{n} r_i\right)^2$. □

Example 8.2 We can find a VMT labeling of a 6-regular graph G on 8 vertices (K_8 without a perfect matching). G can be decomposed into $F_0 = 2C_4$ and $F_1 = K_{4,4}$. We know a VMT labeling of $2C_4$ (see Figure 6.4) and an SPM labeling of $K_{4,4}$ (see Figure 1.6). Using the construction from the proof of Theorem 8.1 we take $b_1 = 16$, $r_1 = 4$, $k_{F_0} = 22$, and $h_{F_1} = 34$. We obtain a VMT labeling of G with the magic constant $k = 22 + (1/2 + 8 + 8) \cdot 4 + (1/4) \cdot 8 \cdot 16 = 120$, see Figure 8.1.



Factor F_0 Factor F_1

Figure 8.1: *Vertex magic total labeling of G in two factors $F_0 = 2C_4$ and $F_1 = K_{4,4}$ with the magic constant $k = 120$.*

Example 8.3 We can find also VMT labelings for nonregular graphs. Let G be K_8 without $2P_4$. G can be decomposed into $F_0 = 2P_4$ and $F_1 = K_{4,4}$. We have a VMT labeling of $2P_4$ in Figure 8.2 (found by trial and error) and an SPM labeling of $K_{4,4}$ (see Figure 1.6). We use the construction from the proof of Theorem 8.1 and we take $b_1 = 14$, $r_1 = 4$, $k_{F_0} = 20$, and $h_{F_1} = 34$. We obtain a VMT labeling of G with the magic constant $k = 20 + (1/2 + 8 + 6) \cdot 4 + (1/4) \cdot 8 \cdot 16 = 110$, see Figure 8.2.

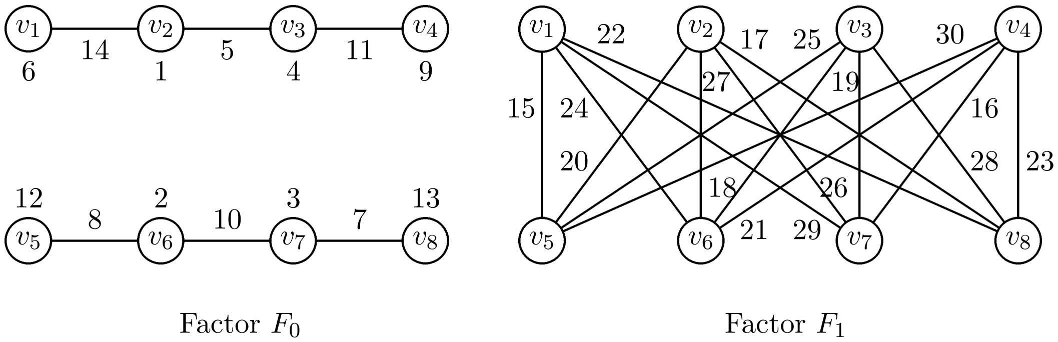


Figure 8.2: *Vertex magic total labeling of G in two factors $F_0 = 2P_4$ and $F_1 = K_{4,4}$ with the magic constant $k = 110$.*

The following theorem is similar to Theorem 8.1. It covers the case when the nonregular factor F_0 has an SPM labeling instead of a VMT labeling.

Theorem 8.4 *Let G be a graph. If G can be decomposed into an SPM factor F_0, a regular VMT factor F_1, and regular SPM factors $F_2, F_3, \ldots, F_n$, then G has a VMT labeling.*

Proof. The proof is analogous to the proof of Theorem 8.1. We denote the magic labelings of $F_0, F_1, \ldots, F_n$ by $\lambda_{F_0}, \lambda_{F_1}, \ldots, \lambda_{F_n}$ and the magic constants by $h_{F_0}, k_{F_1}, h_{F_2}, \ldots, h_{F_n}$. The factors $F_1, F_2, \ldots, F_n$ are regular, we denote their degrees by $r_1, r_2, \ldots, r_n$. F_0 does not need to be regular. Taking $v = |V|$ and $e = |E|$ we define

$$\begin{aligned} b_1 &= |E(F_0)| \\ b_2 &= v + |E(F_0)| + |E(F_1)| \\ b_i &= b_1 + b_2 + \sum_{j=2}^{i-1} |E(F_j)| = v + |E(F_0)| + |E(F_1)| + \frac{1}{2} v \sum_{j=2}^{i-1} r_j \end{aligned}$$

for $i = 3, 4, \ldots, n$. It can be verified that the labeling λ given by

$$\begin{aligned} \lambda(x) &= \lambda_{F_1}(x) \quad \text{for } x \in V \\ \lambda(xy) &= \begin{cases} \lambda_{F_0}(xy) + b_1 & \text{for } xy \in E(F_0) \\ \lambda_{F_1}(xy) + b_2 & \text{for } xy \in E(F_1) \\ \vdots & \\ \lambda_{F_n}(xy) + b_n & \text{for } xy \in E(F_n) \end{cases} \end{aligned}$$

is a bijection to $\{1, 2, \ldots, v+e\}$ and that λ is a VMT labeling of G with the magic constant

$$k = h_{F_0} + k_{F_1} + |E(F_0)|(r_1+1) + \left(\frac{1}{2}(1+vr_1) + v + |E(F_0)|\right)\sum_{i=2}^{n} r_i + \frac{1}{4}v\left(\sum_{i=2}^{n} r_i\right)^2.$$

□

Note 8.5 Examples of nonregular graph with an SPM labeling are given in [DIS].

Let G be a graph. Theorem 8.1 and Theorem 8.4 transform the question about G having a VMT labeling into a question about G having certain decomposition. For some classes of graphs we can guarantee certain decompositions and then the property of being vertex magic total follows. We give a couple of results for compositions of graphs.

The composition of graphs is among the most common operations on graphs.

Definition 8.6 *Let G and H be graphs with the vertex sets $V(G) = U$ and $V(H) = V$. The* composition *or* lexicographic product *of graphs G and H is the graph $G[H]$ with the vertex set $V(G[H]) = U \times V$. Two vertices (u, v) and (u', v') are adjacent in $G[H]$ if $uu' \in E(G)$ or if $u = u'$ and $vv' \in E(H)$.*

Example 8.7 The composition of P_2 and P_3 is the graph $P_2[P_3]$ shown in Figure 8.3.

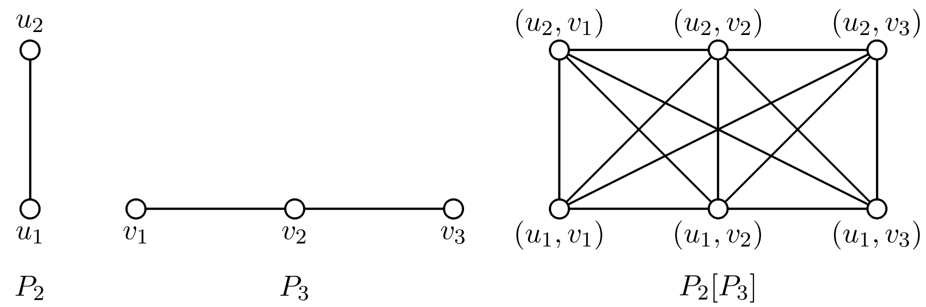


Figure 8.3: *Composition $P_2[P_3]$.*

The composition of two complete graphs is a complete graph: $K_m[K_n] = K_{mn}$. Copies of a graph G can be described as a composition of a null graph on n vertices $\overline{K_n}$ and a graph G: $\overline{K_n}[G] = nG$. Composition of a complete graph and a null graph is a complete multipartite graph: $K_n[\overline{K_p}] = K_{\underbrace{n, n, \ldots, n}_{p}}$.

Note 8.8 Notice that in $G[H]$ every vertex of G is "blown up" into a copy of H and every edge of G is "blown up" into a complete bipartite graph $K_{|H|,|H|}$. We point out that the composition $G[H]$ can be decomposed into two factors: one factor consisting of

$|G|$ copies of H, the second factor consisting of a multipartite graph with each partite set of size $|H|$.

Theorem 8.9 *Let G be a p-regular graph on $2m$ vertices which has a 1-factorization and let H be an r-regular VMT graph on n vertices. If n is even and r is odd, then the composition $G[H]$ has a vertex magic total labeling.*

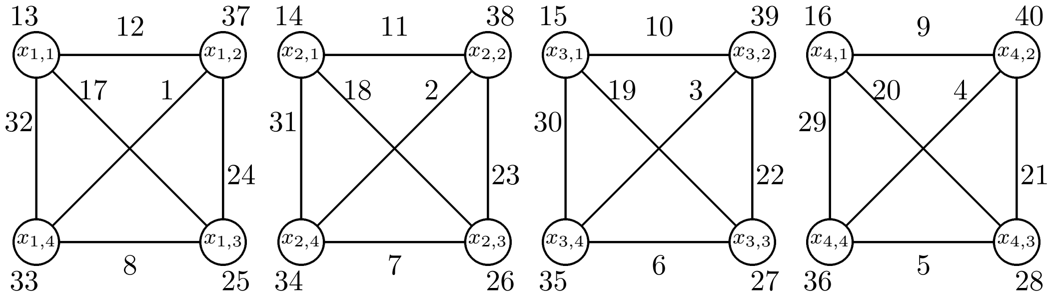
Proof. According to Note 8.8 we can decompose $G[H]$ into a factor F_0 which consists of $2m$ copies of H and a factor F_0' which consists of a multipartite graph. The 1-factorization of G has p factors, thus F_0' is a pn-regular graph. Every edge in a 1-factor of G corresponds to a $K_{n,n}$ subgraph in the factor F_0' because every edge of G is "blown up" into a $K_{n,n}$. Every 1-factor of G has m edges and corresponds to an n-factor of F_0' consisting of m copies of $K_{n,n}$. We denote the p n-factors of F_0' by $F_1, F_2, \ldots, F_p$.

According to the assumption H is an r-regular VMT graph where r is odd. From Theorem 6.1 follows that also nH has a VMT labeling λ_0. We denote the magic constant of λ_0 by k_{λ_0}. Moreover, from Theorem 6.4 follows that every F_i for $i = 1, 2, \ldots, p$ has an SPM labeling ($K_{n,n}$ is n-regular and Class 1). From Theorem 8.1 follows that $G[H]$ has a VMT labeling with the magic constant

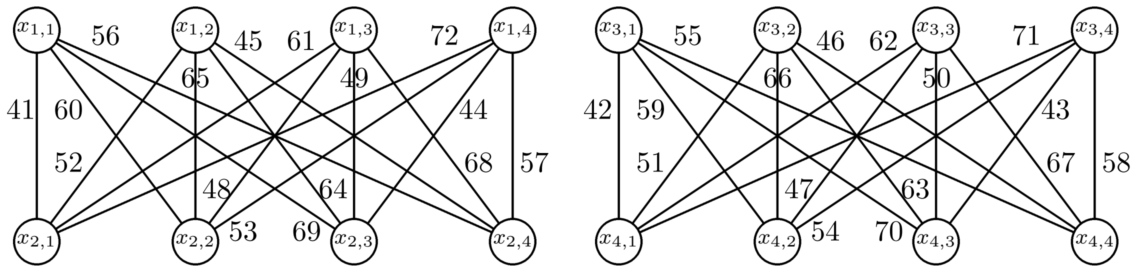
$$\begin{aligned} k &= k_{F_0} + \left(\frac{1}{2} + v + |E(F_0)|\right) \sum_{i=1}^{n} r_i + \frac{1}{4} v \left(\sum_{i=1}^{n} r_i\right)^2 \\ &= k_{\lambda_0} + \left(\frac{1}{2} + 2mn + mnr\right) np + \frac{1}{2} mn \,(np)^2 . \end{aligned}$$

□

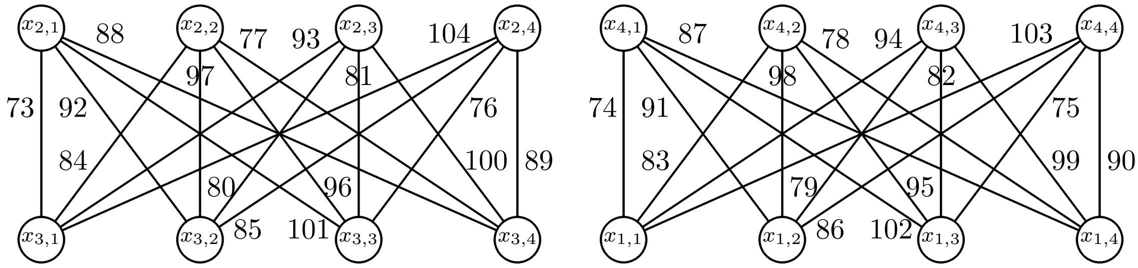
Example 8.10 According to Corollary 8.9 $C_4[K_4]$ has a VMT labeling. The factor F_0 consists of four copies of K_4. A VMT labeling of $4K_4$ with the magic constant $k = 74$ is given in Figure 4.3. A 1-factorization of C_4 is at hand. Based on the 1-factorization of C_4 we decompose F_0' into two factors F_1 and F_2, each consisting of two copies of $K_{4,4}$. An SPM labeling of $2K_{4,4}$ with the magic constant $h = 66$ is given in Figure 6.6. Using the labeling (30) we find a VMT labeling of $C_4[K_4]$. For $k_{\lambda_0} = 74$, $m = 2$, $n = 4$, $p = 2$, and $r = 3$ we get the magic constant $k = 74 + (1/2 + 2\cdot 2\cdot 4 + 2\cdot 4\cdot 3)\cdot 4\cdot 2 + 1/2\cdot 2\cdot 4\cdot (4\cdot 2)^2 = 654$.



Factor F_0 with the magic constant $k_{F_0} = 74$.



Factor F_1 with the magic constant $h_{F_1} = 226$.



Factor F_2 with the magic constant $h_{F_2} = 354$.

Figure 8.4: *Vertex magic total labeling of $C_4[K_4]$ with the magic constant $h = 654$.*

We can relax the restrictions on n and r in Theorem 8.9 because they follow only from constructions which are known so far.

Theorem 8.11 *Let G be a p-regular graph on $2m$ vertices which has a 1-factorization and let H be an r-regular graph on n vertices so that*

(1) $2mH$ has a VMT labeling and

(2) $mK_{n,n}$ has an SPM labeling.

Then the composition $G[H]$ has a vertex magic total labeling.

Proof. The proof is almost identical to the proof of Theorem 8.9. We use the same notation and the same decomposition into factors F_i for $i = 1, 2, \ldots, p$. According to the assumption of this theorem F_0 has a VMT labeling and every factor F_i for $i = 1, 2, \ldots, p$

has an SPM labeling. Thus from Theorem 8.1 follows that $G[H]$ has a VMT labeling with the magic constant $k = k_{\lambda_0} + \left(\frac{1}{2} + 2mn + mnr\right) np + \frac{1}{2}mn\,(np)^2$. $\square$

Example 8.12 Using the construction given in the proof of Theorem 8.11 we can find a VMT labeling of $K_2[C_4]$. The factor F_0 consists of $2C_4$ and the factor $F_1 = K_{4,4}$. We know a VMT labeling for $2C_4$ (given in Figure 6.4) though we do not have a general construction for VMT labelings of $2nC_4$. The magic constant for $k_{\lambda_0} = 22$, $m = 1$, $n = 4$, $p = 1$, and $r = 2$ is $k = 22 + (1/2 + 2 \cdot 1 \cdot 4 + 1 \cdot 4 \cdot 2) \cdot 4 \cdot 1 + 1/2 \cdot 1 \cdot 4 \cdot (4)^2 = 120$.

The VMT labeling of $K_2[C_4]$ given by (30) is identical to the labeling already given in Example 8.2, see Figure 8.1.

8.2 On SPM labelings of graphs

The following result extends a result given in [Iv]. We give a supermagic labeling of a graph which can be factorized into supermagic graphs. While in [Iv] all factors were required to be regular, we allow one of the factors to be nonregular.

Theorem 8.13 *Let G be graph. If G can be decomposed into an SPM factor F_0 and regular SPM factors $F_1, F_2, \ldots, F_n$, then G has an SPM labeling.*

Proof. The proof is similar to the proof of Theorem 8.1. Let $G(V, E)$ be a graph satisfying the conditions above. We denote the SPM labeling of F_0 by λ_{F_0} and the magic constant of λ_{F_0} by h_{F_0}. F_0 does not need to be regular. Then we denote the SPM labeling of each r_i-regular factor F_i by λ_{F_i} and the magic constant of each λ_{F_i} by h_{F_i} for $i = 1, 2, \ldots, n$. While $e = |E|$ we define

$$\begin{aligned} b_1 &= |E(F_0)| \\ b_i &= b_1 + \sum_{j=1}^{i-1} |E(F_j)| = |E(F_0)| + \frac{1}{2} v \sum_{j=1}^{i-1} r_j \end{aligned}$$

for $i = 2, 3, \ldots, n$. We show that the labeling λ given by

$$\lambda(xy) = \begin{cases} \lambda_{F_0}(xy) & \text{for } xy \in E(F_0) \\ \lambda_{F_1}(xy) + b_1 & \text{for } xy \in E(F_1) \\ \lambda_{F_2}(xy) + b_2 & \text{for } xy \in E(F_2) \\ \vdots & \\ \lambda_{F_n}(xy) + b_n & \text{for } xy \in E(F_n) \end{cases} \qquad (31)$$

is an SPM labeling of G.

Again it is easy to observe that λ is a bijection to $\{1, 2, \ldots, e\}$. The first $|E(F_0)|$ labels are used to label the edges of F_0. The next $|E(F_1)|$ labels are assigned to the edges of F_1. Similarly, based on the definition of b_i different labels are used over the factors

F_2, F_3, $\ldots$, F_n. Since for $i = 0, 1, \ldots, n$ each of the labelings λ_{F_i} is an SPM labeling with the magic constant h_{F_i}, it follows from Theorem 3.9 that λ is an SPM labeling with the magic constant

$$h = h_{F_0} + \sum_{i=1}^{n} (h_{F_i} + r_i b_i)$$

which using $h_{F_i} = \frac{1}{2} r_i(1 + |E(F_i)|) = \frac{1}{4} r_i(2 + v r_i)$ for $j = 1, 2, \ldots, n$ and the definition of b_i simplifies to

$$\begin{aligned} h &= h_{F_0} + \frac{1}{2}\sum_{i=1}^{n} r_i + \frac{1}{4} v \sum_{i=1}^{n} r_i^2 + \sum_{i=1}^{n} r_i \left(|E(F_0)| + \frac{1}{2} v \sum_{j=1}^{i-1} r_j \right) \\ &= h_{F_0} + \left(\frac{1}{2} + |E(F_0)| \right) \sum_{i=1}^{n} r_i + \frac{1}{4} v \left(\sum_{i=1}^{n} r_i^2 + 2 \sum_{i=1}^{n} \sum_{j=1}^{i-1} r_i r_j \right) \\ &= h_{F_0} + \left(\frac{1}{2} + |E(F_0)| \right) \sum_{i=1}^{n} r_i + \frac{1}{4} v \left(\sum_{i=1}^{n} r_i \right)^2 . \end{aligned}$$

It follows that λ is a supermagic labeling of G with the magic constant $h = h_{F_0} + \left(\frac{1}{2} + |E(F_0)|\right) \sum_{i=1}^{n} r_i + \frac{1}{4} v \left(\sum_{i=1}^{n} r_i\right)^2$. $\square$

Examples of nonregular supermagic graphs can be found in [DIS].

We can find SPM labelings for certain compositions of graphs based on Theorem 8.13.

Theorem 8.14 *Let G be a p-regular graph on $2m$ vertices which has a 1-factorization and let H be an r-regular Class 1 SPM graph on n vertices where r is even and $n \geq 3$. If n is even or m is odd, then the composition $G[H]$ has a supermagic labeling.*

Proof. The proof is analogous to the proof of Theorem 8.9. We use the same notation and the same decomposition into factors F_i for $i = 1, 2, \ldots, p$.

According to the assumption H is an r-regular Class 1 graph where r is even. From Theorem 6.4 follows that also nH has an SPM labeling λ_0. We denote the magic constant of λ_0 by h_{λ_0}. Moreover, from Theorem 6.4 it also follows that every F_i for $i = 1, 2, \ldots, p$ has an SPM labeling ($K_{n,n}$ is n-regular and Class 1, $n \geq 3$).

From Theorem 8.13 follows that $G[H]$ has an SPM labeling with the magic constant

$$h = h_{F_0} + \left(\frac{1}{2} + |E(F_0)| \right) \sum_{i=1}^{n} r_i + \frac{1}{4} v \left(\sum_{i=1}^{n} r_i \right)^2$$

which using $h_{F_0} = \frac{1}{4} r(2 + vr)$, $|E(F_0)| = \frac{1}{2} vr$, $\sum_{i=1}^{n} r_i = pn$, and $v = mn$ simplifies to

$$\begin{aligned} h &= \frac{1}{4} r\,(2 + mnr) + \left(\frac{1}{2} + \frac{1}{2} mnr \right) pn + \frac{1}{4} mn (pn)^2 \\ &= \frac{1}{2} (r + pn)\,(1 + mn(r + pn)) . \end{aligned}$$

$\square$

A theorem similar to Theorem 8.14 was published in [Iv].

Theorem 8.15 (J. Ivančo [Iv]) *Let G_1 be a regular graph on m vertices and G_2 a regular graph on n vertices such that*

(1) $n \geq 3$;

(2) mG_2 is supermagic or totally disconnected;

(3) $n \equiv 0 \pmod 2$ or $n|E(G_1)| \equiv 1 \pmod 2$.

Then $G_1[G_2]$ is supermagic.

Even though many graphs satisfy both theorems, none of them is stronger. The composition $C_5[\overline{K_3}]$ is SPM according to Theorem 8.15 but it does not satisfy the conditions of Theorem 8.14. On the other hand $C_6[\overline{K_3}]$ is SPM according to Theorem 8.14, see Example 8.19, but it does not satisfy the conditions of Theorem 8.15.

Unlike by a VMT labeling the magic constant of an SPM labeling of a regular graph $G(V, E)$ depends only on the number of vertices $|V|$ and the degree $\Delta(G)$. Thus various SPM labeling of the same graph yield the same magic constant.

Example 8.16 Using the construction given in the proof of Theorem 8.14 we can find an SPM labeling of $C_4[K_{4,4}]$. The factor F_0 consists of $4K_{4,4}$, the factor $F_1 = 2K_{8,8}$ and the factor $F_2 = 2K_{8,8}$. We can construct an SPM labelings for $4K_{4,4}$ and for $2K_{8,8}$ using Theorem 6.4 since $K_{4,4}$ and $K_{8,8}$ have SPM labelings. The magic constant for $m = 2$, $n = 8$, $p = 2$, and $r = 4$ is $h = \frac{1}{2} \cdot (4 + 2 \cdot 8) \cdot (1 + 2 \cdot 8 \cdot (4 + 2 \cdot 8)) = 3210$.

Note 8.17 If we try to relax some conditions of Theorem 8.14 similarly as in Theorem 8.11, we realize that this is not possible. According to Theorem 6.5, if we need an SPM labeling of $2mH$, then the degree r of H must be even. Moreover, since every factor F_i for $i = 1, 2, \ldots, p$ is $mK_{n,n}$, must be either n even or m odd. Therefore, the restrictions of Theorem 8.14 cannot be relaxed any more. Yet, this does not mean that $G[H]$ for m even and n odd does not have an SPM labeling.

From many classes of graphs which can be shown to have an SPM labeling using Theorem 8.14 we picked one example. Compare the following corollary to the known results mentioned in Table 1.2.

Corollary 8.18 *Let m and n be integers such that m is odd or n is even and $n \geq 3$, then the composition $C_{2m}[\overline{K_n}]$ has an SPM labeling.*

Proof. The graph $\overline{K_n}$ has a "trivial" SPM labeling and a 1-factorization of C_{2m} is obvious. The statement follows from Theorem 8.14. □

Example 8.19 According to Corollary 8.18 $C_6[\overline{K_3}]$ has an SPM labeling. A 1-factorization of C_6 is at hand, see Figure 8.5. The edges of F_1 are drawn in "thick". The factor

F_0 is a null graph on 18 vertices. Based on the 1-factorization of C_6 we decompose $C_6[\overline{K_3}]$ into two factors F_1 and F_2, each consisting of three copies of $K_{3,3}$. An SPM labeling of $K_{3,3}$ is in Figure 8.5. From Theorem 6.4 it follows that $3K_{3,3}$ also has an SPM labeling. Using the labeling (31) we find an SPM labeling of $C_6[\overline{K_3}]$. The magic constant for $m = 3$, $n = 3$, $p = 2$, and $r = 0$ is $h = \frac{1}{2} \cdot (2 \cdot 3) \cdot (1 + 3 \cdot 3 \cdot (2 \cdot 3)) = 165$, see Figure 8.6.

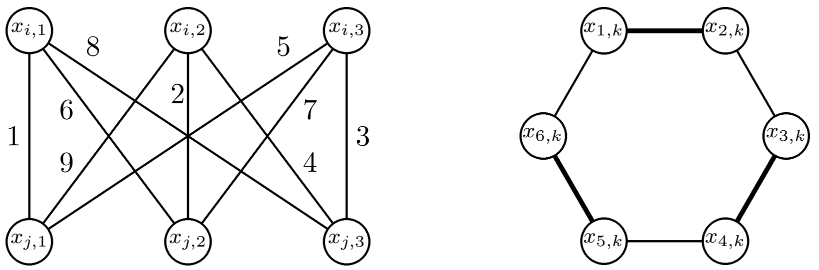
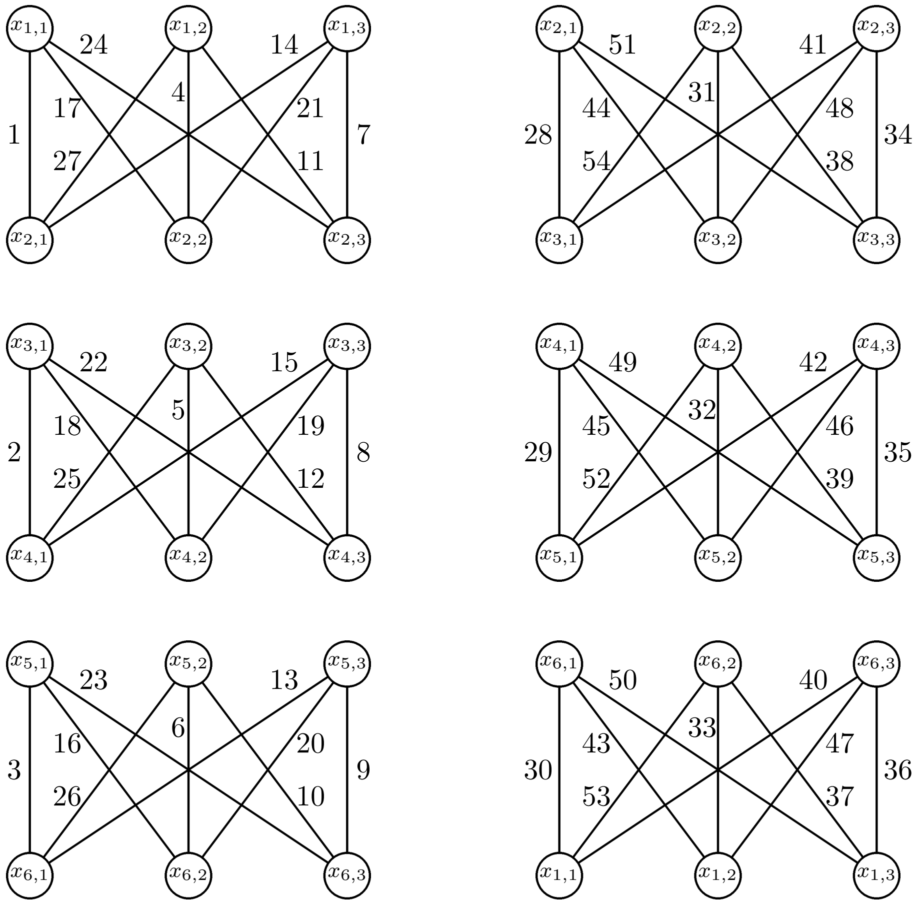


Figure 8.5: *SPM labeling of* $K_{3,3}$ *with the magic constant* 15*; a* 1*-factorization of* C_6*.*



Factor F_1 with $h_{F_1} = 42$. Factor F_2 with $h_{F_2} = 123$.

Figure 8.6: *Supermagic labeling of* $C_6[\overline{K_3}]$ *with the magic constant* $h = 165$*.*

8.3 On VAMT labelings of graphs

For (s,d)-VAMT labelings we can give analogous results as for VMT or SPM labelings in the previous sections.

Theorem 8.20 *Let G be a graph with v vertices and e edges. If G can be decomposed into an (s,d)-VAMT factor F_0 and an r_1-regular SPM factor F_1, an r_2-regular SPM factor F_2, ..., an r_n-regular SPM factor F_n, then G has an $\left(s+\left(\frac{1}{2}+v+|E(F_0)|\right)\sum_{i=1}^{n} r_i+\frac{1}{4}v\left(\sum_{i=1}^{n} r_i\right)^2, d\right)$-VAMT labeling.*

Proof. Again the proof is similar to the proof of Theorem 8.1. Let $G(V,E)$ be a graph satisfying the conditions above. We denote the (s,d)-VAMT labeling of F_0 by λ_{F_0}. Without loss of generality we can denote the vertices $v_1, v_2, \dots, v_v$ so that the weights form an arithmetic progression

$$\lambda_{F_0}(v_i) = s + (i-1)d$$

for $i = 1, 2, \dots, n$. Notice that F_0 does not need to be regular. Then we denote the SPM labeling of each r_i-regular factor F_i by λ_{F_i} and the magic constant of each λ_{F_i} by h_{F_i} for $i = 1, 2, \dots, n$. While $v = |V|$ and $e = |E|$ we define

$$\begin{aligned} b_1 &= v + |E(F_0)| \\ b_i &= b_1 + \sum_{j=1}^{i-1} |E(F_j)| = v + |E(F_0)| + \frac{1}{2}v\sum_{j=1}^{i-1} r_j \end{aligned}$$

for $i = 2, 3, \dots, n$. Consider the labeling λ given by

$$\begin{aligned} \lambda(x) &= \lambda_{F_0}(x) \qquad \text{for } x \in V \\ \lambda(xy) &= \begin{cases} \lambda_{F_0}(xy) + b_1 & \text{for } xy \in E(F_0) \\ \lambda_{F_1}(xy) + b_2 & \text{for } xy \in E(F_1) \\ \vdots & \\ \lambda_{F_n}(xy) + b_n & \text{for } xy \in E(F_n). \end{cases} \end{aligned} \tag{32}$$

It can be easily observed that λ is a bijection to $\{1, 2, \dots, v+e\}$.

Since for $i = 1, \dots, n$ each of the labelings λ_{F_i} is an SPM labeling with the magic constant h_{F_i}, it follows from Theorem 3.9 that the weight of every vertex for $i = 1, 2, \dots, n$ is

$$w_\lambda(v_i) = s + (i-1)d + \sum_{i=1}^{n} (h_{F_i} + r_i b_i)$$

which using similar steps as in the proof of Theorem 8.1 simplifies to

$$= s + (i-1)d + \left(\frac{1}{2} + v + |E(F_0)|\right)\sum_{i=1}^{n} r_i + \frac{1}{4}v\left(\sum_{i=1}^{n} r_i\right)^2.$$

So λ is an $(s + \left(\frac{1}{2} + v + |E(F_0)|\right) \sum_{i=1}^{n} r_i + \frac{1}{4} v \left(\sum_{i=1}^{n} r_i\right)^2, d)$-vertex antimagic total labeling of G. □

The following theorem covers the case when the nonregular factor F_0 has an SPM labeling instead of a VAMT labeling. As mentioned in Note 8.5 we can have nonregular SPM graphs.

Theorem 8.21 *Let G be a graph. If G can be decomposed into an SPM factor F_0 with the magic constant h_{F_0}, an r_1-regular (s, d)-VAMT factor F_1, an r_2-regular SPM factor F_2, an r_3-regular SPM factor F_3, ..., an r_n-regular SPM factor F_n, then G has an $(h_{F_0} + s + |E(F_0)|(r_1 + 1) + \left(\frac{1}{2}(1 + vr_1) + v + |E(F_0)|\right) \sum_{i=2}^{n} r_i + \frac{1}{4} v \left(\sum_{i=2}^{n} r_i\right)^2, d)$-VAMT labeling.*

We omit the proof because it is just a modification of the proofs of Theorems 8.4 and 8.20.

Example 8.22 Using Theorem 8.20 we can find a vertex antimagic total labeling of the graph G from Example 8.2 (a K_8 without a perfect matching). G can be decomposed into a 2-regular factor $F_0 = 2C_4$ and a 4-regular factor $F_1 = K_{4,4}$. Using Theorem 2.6 we construct a $(26, 1)$-VAMT labeling of F_0, see Figure 8.7. An SPM labeling of F_1 is given in Figure 1.6. Taking $s = 26$, $v = 8$, $E_{F_0} = 8$, and $r_1 = 4$ we get $s' = 26 + (1/2 + 8 + 8) \cdot 4 + 1/4 \cdot 8 \cdot 4^2 = 124$, thus we get a $(124, 1)$-VAMT labeling of G, see Figure 8.7.

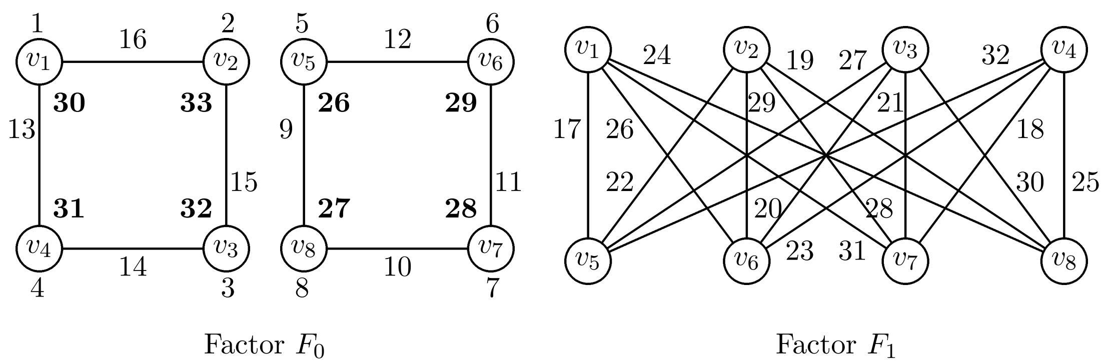


Figure 8.7: *A $(124, 1)$-vertex antimagic total labeling of G in two factors $F_0 = 2C_4$ and $F_1 = K_{4,4}$.*

8.4 Remarks

The results in this chapter turned the problem of finding a VMT, SPM or a VAMT labeling into a decomposition problem. If we decompose graph G into regular factors (we allow one factor to be nonregular) which are known to have certain magic-type labeling (usually supermagic labeling), we can construct a magic-type labeling of G. On the other hand it may be difficult to determine if a graph can be decomposed into specific factors.

Magic-type labelings of compositions were given, since in this special case properties of having a certain factorization are easily verified. Also some results given in Chapter 4 and Chapter 7 could be obtained as special cases of theorems from this chapter.

Even though some of the results were already published in [Iv], magic-type labelings are presented in this chapter in a more general context. Often, based on one magic-type labeling we construct other magic-type labelings or at least use common ideas in constructions of various magic-type labelings.

9 Conclusion

Magic graph labelings are a natural extension of magic squares, a well known topic of mathematical recreation. An introduction to certain magic-type labelings was given in Chapter 1. For a detailed survey on graph labelings we refer to [Gal].

In this thesis we focus on three magic-type labelings: vertex magic total labelings (VMT), supermagic labelings (SPM) and (s,d)-vertex antimagic total labelings (VAMT). We try to keep a general point of view and show how constructions and common ideas are applied to construct not just one, but several magic-type labelings. The organization of topics into chapters which are dedicated to a certain method fulfills this approach.

Often one magic-type labeling of a graph G can be used to construct a different magic-type labeling of G. In Chapter 2 we present a general construction of $(s,1)$-VAMT and $(s,2)$-VAMT labelings of 2-regular graphs. These labelings are used to construct more complex VMT labelings in Chapters 4 and 8. The notion of magic-type labelings was generalized in Chapter 3 to prepare a tool used to extend a certain magic-type labeling of a small graph to construct the same or a different magic-type labeling of a larger graph. It is not without interest that a similar (in some sense complementary) approach McQuillan used in [McQ] and [MS]. Several transformations are used for "meta proofs" of constructions in later chapters. VMT labelings of several classes of Cartesian products of regular graphs are given in Chapter 4. The magic labeling of $G \,\square\, H$ is often based on magic-type labelings of G and/or H.

In the second part of the thesis the idea of Kotzig arrays is exploited to construct several magic-type labelings. In [Wal2] Wallis introduced Kotzig arrays. The name was given after an unpublished paper by Kotzig. Wallis used Kotzig arrays as a tool how to "clone" VMT labelings of a graph G to obtain a VMT of nG. Kotzig arrays proved to be useful, they allowed a uniform approach not only for VMT, but also for SPM labelings. Even though some results for SPM labeling from Chapters 6 and 8 were already published in [Iv], we gave a unified and simple construction for several magic-type labeling always using a single labeling based on Kotzig arrays and we extended some of the results from [Iv] also for nonregular graphs.

Among the most general methods for constructing magic-type labelings are methods based on decompositions. In Chapter 8 we generalize some results on magic-type labelings of regular graphs from previous chapters, moreover we were able to give methods of constructing VMT, SPM and VAMT labelings for certain nonregular graphs. VMT and SPM labeling were found for compositions of certain regular graphs.

There is no necessary and sufficient condition known for a graph to have a VMT labelings, but a conjecture[5] by MacDougall says that any regular graph with the exception

[5]About MacDougall's conjecture is written in Chapter 8.

of K_2 and $2K_3$ has a VMT labeling. All results in this thesis support the conjecture, yet it seems that the proof of the conjecture is a difficult problem. We do not have a general construction even for 2-regular graphs.

Kotzig arrays and graph decompositions into certain regular graphs play a crucial role in most of the constructions in this thesis. Yet these methods have their limits and thus more different methods are needed. We present some open problems which seem to be beyond limits of the methods presented in this thesis.

Open problems

Problem 1 Find a VMT labeling of $C_{2m} \square C_{2n}$.

Problem 2 Find a VMT labeling of a 2-regular graph other than nC_m.

Problem 3 Are there other general methods besides decompositions and Kotzig arrays for finding magic-type labelings of large graphs using magic-type labelings of small graphs?

References

[BBDMSS] M. Bača, F. Bertault, J.A. MacDougall, M. Miller, R. Simanjuntak, and Slamin, Vertex-antimagic total labelings of graphs, *Discussiones Mathematicae Graph Theory*, **23** (2003), 67–83.

[BH] M. Bača, I. Holländer, On (a, d)-antimagic prisms, *Ars. Comb.*, **48** (1998), 297–306.

[BMS] M. Bača, M. Miller, and Slamin, Every generalized Petersen graph has a vertex-magic total labeling, *Int. J. Comput. Math.*, 11th Australasian Workshop on Combinatorial Algorithms (Hunter Valley, 2000), **79** (2002), 1259–1263.

[CD] C. J. Colbourn and J. H. Dinitz (Eds.), The CRC Handbook of Combinatorial Designs, Boca Raton, *CRC Press*, (1996).

[DMSW] J.A. MacDougall, M. Miller, Slamin, and W. D. Wallis, Vertex-magic total labelings of graphs, *Util. Math.*, **61** (2001), 3–21.

[DIS] S. Drajnová, J. Ivančo, A. Semančová, Numbers of edges in supermagic graphs, *preprint*

[DMSW2] J. A. MacDougall, M. Miller, Slamin, and W.D. Wallis, Problems in magic total labellings, *Proceedings of AWOCA'99* (1999), 19–25.

[FKK1] D. Fronček, P. Kovář, and T. Kovářová, Vertex magic total labeling of Cartesian products of cycles, *Proceedings of Conference MIGHTY XXXVI*, Oshkosh (2003).

[FKK2] D. Fronček, P. Kovář, and T. Kovářová, Vertex magic total labeling of Cartesian products of some vertex magic total regular graphs and odd cycles, *Proceedings of Conference MIGHTY XXXVI*, Oshkosh (2003).

[FKK3] D. Fronček, P. Kovář, and T. Kovářová, Vertex magic total labeling of products of cycles, *accepted for publication.* (Australasian Journal of Combinatorics)

[Gal] J. A. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, **DS 6** (2003).

[Iv] J. Ivančo, On supermagic regular graphs, *Mathematica Bohemica*, **125** (2000), 99–114.

[Kov1] P. Kovář, Vertex magic total labeling of Cartesian products of some vertex magic total regular graphs and even cycles, *Proceedings of Conference MIGHTY XXXVII*, Valparaiso (2003).

[Kov2a] P. Kovář, Vertex magic total labeling of Cartesian products of certain regular VMT graphs and certain regular supermagic graphs, *Proceedings of 17th MCCCC*, Las Vegas (2003).

[Kov2] P. Kovář, Vertex magic total labeling of products of regular VMT graphs and regular supermagic graphs, *submitted.* (JCMCC)

[Kov3] P. Kovář, Unified approach to magic labelings of copies of regular graphs, *to appear.* (Congressus Numerantium)

[MsP] P. Kovář, On magic graph labelings, *Master Thesis*, University of Minnesota Duluth, (2004).

[MsT] T. Kovářová, Magic labelings of graphs, *Master Thesis*, University of Minnesota Duluth, (2004).

[KR] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.*, **13** (1970), 451–461.

[LM] Y. Lin and M. Miller, Vertex-magic total labelings of complete graphs, *Bull. Inst. Combin. Appl.*, **33** (2001), 68–76.

[McQ] D. McQuillan, Vertex-magic cubic graphs, *JCMCC*, **48** (2004), 103–106.

[MS] D. McQuillan, Katy Smith, A spectrum problem for odd complete graphs, *preprint.*

[Sedl] J. Sedláček, On magic graphs, *Math. Slov.*, **26** (1976), 329–335.

[SlM] Slamin and M. Miller, On two conjectures concerning vertex-magic total labelings of generalized Petersen graphs, *Bull. Inst. Combin. Appl.*, **32** (2001), 9–16.

[Stew1] B. M. Stewart, Magic graphs, *Canadian J. Math.*, **18** (1966), 1031–1059.

[Stew2] B. M. Stewart, Supermagic complete graphs, *Canadian J. Math.*, **19** (1967), 427–438.

[Wal1] W. D. Wallis, Magic Graphs, *Birkhäuser*, Boston MA, (2001).

[Wal2] W. D. Wallis, Vertex magic labelings of multiple graphs, *Congressus Numerantium*, **152** (2001), 81–83.

[West] D. B. West, Introduction to graph theory – 2nd ed., *Prentice-Hall*, Upper Saddle River NJ, (2001).

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