**VŠB – Technical University of Ostrava**
**Faculty of Electrical Engineering and Computer Science**

Ph.D. Thesis

# Spanning Tree Factorizations of Complete Graphs

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# Abstract

A decomposition of a given graph $U$ is a set of subgraphs such that each edge of $U$ appears in exactly one subgraph of the set. The decomposition is called a factorization if each of the subgraphs is a factor of $U$. A factor of $U$ is each connected subgraph containing all vertices of $U$.

We investigate factorizations of complete graphs $K_{4n}$ into isomorphic spanning trees. Particularly, we show that for every integer $d$, such that $3 \leq d \leq 4n-1$, there exists a spanning tree with diameter $d$ that factorizes $K_{4n}$. The question of existence of a factorization of $K_{4n+2}$ into isomorphic spanning trees with a given diameter $d$ was positively answered by D. Fronček [6]. Further, in this thesis we examine factorizations of $K_{4n}$ into caterpillars with diameter 4. Presented results together with the results of P. Eldergill [4], D. Fronček [7], and M. Kubesa [16] give a complete classification of caterpillars with diameter 4 that factorize $K_{2n}$.

The methods for complete graph decompositions are based mainly on graph labelings. In general, a labeling of a graph is an assignment of numbers (usually nonnegative integers) to vertices, or edges, or both. For the purpose of decompositions of $K_{4n}$ we introduce two methods based on new types of vertex labelings. First, a fixing labeling and a $2n$-cyclic labeling which allow decompositions of $K_{2nk}$, where $k$ is odd and $n, k > 1$. We show that if a graph $G$ with $2nk-1$ edges has one of these labelings, then there exists a $G$-decomposition of $K_{2nk}$ into $nk$ copies of $G$. Second, a swapping labeling which is used for decompositions of $K_{4n}$, where $n$ is any positive integer. A swapping labeling of a graph $G$ with $4n-1$ edges guarantees the existence of a $G$-decomposition of $K_{4n}$ into $2n$ copies of $G$.

Fixing labelings and swapping labelings are further generalizations or modifications of the blended $\rho$-labeling introduced by D. Fronček [6] as a tool for spanning tree factorizations of $K_{4n+2}$.

# Contents

# Chapter 1

# Introduction

All graphs we deal with are finite and simple. Undefined graph theory terms can be found in any introductory graph theory textbook. We refer the reader for instance to [26]. As isomorphic decompositions of complete graphs are the main topic of this thesis we state the definition of a graph decomposition first.

**Definition 1.1** *Let $U$ be a graph on $n$ vertices. A* decomposition of the graph *$U$ is a family of pairwise edge disjoint subgraphs $\mathcal{D} = \{G_0, G_1, \ldots, G_s\}$ such that every edge of $U$ belongs to a member of $\mathcal{D}$. If each subgraph $G_r$ is isomorphic to a graph $G$ we speak about $G$-*decomposition of $U$. *If $G$ has exactly $n$ vertices and none of them is isolated, then $G$ is a* factor *of $U$ and such a $G$-decomposition is called a $G$-*factorization. *The decomposition is* cyclic *if there exists an ordering $(x_1, x_2, \ldots, x_n)$ of the vertices of $U$ and isomorphisms $\phi_r : G_0 \longrightarrow G_r$, $r = 0, 1, 2, \ldots, s$, such that $\phi_r(x_i) = x_{i+r}$ for each $i = 1, 2, \ldots, n$. Subscripts are taken modulo $n$.*

The topic of graph decompositions is very wide with many aspects and was intensively studied over past 4 decades. A lot of research was inspired by Ringel's conjecture from 1963 [23]. Ringel conjectured that the complete graph $K_{2n+1}$ can be decomposed into $2n+1$ copies of any tree $T$ with $n$ edges. Decompositions of complete graphs and complete bipartite graphs received special attention. However, most of the papers deal with decompositions into smaller isomorphic graphs or not necessarily isomorphic factorizations into factors with given diameter (see for instance [1, 2, 22]). The area of isomorphic spanning tree factorizations of complete graphs that are of our main interest remained almost unexplored. It is a part of graph theory folklore that there exists a factorization of $K_{2n}$ into Hamiltonian paths $P_{2n}$. It is also easy to observe that a cyclic factorization of

$K_{2n}$ into symmetric double stars is possible. (A symmetric double star is a graph obtained by connecting the central vertices of two stars $K_{1,n-1}$ by an edge.) Until recently, almost nothing was published about other classes of spanning trees.

The methods for decompositions of complete graphs are mainly based on graph labelings. Typically a vertex labeling of a graph is a mapping which assigns distinct nonnegative integers to the vertices of a graph. An edge label can be induced from the labels of the endvertices. There are many different ways how to define such an edge label. However, for the purpose of a graph decomposition the edge label is usually defined naturally as the "length" of the edge. An extensive survey of the results published on the topic of graph labelings is given by Gallian in [10].

The most popular labelings, which are used as tools for isomorphic decompositions of complete graphs, are $\rho$-labelings and graceful labelings introduced by A. Rosa [25] in 1967. The existence of a $\rho$-labeling or a graceful labeling of a graph $G$ with $n$ edges guarantees a cyclic $G$-decomposition of the complete graph $K_{2n+1}$ into $2n+1$ copies of $G$, as was proved by A. Rosa [25]. Especially graceful labelings become very popular because of the famous Graceful Tree Conjecture by Kotzig and Ringel from 1964 [26]. The conjecture is that every tree has a graceful labeling. Since then many classes of trees were investigated to have a graceful labeling but the conjecture is open till today. The first wide family of trees that was proved to have the labeling are caterpillars (Rosa [25]). A *caterpillar* is a tree such that a path is obtained after removal of all its endvertices. The definition can be slightly changed to obtain other class of trees called lobsters. A lobster is a tree such that the removal of all its endvertices leaves a caterpillar. There are some partial results on the gracefulness of lobsters but the general result is not known. All trees with at most four endvertices were proved to be graceful by Huang, Kotzig and Rosa [13]. For the survey of the results on trees with the graceful labeling we recommend again [10].

Graceful or $\rho$-labelings were often used to construct new types of labelings, which in some sense generalize their properties. Among them are: $\rho$-symmetric graceful labelings or symmetric graceful labelings introduced in 1997 by Eldergill [4]. Eldergill gave a necessary and sufficient condition for the existence of a cyclic factorization of $K_{2n}$ into symmetric spanning trees. By a *symmetric tree* we understand a tree with an automorphism $\psi$ and an edge $(x, y)$ such that $\psi(x) = y$ and $\psi(y) = x$.

Methods for factorizations of $K_{4n+2}$ into a wider class of trees are due to D. Fronček [5, 6], and are based on blended $\rho$-labelings or flexible $q$-labelings. The only methods known till recently for factorizations of $K_{4n}$ are Eldergill's symmetric labelings and switching blended labeling introduced by D. Fronček and M. Kubesa in [9]. Both of them require certain strong types of automorphisms which reduce the class of trees permissible for factorizations.

The general problem of finding a factorization for a given spanning tree is far from being solved. Using the methods mentioned above some special classes of spanning trees for factorizations of $K_{4n+2}$ were described [5, 17], and a construction of a spanning tree of any diameter that factorizes $K_{4n+2}$ was given by D. Fronček [6]. The most general existing result is a classification of caterpillars on $4n+2$ vertices with diameter 4 which is due to D. Fronček [7] and M. Kubesa [17, 16]. They achieved a significant progress also in classification of caterpillars of diameter 5, but the classification is not complete yet [16, 18, 19, 20, 21]. The constructions of labelings of caterpillars with diameter 5 become very technical but not a trivial problem, since many subclasses need to be considered separately.

There are two other problems closely related to complete graph decompositions, namely problems on graph coverings or packing of graphs.

If Definition 1.1 of a decomposition of a graph $U$ is relaxed so that the subgraphs $G_0, G_1, \ldots, G_s$ do not have to be edge disjoint we obtain a covering of the graph $U$. An orthogonal double cover (ODC) of the complete graph $K_n$ is a family of subgraphs $G_r$ for $r = 1, 2, \ldots, n$, with $n-1$ edges each, such that every edge of $K_n$ is covered precisely twice and any two subgraphs intersect in exactly one edge. The problems on ODCs of complete graphs have been intensively studied over past 25 years. For a survey of the topic see [11]. Especially the methods used for ODCs of $K_n$ by trees are very similar to those used for spanning tree decompositions. The main tool are so called orthogonal labelings defined in [12], which can be considered as graceful type labelings. Again several classes of trees were investigated for the existence of ODCs and the results led Gronau, Mullin and Rosa [12] to the conjecture that for any tree $T$ on $n \geq 2$ vertices different from the path on 4 vertices there exists an ODC of $K_n$ by $T$. The conjecture is of course open, and the difficulty of the existence problem of ODCs for trees in general can be anticipated if we consider that even for paths the question remains unsolved.

Alternatively, Definition 1.1 of a decomposition $\mathcal{D} = \{G_0, G_1, \ldots, G_s\}$ of a graph $U$ can be modified so that not every edge of $U$ has to belong to a member of $\mathcal{D}$. Then we speak about *packing* of $G_0, G_1, \ldots, G_s$ into the graph $U$. Many papers on packing problems for complete graphs were motivated by well known conjecture of Bollobás and Eldridge. They conjectured that there exists a packing of $G_0, G_1, \ldots, G_s$ into $K_n$ if $|E(G_r)| \leq n - s - 1$, where $r = 0, 1, \ldots, s$. There are results concerning special cases of the conjecture (for a survey see [27]). For instance in [3] Brandt and Woźniak consider packing of $k$ copies of the same tree into $K_n$ for $k = \lfloor \frac{n}{2} \rfloor$. As the main tool to find a packing they use so called distinct length labelings, which also have their origin in Rosa's graceful or $\rho$-labelings. Because of the similarity of used methods we believe that some of the ideas of the methods for spanning tree decompositions introduced further could be used or modified to answer some questions on ODCs by trees or packing of trees into $K_n$ as well.

The aim of this thesis is to develop methods suitable for factorizations of $K_{4n}$ into spanning trees which would enable to achieve complementary results to those known for $K_{4n+2}$ or allow further investigation on spanning tree factorizations of $K_{2n}$ in general. Our methods are based on new types of labelings, namely a 2n-cyclic labeling, a fixing labeling, and a swapping labeling. First two of them are further generalizations or extensions of the blended $\rho$-labeling introduced by D. Fronček [6] and can be used for decomposition of $K_{2nk}$, where $n, k > 1$ and $k$ is odd. It means the case when the number of vertices of a complete graph is a power of two is not covered. A swapping labeling can be used for decompositions of $K_{4n}$, where $n$ is a positive integer. In combination the new methods enabled us to find constructions of spanning trees with given diameter $d$ for factorizations of $K_{4n}$, where $3 \leq d \leq 4n - 1$, and especially to complete the classification of caterpillars with diameter 4 for spanning tree factorization of any $K_{2n}$. Most of the results were submitted for publication in [8, 14, 15].

# Chapter 2

# Known methods

Here we give an overview of previously known methods and labelings, which form the base for our own approach introduced in following chapters.

## 2.1 Basic counting

An obvious necessary condition for the existence of a $G$-decomposition of $K_n$ is that the number of edges of $G$ divides the number of edges of $K_n$. The number of edges of $K_n$ is

$$|E(K_n)| = \binom{n}{2} = \frac{n(n-1)}{2}.$$

Since the number of edges of a spanning tree of $K_n$ is $n-1$, it follows that we consider decompositions of $K_n$ into $\frac{n}{2}$ copies of a graph with $n-1$ edges. Obviously, such a decomposition of $K_n$ is impossible when the number of vertices $n$ is odd. Therefore we deal only with complete graphs $K_{2n}$ with an even number of vertices. Since the number of edges of a spanning tree on $2n$ vertices is $2n-1$ we can in general investigate isomorphic decompositions of $K_{2n}$ into $n$ copies of a graph $G$ with $2n-1$ edges.

Another easily observed necessary condition for the existence of the spanning tree factorization of $K_{2n}$ is that the largest degree of a vertex of a spanning tree is at most $n$. This condition is called the *Degree Condition* in [17]. Proof is very simple but for the completeness we state our own version here.

**Lemma 2.1** (Degree Condition) *Let $T$ be a tree on $2n$ vertices such that there is a $T$-factorization of $K_{2n}$. Then for each vertex $v$ in $T$ is* $\deg(v) \leq n$.

*Proof.* There are $n$ factors $T_0, T_1, \ldots, T_{n-1}$ all isomorphic to $T$ in $K_{2n}$. Let $u$ be a vertex of $K_{2n}$, and by $\deg_i(u)$ we denote the degree of $u$ in the factor $T_i$, where $i = 0, 1, \ldots, n-1$. Then

$$\begin{aligned} \deg(u) &= \sum_{i=0}^{n-1} \deg_i(u), \quad \text{and} \\ 2n-1 &= \sum_{i=0}^{n-1} \deg_i(u). \qquad (2.1) \end{aligned}$$

Without loss of generality we assume that $\deg_0(u) = \Delta(T)$. Since in any other factor the degree of $u$ is at least one, the following holds:

$$\Delta(T) + \sum_{i=1}^{n-1} \deg_i(u) \geq \Delta(T) + n - 1. \qquad (2.2)$$

From (2.1) and (2.2) together we obtain

$$2n - 1 \geq \Delta(T) + n - 1.$$

Therefore is

$$\Delta(T) \leq n,$$

which implies Degree Condition,

$$\deg(v) \leq n \quad \text{for any } v \in T.$$

□

## 2.2 Basic labelings

As we already mentioned, two fundamental types of vertex labelings are the $\rho$-labeling and the graceful labeling (also called $\rho$ or $\beta$-valuations) defined by A. Rosa.

**Definition 2.2** *Let $G$ be a graph with $n$ edges and the vertex set $V(G)$ and let $\lambda$ be an injection $\lambda : V(G) \longrightarrow S$ where $S$ is a subset of the set $\{0, 1, 2, \ldots, 2n\}$. The* length *of an edge $(x, y)$ is defined as $\ell(x, y) = \min\{|\lambda(x) - \lambda(y)|, 2n + 1 - |\lambda(x) - \lambda(y)|\}$. If the set of all lengths of $n$ edges is equal to $\{1, 2, \ldots, n\}$ and $S \subseteq \{0, 1, 2, \ldots, 2n\}$, then $\lambda$ is a $\rho$-*labeling*; if $S \subseteq \{0, 1, 2, \ldots, n\}$ instead, then $\lambda$ is a* graceful labeling.

Every graceful labeling is indeed also a $\rho$-labeling, and a graph which admits a graceful labeling is called *graceful.* The following theorem shows how these labelings are related to decompositions of complete graphs.

**Theorem 2.3** (Rosa, 1967) *A cyclic decomposition of the complete graph $K_{2n+1}$ into $2n+1$ isomorphic copies of a graph $G$ with $n$ edges exists if and only if there exists a $\rho$-labeling of a graph $G$.*

The idea of the proof is the following. We assign the elements of the additive group $Z_{2n+1}$ to the vertices of $K_{2n+1}$. The lengths of the edges of $K_{2n+1}$ are assigned as in the definition of the $\rho$-labeling. Then we obtain $2n+1$ edges of each length $i$ for $i = 1, 2, \ldots, n$. The first copy of $G$ in $K_{2n+1}$ can be found by unifying the vertices of $G$ and $K_{2n+1}$ which have the same label. Since in $G$ there is exactly one edge of each length, by rotating $G$ $2n$-times we obtain a cyclic $G$-decomposition of $K_{2n+1}$.

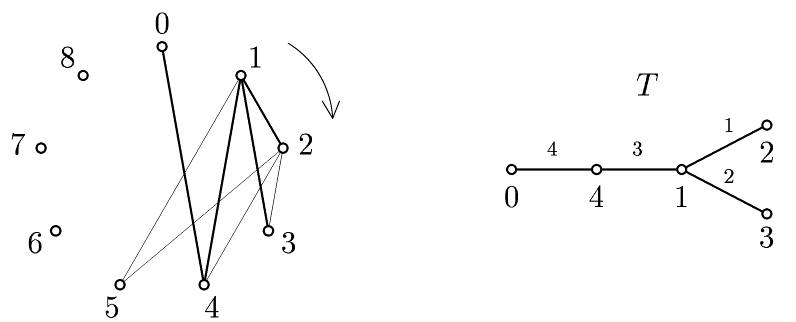



Figure 2.1: *Cyclic decomposition of $K_9$ into 9 copies of $T$ with graceful labeling.*

Among labelings that form the base for our own methods belongs the labeling defined by D. Fronček in [6]. This labeling is a generalization of the bigraceful labeling introduced earlier by Ringel, Llado, and Serra [24].

**Definition 2.4** *Let $G$ be a bipartite graph with $n$ edges and the vertex set $V(G) = V_0 \cup V_1$. Let $\lambda$ be an injection $\lambda : V_i \longrightarrow S_i$, where $S_i$ is a subset of the set $\{0_i, 1_i, \ldots, (n-1)_i\}$, $i = 0, 1$. The* length *of an edge $(x_0, y_1)$ for $x_0 \in V_0$ and $y_1 \in V_1$ with $\lambda(x_0) = a_0$ and $\lambda(y_1) = b_1$ is defined as $\ell_{01}(x_0, y_1) = b - a \pmod n$. If the set of all lengths of $n$ edges is equal to $\{0, 1, 2, \ldots, n-1\}$, then $\lambda$ is a* bipartite $\rho$-labeling*.*

As shown in [6], the existence of a bipartite $\rho$-labeling of a graph $G$ with $n$ edges guarantees a bi-cyclic decomposition of the bipartite complete graph $K_{n,n}$ into $n$ isomorphic copies of $G$. An example is shown in Figure 2.2.

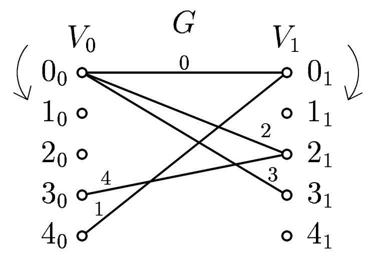



Figure 2.2: *Bipartite $\rho$-labeling of G with 5 edges.*

## 2.3 Symmetric labelings

Here we state the notions related to decomposition of $K_{2n}$ into symmetric graphs introduced by Eldergill [4]. To simplify our notation we will from now on occasionally unify a vertex with its label. It means that rather than "the vertex $x$ such that $\lambda(x) = i$", we will say just "the vertex $i$".

**Definition 2.5** *A connected graph $G$ with an edge $(x, y)$ (called a* bridge*) is* symmetric *if there is an automorphism $\psi$ of $G$ such that $\psi(x) = y$ and $\psi(y) = x$. The isomorphic components of $G - (x, y)$ are called* banks *and denoted by $H, H'$, respectively. A labeling of a symmetric graph $G$ with $2n+1$ edges and banks $H, H'$ is $\rho$-*symmetric graceful *if $H$ has a $\rho$-labeling and $\psi(i) = i + n \pmod{2n}$ for each vertex $i$ in $H$. A labeling of a symmetric graph $G$ with $2n - 1$ edges is* symmetric graceful *if it is $\rho$-symmetric graceful and the bank $H$ is moreover graceful. A graph which admits a $\rho$-symmetric graceful labeling or a symmetric graceful labeling is called $\rho$-*symmetric graceful *or a* symmetric graceful*, respectively.*

Eldergill proved the following theorem for symmetric trees. Since the assumption that the graph must be acyclic was never used, the theorem is true for symmetric graphs in general.

**Theorem 2.6** (Eldergill) *Let $G$ be a symmetric graph with $2n - 1$ edges. Then there exists a cyclic $G$-decomposition of $K_{2n}$ if and only if $G$ is $\rho$-symmetric graceful.*

One can easily observe how the construction of a $\rho$-symmetric graceful labeling is based on the $\rho$-labeling or graceful labeling. In a graph with $n-1$ edges that has either a graceful or a $\rho$-labeling there is only one edge of each length $1, 2, \ldots, n-1$, while in a graph with $2n - 1$ edges which is symmetric graceful or $\rho$-symmetric graceful there are two edges of each length $1, 2, \ldots, n-1$ except for just one edge of the maximum length $n$. Since any graceful graph with $n-1$ edges yields a symmetric graceful graph with $2n-1$ edges, one can find an infinite class of symmetric graceful graphs whenever an infinite class of graceful graphs is known. Again an example is given in Figure 2.3.

Eldergill's method is too restrictive, allowing decompositions only into symmetric graphs. For instance symmetricity restrict decompositions only to graphs with an odd diameter. To answer the question about factorizations into spanning trees with more general structure a more powerful decomposition method was needed.

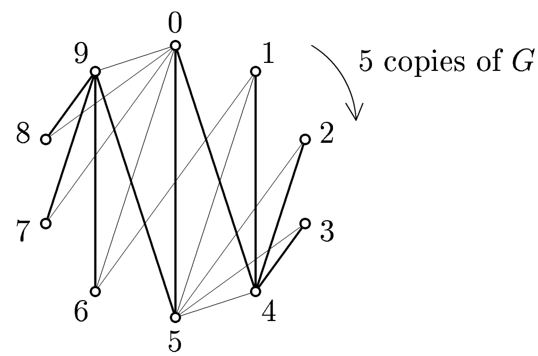



Figure 2.3: *Symmetric graceful labeling of G with 9 edges.*

## 2.4 Blended type labelings

To find a more general method, D. Frončev defined in [6] a blended $\rho$-labeling. As one of our new labelings used for decompositions of $K_{2nk}$ is just a straightforward extension of the blended $\rho$-labeling, we state its definition here.

**Definition 2.7** *Let $G$ be a graph with $4n+1$ edges, $V(G) = V_0 \cup V_1$, $V_0 \cap V_1 = \emptyset$, and $|V_0| = |V_1| = 2n+1$. Let $\lambda$ be an injection, $\lambda : V_i \longrightarrow \{0_i, 1_i, \ldots, (2n)_i\}$, $i = 0, 1$.*

*The* pure length *of an edge $(x_i, y_i)$ with $x_i, y_i \in V_i$, where $i \in \{0, 1\}$, for $\lambda(x_i) = a_i$ and $\lambda(y_i) = b_i$ is defined as*

$$\ell_{ii}(x_i, y_i) = \min\{|a-b|, 2n+1-|a-b|\}.$$

*The* mixed length *of an edge $(x_0, y_1)$ with $x_0 \in V_0$, $y_1 \in V_1$, for $\lambda(x_0) = a_0$ and $\lambda(y_1) = b_1$, is defined as*

$$\ell_{01}(x_0, y_1) = b - a \mod (2n+1).$$

*Then $G$ has a* blended $\rho$-labeling *(briefly* blended labeling*) if*

*(1)* $\{\ell_{ii}(x_i, y_i) | (x_i, y_i) \in E(G)\} = \{1, 2, \ldots, n\}$ for $i = 0, 1$

*(2)* $\{\ell_{01}(x_0, y_1) | (x_0, y_1) \in E(G)\} = \{0, 1, \ldots, 2n\}$.

The edges $(x_i, y_i)$ for $i = 0, 1$ with the pure length $\ell_{ii}$ are called *pure edges* and the edges $(x_0, y_1)$ with the mixed length $\ell_{01}$ are called *mixed edges.* A graph $G$ with a blended labeling can be split into three subgraphs as follows. Subgraphs of $G$ induced on vertices of $V_0$ and $V_1$ are denoted by $H_0$, $H_1$ respectively, and $H_{01}$ denotes a bipartite subgraph with partite sets $V_0$, $V_1$. If a blended labeling is restricted to these subgraphs, the labelings of $H_0$ and $H_1$ can be viewed as the usual $\rho$-labelings in case that the subscripts of the labels are omitted. A $\rho$-labeling guarantees a cyclic decomposition of the complete graph $K_{2n+1}$ into $n$ copies of $H_0$ or $H_1$. The labeling of the subgraph $H_{01}$ is then a bipartite $\rho$-labeling which allows a bi-cyclic decomposition of the complete bipartite graph $K_{2n+1,2n+1}$ into $2n + 1$ isomorphic copies of $H_{01}$. See an example in Figure 2.4.

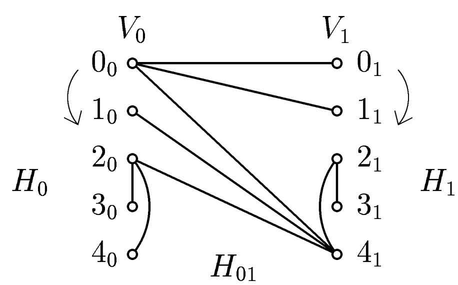



Figure 2.4: *Blended $\rho$-labeling of a tree on* 10 *vertices.*

Blended labelings are suitable for decompositions of $K_{4n+2}$.

**Theorem 2.8** (Fronček) *Let $G$ with $4n + 1$ edges have a blended $\rho$-labeling. Then there exists a bi-cyclic decomposition of $K_{4n+2}$ into $2n + 1$ copies of $G$.*

D. Fronček gives constructions of several special infinite classes of non-symmetric trees that admit blended $\rho$-labelings. Also based on blended labelings he found factorizations of $K_{4n+2}$ into spanning trees of any possible diameter [6].

However, a blended $\rho$-labeling cannot be used for decompositions of complete graphs with $4n$ vertices. A cyclic decomposition in each of the partite sets separately as in the method based on the blended labeling is not possible when decomposing $K_{4n}$. By splitting vertices of $K_{4n}$ into two equal partite sets $V_i, i = 0, 1$, the number of vertices in a partite set is even, namely $|V_i| = 2n$, and a cyclic decomposition of $K_{2n}$ into $2n$ copies of a graph $H_i$ does not exists. It is so because the basic condition that the number of edges of $K_{2n}$ is divisible by the number of copies of $H_i$ is not satisfied.

The known method which allows decomposition of $K_{4n}$ into other than symmetric graphs is based on a switching blended labeling. This labeling is a modification of the blended labeling and was defined by D. Frončková and M. Kubesa in [9].

**Definition 2.9** *Let $T$ be a tree on $4n$ vertices such that $V(T) = V_0 \cup V_1$, $V_0 \cap V_1 = \emptyset$ with $|V_0| = |V_1| = 2n$. Let $\lambda$ be an injection, $\lambda : V_i \longrightarrow \{0_i, 1_i, 2_i, \dots, (2n-1)_i\}$, $i = 0, 1$. The pure length $\ell_{ii}$, for $i \in \{0, 1\}$ and the mixed length $\ell_{01}$ of an edge are defined as for the blended labeling. The tree $T$ has a* switching blended labeling *(or just* switching labeling *for short) if*

*(1)* $\{\ell_{00}(x_0, y_0) | (x_0, y_0) \in E(T)\} = \{1, 2, \dots, n\}$,

*(2)* $\{\ell_{11}(x_1, y_1) | (x_1, y_1) \in E(T)\} = \{1, 2, \dots, n-1\}$,

*(3)* $\{\ell_{01}(x_0, y_1) | (x_0, y_1) \in E(T)\} = \{0, 1, 2, \dots, 2n-1\}$, *and*

*(4) there exists an automorphism $\varphi$ of $T - (x_0, (x+n)_0)$, where $(x_0, (x+n)_0)$ is the unique edge of the pure length $n$ in $T$, such that $\varphi(x_0) = y_1$ and $\varphi((x+n)_0) = (y+n)_1$ for some $y_1 \in V_1$.*

In [9] the following theorem is proved.

**Theorem 2.10** (Frončková, Kubesa) *Let $T$ be a tree with $4n$ vertices with a switching blended labeling $\lambda$. Then there is a $T$-factorization of $K_{4n}$ into $2n$ copies of $T$.*

Switching blended labeling is still too restrictive, since it requires certain "strong" type of automorphism, which does not exist for some classes of trees. We will show that trees with diameter 4 do not allow a switching blended labeling at all. Therefore, we develop new techniques for decompositions of complete graphs with an even number of vertices, especially those which allow us to consider more general classes of spanning trees for factorizations of $K_{4n}$.

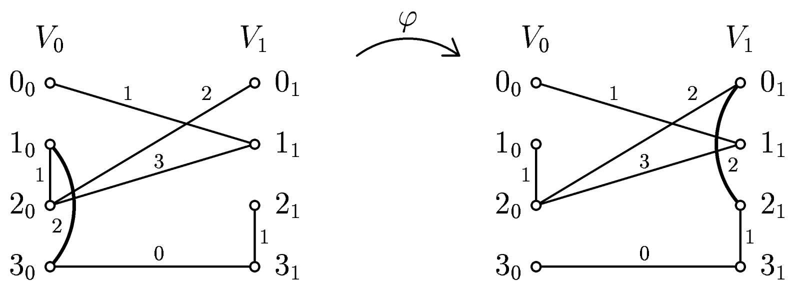



Figure 2.5: *Switching blended labeling of a tree on* 8 *vertices.*

# Chapter 3

# $\mathbf{K_{2nk}}$ decompositions

The methods introduced in this chapter are suitable for isomorphic decompositions of complete graphs on $2nk$ vertices, where $n, k$ are positive integers such that $n, k > 1$ and $k$ is odd. The methods are strongly based on Eldergill's cyclic decomposition of $K_{2n}$ into symmetric graphs and at the same time they generalize the properties of the blended labeling.

## 3.1 Notation and definitions

We introduce the notation using permutations, which will better suit our further needs. A *permutation* $\pi$ of a set $A$ is a bijection $\pi : A \longrightarrow A$. It is a well known fact that all permutations of a set $A$ form a group under composition. By $\iota$ we will denote the identity of a permutation group. By $\alpha_n$ we mean the *cyclic permutation* on the set $A = \{0_i, 1_i, 2_i, \dots, (n-1)_i\}$ defined as $\alpha_n(a_i) = (a+1)_i \pmod n$ for any $a_i \in A$, where $i$ is some integer.

**Definition 3.1** *Let $G$ be a graph with a vertex labeling $\lambda$, where $\lambda$ is an injection from $V(G)$ to $A$, and let $\pi$ be a permutation on the set of labels $A$. We define a* permutation of $G$ *to be a copy of $G$ with the vertex labeling $\lambda_\pi : V(G) \longrightarrow A$ such that $\lambda_\pi(u) = \pi(\lambda(u))$ and denote it by $\pi[G]$. If the set of labels is $A = \{0_i, 1_i, 2_i, \dots, (n-1)_i\}$ and $\pi = \alpha_n$ then $\alpha_n^r[G]$ is called a* rotation *of $G$ for any $r = \{1, 2, \dots, n\}$.*

Since we usually identify a vertex with its label we will talk about permutations of vertices rather than permutations of labels. By the permutation $\pi(a)$ of a vertex $a$ we mean the permutation of the label $a$ assigned to the vertex $x$ by a labeling $\lambda(x) = a$.

**Definition 3.2** *Let $U$ be a graph with the vertex set $V(U) = \bigcup_{i=0}^{m-1} V_i$, where $V_i = \{0_i, 1_i, 2_i, \ldots, (k-1)_i\}$ for $i = 0, 1, 2, \ldots, m-1$. Let $\pi$ be a permutation of the vertices of $U$ such that $\pi = \pi_0\pi_1\pi_2 \ldots \pi_{m-1}$, where $\pi_i$ is the permutation of the set $V_i$ and $\pi_i = \alpha_k$, for each $i = 0, 1, 2, \ldots, m-1$. A $G$-decomposition $G_0, G_1, G_2, \ldots, G_s$ of a graph $U$ is called $m$-*cyclic *if $G_r = \pi^r[G_0]$ for any $r = 1, 2, \ldots, s$.*

A cyclic decomposition is just a special case of the previous definition. It means that a cyclic $G$-decomposition of $K_{2n}$ with the vertex set $Z_{2n}$, where $Z_{2n}$ is the additive group modulo $2n$, is obtained by permuting the vertices of $G_0 = G$ by the cyclic permutation $\alpha_{2n}$. (In this case the subscript is omitted so that $A = Z_{2n}$ and $\alpha_{2n}(a) = a + 1 \pmod{2n}$, where $a \in Z_{2n}$.) Then each copy of the graph $G$ is a *rotation* $G_r = \alpha_{2n}^r[G_0]$.

A *bi-cyclic* $G$-decomposition of $K_{n,n}$ with partite sets $V_i = \{0_i, 1_i, \ldots, (n-1)_i\}$, for $i = 0, 1$ is obtained when the vertices of $G_0 = G$ are permuted by the permutation $\pi = \pi_0\pi_1$ composed of two cyclic permutations $\pi_i = \alpha_n$, for $i = 0, 1$.

Further we will also make use of a slightly more general definition of the pure length and the mixed length of an edge than is given in Definition 2.7 of the blended labeling. We will allow the vertex set of a graph $G$ to be split into more than only two partite sets. Then the edges connecting vertices between two different partite sets will be assigned the mixed lengths, while the edges connecting vertices inside a partite set will be assigned the pure lengths.

**Definition 3.3** *Let $G$ be a graph with the vertex set $V(G) = \bigcup_{i=0}^{m-1} V_i$, where $|V_i| = k$ and $V_i \cap V_j = \emptyset$ for $i \neq j$. Let $\lambda$ be an injection, $\lambda : V_i \to \{0_i, 1_i, 2_i, \ldots, (k-1)_i\}$, for $i = 0, 1, \ldots, m-1$.*

*The* pure length *of an edge $(x_i, y_i)$ with $x_i, y_i \in V_i$, for $\lambda(x_i) = a_i$ and $\lambda(y_i) = b_i$ is defined as*

$$\ell_{ii}(x_i, y_i) = \min\{|a-b|, k - |a-b|\}.$$

*The* mixed length *of an edge $(x_i, y_j)$, where $i < j$, with $x_i \in V_i$ and $y_j \in V_j$, for $\lambda(x_i) = a_i$ and $\lambda(y_j) = b_j$ is defined as*

$$\ell_{ij}(x_i, y_j) = b - a \pmod{k}.$$

## 3.2 Multicyclic decomposition of $K_{2nk}$

Here we give a method of factorization of the complete graph on $2nk$ vertices into $n$ isomorphic "locally dense" factors. The method is based on cyclic factorization of $K_{2n}$ into symmetric trees. The idea of the method is the following: We take a tree $T$ on $2n$ vertices with a symmetric graceful labeling, which allows a factorization of $K_{2n}$. Then we "blow up" this tree to construct a bigger graph $U$ on $2nk$ vertices (for any $k > 1$), which is a connected factor of $K_{2nk}$ and show that there is a $U$-factorization of $K_{2nk}$.

In the next section we develop a method for further decomposition of a graph $U$ into $k$ isomorphic copies of a graph $G$ with $2nk-1$ edges (for $k$ odd). Finally, by decomposing each copy of the graph $U$ into $k$ isomorphic copies of $G$ we obtain a $G$-decomposition of $K_{2nk}$ into $nk$ isomorphic copies of $G$.

The construction of the graph $U = U(T, s; k)$ can be described in two steps. First we obtain the graph $T[\overline{K}_k]$ by blowing up each vertex $i$ of the tree $T$ into the set $V_i$ with $k$ vertices and each edge $(i, j)$ of $T$ into all $k^2$ edges between the vertices of the partite sets $V_i$ and $V_j$. Then we choose a vertex $s$ in $T$ and its symmetric image $\psi(s) = s + n$ and add all edges into the corresponding partite sets $V_s$ and $V_{s+n}$ so that we have two complete graphs $K_k$ in addition to the edges of $T[\overline{K}_k]$. For convenience we use the following notation: $K_{V_i}$ denotes the complete graph on the vertices of the vertex set $V_i$ and $K_{V_i,V_j}$ denotes the complete bipartite graph on the vertices of the partite sets $V_i, V_j$.

**Definition 3.4** *Let $T$ be a symmetric tree on $2n$, $n \geq 1$, vertices with a $\rho$-symmetric graceful labeling. We define the graph $U(T, s; k)$ with the underlying tree $T$, where $s$ is the label of any vertex of $T$, $0 \leq s \leq n-1$, and $k$ is a positive integer, to have the vertex set*

$$V(U(T, s; k)) = \bigcup_{i=0}^{2n-1} V_i, \ |V_i| = k, \ V_i \cap V_j = \emptyset \text{ for } i \neq j,$$

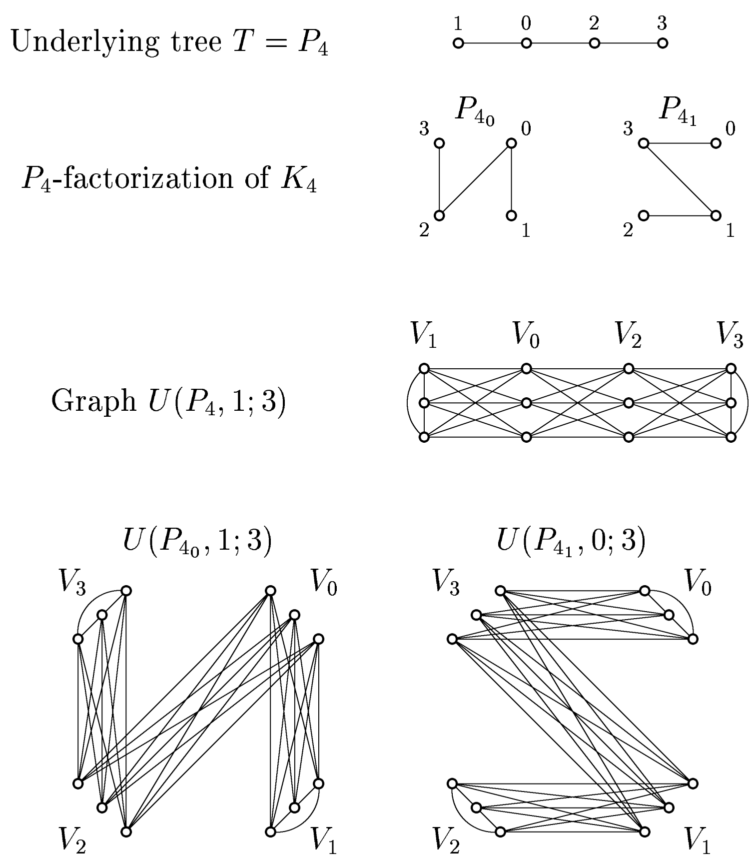
*and the edge set*

$$\begin{aligned} E(U(T, s; k)) &= \{(x, y) | x \in V_i, \, y \in V_j \, \wedge \, (i, j) \in E(T)\} \\ &\cup \{(x, y) | x, y \in V_s\} \, \cup \, \{(x, y) | x, y \in V_{s+n}\}. \end{aligned}$$

In other words, the graph $U(T, s; k)$ is a union of $2n-1$ complete bipartite graphs $K_{V_i,V_j}$ on the vertices of the partite sets $V_i, V_j$ whenever $i$ is adjacent to $j$

in $T$ and two complete graphs $K_{V_s}$ and $K_{V_{s+n}}$ on the vertices of the vertex sets $V_s, V_{s+n}$ for the chosen vertex with label $s$ in $T$. Each vertex set $V_i$ is of size $k$ and the subscript $i$ is the label of the corresponding vertex in $T$. It is easy to observe that $K_{2nk}$ can be decomposed into $n$ isomorphic copies of $U(T, s; k)$ (see Figure 3.1) and we will give a proof of this fact.

One can also notice that similar approach can be used for other $G$-decompositions of $K_{2n}$. For instance, we can blow up any graph $G$ for which there exists a bi-cyclic $G$-decomposition of the complete graph $K_{2n}$ into $n$ copies of $G$. Recall that bi-cyclic decompositions are based on blended labelings. Even more general types of decomposition can be probably used—one must be just careful about the choice of the two particular vertices in $G$ that correspond to the complete graphs $K_k$ in $U$.




$U(P_4, 1; 3)$-factorization of $K_{12}$

Figure 3.1: *$U(T, s; k)$-factorization of $K_{2nk}$.*

**Lemma 3.5** *Let $T$ be a tree on $2n$ vertices with a $\rho$-symmetric graceful labeling. Then there is a $U(T, s; k)$-factorization of $K_{2nk}$ into $n$ isomorphic copies of $U(T, s; k)$ for any $k \geq 1$.*

*Proof.* When $T$ is a $\rho$-symmetric graceful tree on $2n$ vertices, then according to Theorem 2.6 there is a cyclic $T$-factorization of $K_{2n}$ with the factors $T_0, T_1, \ldots, T_{n-1}$. By Definition 3.4, the graph $U(T, s; k)$ with the underlying tree $T$ is a connected factor of $K_{2nk}$. From each copy of $T$ we obtain an isomorphic copy of $U(T, s; k)$. We may assume that $T = T_0$ and construct the graph $U(T_0, s; k)$. Every other factor $T_r$ for $r = 1, \ldots, n-1$ is the rotation $\alpha_{2n}^r[T_0]$. Using the same permutation $\alpha_{2n}^r$ for the subscripts of the partite sets of $U(T_0, s; k)$ we get the remaining factors $U(T_r, s+r; k)$. Together $U(T_0, s; k), U(T_1, s+1; k), \ldots, U(T_{n-1}, s+n-1; k)$ form a $U(T, s; k)$-factorization of $K_{2nk}$. We just need to convince ourselves that each edge of $K_{2nk}$ belongs to exactly one copy of $U(T, s; k)$.

The vertices of the complete graph $K_{2nk}$ can be split into $2n$ partite sets $V_i$ for $i = 0, 1, \ldots, 2n-1$ with $k$ vertices in each of them. Then we can view the edge set of $K_{2nk}$ as a union of the edge sets of $n(2n-1)$ complete bipartite graphs $K_{V_i,V_j}$, $i \neq j$ and $2n$ complete graphs $K_{V_i}$ on $k$ vertices of each of the partite sets $V_i$.

Since there is a $T$-factorization of $K_{2n}$, each edge $(i, j)$ of $K_{2n}$ belongs to exactly one factor $T_r$. By the definition of $U(T, s; k)$, the edge $(i, j) \in E(T_r)$ corresponds to the complete bipartite graph $K_{V_i,V_j}$ in $U(T_r, s+r; k)$. Then each complete bipartite graph $K_{V_i,V_j}$ also belongs to exactly one factor of $K_{2nk}$, in particular, to $U(T_r, s+r; k)$.

Now we check the complete graphs $K_{V_i}$ for $i = 0, 1, \ldots, 2n-1$. In $T_0$ the vertex $s$ and its symmetric image $s+n \pmod{2n}$ are chosen to add $K_{V_s}$ and $K_{V_{s+n}}$ into $U(T_0, s; k)$. In $T_r$ the corresponding vertices are $\alpha_{2n}^r(s) = s+r \pmod{2n}$ and $\alpha_{2n}^r(s+n) = s+n+r \pmod{2n}$. So we have two different vertices in each $T_r$ for $r = 0, 1, \ldots, n-1$.

Suppose now that while making copies of $T$ we obtain the same image of the vertex $s$ or $s+n$ in two different factors $T_r$ and $T_t$. Because our $T$-factorization is cyclic, we can assume without loss of generality (WLOG) that $r = 0$ and $t \in \{1, 2, \ldots, n-1\}$.

(i) Firstly, let

$$\begin{aligned} \alpha_{2n}^t(s) &= \alpha_{2n}^0(s), && \text{then} \\ s+t &\equiv s \pmod{2n}, && \text{and} \\ t &= 0, \end{aligned}$$

which contradicts our assumption that $t \neq 0$.

(ii) If

$$\begin{aligned} \alpha_{2n}^t(s+n) &= \alpha_{2n}^0(s+n), && \text{then} \\ s+n+t &\equiv s+n \pmod{2n}, && \text{and} \\ t &= 0, \end{aligned}$$

we again get the same contradiction.

(iii) Finally, if

$$\begin{aligned} \alpha_{2n}^t(s+n) &= \alpha_{2n}^0(s), && \text{then} \\ s+n+t &\equiv s \pmod{2n}, && \text{and} \\ n+t &\equiv 0 \pmod{2n}, \end{aligned}$$

which is impossible, since we have assumed that $t \in \{1, 2, \ldots, n-1\}$.

Therefore the images of the vertices $s$ and $s+n$ appear in $2n$ different vertices of $K_{2n}$ and each of them is in exactly one factor $T_r$. This means that also each corresponding complete graph $K_{V_i}$ for $i = 0, 1, \ldots, 2n-1$ is in exactly one factor $U(T_r, s; k)$.

Since all complete graphs $K_{V_i}$ and all complete bipartite graphs $K_{V_i,V_j}$ are pairwise edge disjoint, then also each edge of $K_{2nk}$ is in exactly one $U(T_r, s; k)$, and so $U(T_0, s; k), U(T_1, s; k), \ldots, U(T_{n-1}, s; k)$ give a $U(T, s; k)$-factorization of $K_{2nk}$.

□

## 3.3 $2n$-cyclic blended labeling

Now we find a decomposition of the graph $U(T, s; k)$ into $k$ isomorphic copies of a graph $G$ with $2nk - 1$ edges, and consequently we obtain also a $G$-decomposition of $K_{2nk}$ into $nk$ isomorphic copies of $G$. Hence, we need to explore the properties of a graph $G$ that would decompose $U(T, s; k)$. To characterize such a graph $G$ we introduce a new type of labeling.

The labeling is in fact a generalization of the blended $\rho$-labeling. The main idea is that we split the graph $U(T, s; k)$ into two copies of the complete graph

$K_k$ and $2n-1$ copies of the complete bipartite graph $K_{k,k}$. Each of these graphs is then decomposed separately using known methods based on known vertex labelings. The complete graphs $K_k$ are both decomposed cyclically into $k$ copies of a graph with $(k-1)/2$ edges, which requires $k$ to be odd. Each complete bipartite graph $K_{k,k}$ is then decomposed bi-cyclically into $k$ copies of a graph with $k$ edges.

**Definition 3.6** *Let $G$ be a graph with $2nk-1$ edges, for $k$ odd and $k,n>1$, and the vertex set $V(G)=\bigcup_{i=0}^{2n-1} V_i$, where $|V_i|=k$ and $V_i \cap V_j = \emptyset$ for $i \neq j$. Let $\lambda$ be an injection, $\lambda : V_i \to \{0_i, 1_i, 2_i, \ldots, (k-1)_i\}$, for $i=0,1,\ldots,2n-1$. By $H_{ij}$ we denote the bipartite subgraph of $G$ induced on the vertices of the partite sets $V_i$ and $V_j$ with edges of mixed length $\ell_{ij}$, and by $H_i$ we denote the subgraph of $G$ induced on the vertices of $V_i$ with edges of pure length $\ell_{ii}$.*

*We say that $G$ has a $2n$-*cyclic blended labeling *(shortly $2n$-*cyclic labeling*) if there exists an underlying tree $T$ on $2n$ vertices with a $\rho$-symmetric graceful labeling such that the following holds:*

*(1) For some vertex $s \in T$ and its symmetric image $t = s+n \pmod{2n}$ is*
$\{\ell_{ss}(x_s,y_s) | (x_s,y_s) \in E(H_s)\} = \{1,2,\ldots,(k-1)/2\}$, *and*
$\{\ell_{tt}(x_t,y_t) | (x_t,y_t) \in E(H_t)\} = \{1,2,\ldots,(k-1)/2\}$,

*(2) and for each edge $(i,j) \in E(T)$ is*
$\{\ell_{ij}(x_i,y_j) | (x_i,y_j) \in E(H_{ij})\} = \{0,1,2,\ldots,k-1\}$.

Similarly as a graph with a blended $\rho$-labeling, a graph $G$ with a $2n$-cyclic blended labeling is split into two subgraphs $H_s$ and $H_t$ on the vertices of the partite sets $V_s$ and $V_t$ with pure edges, and $2n-1$ subgraphs $H_{ij}$ for each $(i,j) \in E(T)$ with mixed edges. The labelings induced by $\lambda$ on the vertices of $H_s$ or $H_t$ are $\rho$-labelings (if we omit the subscripts of the labels), and the labeling induced on the vertices of any $H_{ij}$ is a bipartite $\rho$-labeling. (To have exactly a bipartite $\rho$-labeling we shall substitute $0,1$ for $i,j$).

Further we show that a graph $G$ with a $2n$-cyclic labeling allows a $2n$-cyclic decomposition of the graph $U(T,s;k)$. We get the decomposition by permuting the vertices of $U(T,s;k)$ by the permutation composed of $2n$ cycles, each of them of length $k$, so that the vertices of each of the partite sets $V_i$ permute separately. Then the $\rho$-labelings of the subgraphs $H_s$ and $H_t$ guarantee, according to Theorem 2.3, a cyclic decomposition of $K_{V_s}$ and $K_{V_t}$ into $k$ copies of $H_s$ and $H_t$, respectively. Similarly, the bipartite $\rho$-labelings of subgraph $H_{ij}$ guarantee a bi-cyclic decomposition of each of the complete bipartite graphs $K_{V_i,V_j}$ into $k$ copies of $H_{ij}$. From these facts the existence of the decomposition of $U(T,s;k)$ is almost evident. However in the proof we decided to follow the general idea of a $G$-decomposition, which is to find isomorphic and edge disjoint copies of $G$ and show that each edge of the decomposed graph is covered.

**Lemma 3.7** *Let a graph $G$ with $2nk-1$ edges, for $k,n>1$ and $k$ odd, have a $2n$-cyclic blended labeling. Then there exists a $2n$-cyclic $G$-decomposition of $U(T,s;k)$ into $k$ isomorphic copies of $G$.*

*Proof.* Let a graph $U(T,s;k)$ have the vertex set $V(U(T,s;k)) = \bigcup_{i=0}^{2n-1} V_i$, where $V_i \cap V_j = \emptyset$ for $i \neq j$ and $V_i = \{0_i, 1_i, 2_i, \dots, (k-1)_i\}$, $i = 0,1,2,\dots, 2n-1$. Let $\pi$ be the permutation on the vertex set of $U(T,s;k)$ such that $\pi = \pi_0\pi_1\dots\pi_{2n-1}$, where $\pi_i$ is the cyclic permutation $\alpha_k$ on the vertices of $V_i$ for each $i = 0,1,2,\dots,2n-1$.

Now suppose that $G_0 = G$ and $G_r = \pi^r[G]$ for $r = 1,2,\dots,k-1$. Then $G_0, G_1, G_2, \dots, G_{k-1}$ are $k$ isomorphic copies of $G$ on vertices of $U(T,s;k)$. We show that $G_0, G_1, G_2, \dots, G_{k-1}$ is a $G$-decomposition of $U(T,s;k)$.

Permutations $\pi^r$ preserve the lengths of the edges. In particular, if $(x_t, (x+a)_t)$, $t \in \{s, s+n\}$ is an edge of a pure length $a$, $1 \leq a \leq \frac{k-1}{2}$, in $G_0$, then $(\pi^r(x_t), \pi^r((x+a)_t)) = ((x+r)_t, (x+a+r)_t)$ is the edge of the pure length $a$ in $G_r$, and if $(x_i, (x+b)_j)$ is an edge of a mixed length $b$, $0 \leq b \leq k-1$, in $G_0$, then $(\pi^r(x_i), \pi^r((x+b)_j)) = ((x+r)_i, (x+b+n)_j)$ is the edge of the mixed length $b$ in $G_r$.

In $U(T,s;k)$ we have $k$ edges of each pure length $\ell_{tt} \in \{1,2,\dots\frac{k-1}{2}\}$, where $t \in \{s, s+n\}$, and $k$ edges of each mixed length $\ell_{ij} \in \{0,1,2,\dots,k-1\}$ for each $(i,j) \in T$. In $G$ we have one edge of each pure length $\ell_{tt}$, where $t \in \{s, s+n\}$, and one edge of each mixed length $\ell_{ij}$ for each $(i,j) \in T$. Because the lengths of the edges are preserved, while making $k$ isomorphic copies of $G$ we obtain $k$ copies of the edge of each mixed or pure length. If they are all different we obtained the decomposition.

Suppose now that the same edge $(x_t, (x+a)_t)$ of the pure length $\ell_{tt} = a$ is in two different copies of $G$, $G_r$ and $G_p$. We can again WLOG assume that $r = 0$. But if $(x_t, (x+a)_t) \in G_p$, then $(x_t, (x+a)_t) = ((y+p)_t, (y+p+a)_t)$ for some $y$ since each edge of $G_p$ arises from an edge of $G_0$ by adding $p$ to both its endvertices. Hence, $(y_t, (y+a)_t) \in G_0$. However, $(x_t, (x+a)_t)$ is the only edge of the pure length $\ell_{tt} = a$ in $G$, which yields $x = y$ and therefore $p = 0$. This contradicts our original assumption that $G_p$ is different from $G_0$. Similarly we suppose that an edge $(x_i, (x+b)_j)$ of a mixed length $\ell_{ij} = b$ is in two different copies of $G$, $G_0$ and $G_p$, where $p \in \{1, 2, \ldots, k-1\}$. If $(x_i, (x+b)_j) \in G_p$, then $(x_i, (x+b)_j) = ((y+p)_i, (y+p+a)_j)$ for some $y$ for the same reasons as above, and $(y_i, (y+b)_j) \in G_0$. From the uniqueness of the edge of the mixed length $\ell_{ij} = b$ in $G_0$ we again get $x = y$ and $p = 0$, which is a contradiction.

Thus in $k$ copies of $G$ we have all $k(2nk-1)$ different edges of $U(T, s; k)$, and so $G_0, G_1, G_2, \ldots, G_{k-1}$ form a $2n$-cyclic decomposition of $U(T, s; k)$. □

Finally we can state the theorem, which is just a direct consequence of the previous two lemmas.

**Theorem 3.8** *Let $G$ with $2nk-1$ edges be a graph that allows a $2n$-cyclic blended labeling for $k$ odd and $k, n > 1$. Then there exists a $G$-decomposition of $K_{2nk}$ into $nk$ copies of $G$.*

*Proof.* By Lemma 3.5 the complete graph $K_{2nk}$ can be factorized into $n$ copies of $U(T, s; k)$, and by Lemma 3.7 the graph $U(T, s; k)$ can be decomposed into $k$ copies of $G$ if $G$ has a $2n$-cyclic blended labeling. Therefore, $K_{2nk}$ is decomposable into $nk$ isomorphic copies of $G$. □

We conclude this section with a simple example of a $2n$-cyclic labeling for a tree of a small order. For $K_{2nk}$ decomposition we choose the smallest case which is obtained when $k = 3$ and $n = 2$. As we already mentioned Eldergill's method enables factorizations only into symmetric spanning trees which all have odd diameter. For instance with our method we can easily find a factorization of $K_{12}$ into a spanning trees with the largest even diameter, which is $d = 10$.

**Construction 3.9** To find a 4-cyclic labeling of any spanning tree $G$ of $K_{12}$ we must have also an underlying tree $T$ with 4 vertices and a $\rho$-symmetric graceful labeling. The only symmetric (with respect to an edge) tree on 4 vertices is the path $P_4$. We use the symmetric graceful labeling of $P_4$ given in Figure 3.1. An example of a 4-cyclic labeling of a spanning tree $G$ with the vertex set $V(G) = \bigcup_{i=0}^{3} V_i$, where $V_i = \{0_i, 1_i, 2_i\}$, and diameter $d = 10$ is in Figure 3.2.

In each of the partite sets $V_1$ and $V_3$ there is one pure edge of the length $\ell_{11} = \ell_{33} = 1$. In each of the three pairs of the partite sets $V_1, V_0$ and $V_0, V_2$ and $V_2, V_3$ corresponding to the three edges of $P_4$ there are always mixed edges of all the lengths $0, 1, 2$. If we permute $G$ by the permutation $\pi = \pi_0\pi_1\pi_2\pi_3$, where $\pi_i$ is the cyclic permutation $\alpha_4$ on the vertices of $V_i$ for $i = 1, 2, 3, 4$, we obtain a 4-cyclic

$G$-factorization of the graph $U(P_4, 1; 3)$. In Figure 3.1 a $U(P_4, 1; 3)$-factorization of $K_{12}$ is shown. Then according to Theorem 3.8 there is a $G$-factorization of $K_{12}$.

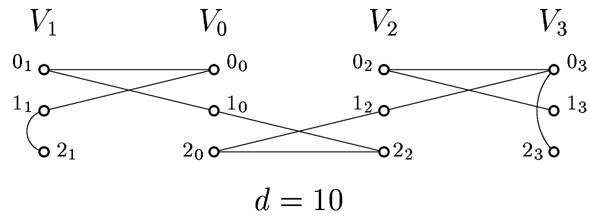



Figure 3.2: *4-cyclic blended labeling of the spanning tree $G$ of $K_{12}$ with $d = 10$.*

## 3.4 Fixing blended labeling

We will now relax the definition of a $2n$-cyclic blended labeling to obtain another labeling, which will allow us to find more general constructions of spanning trees for decompositions of $K_{2nk}$, where $k$ is odd.

To decompose the bipartite complete graph $K_{n,n}$ into $n$ copies of $G$ with $n$ edges we relied so far on the existence of a bipartite $\rho$-labeling. A bipartite $\rho$-labeling guarantees that the edges of the bipartite graph $G$ with the partite sets $|V_0| = |V_1| = n$ have all different lengths $0, 1, \ldots, n-1$ which enables bi-cyclic decomposition. Now suppose that $G$ has the following property. The degree of each vertex $x_0 \in V_0$ is $\deg(x_0) = 1$. In other words each vertex in $V_0$ has exactly one neighbor in $V_1$. It is not required that the edges have different mixed lengths. Then by permuting the vertices of $V_1$ by the cyclic permutation $\alpha_n$, while the vertices of $V_0$ are fixed under identity permutation $\iota$ we obtain the decomposition of $K_{n,n}$ into $n$ isomorphic copies of $G$. This idea is used in the following definition of the fixing labeling. For an illustration see Figure 3.3.

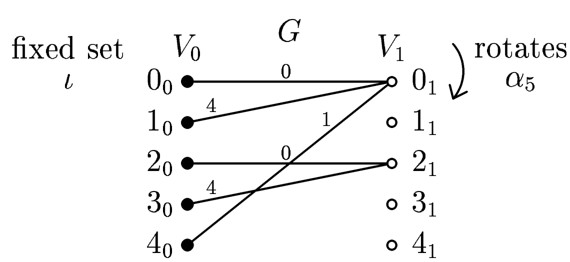



Figure 3.3: *Decomposition of $K_{5,5}$ into 5 isomorphic copies of $G$.*

**Definition 3.10** *Let $G$ be a graph with $2nk-1$ edges, for $k$ odd and $k,n>1$, and vertex set $V(G)=\bigcup_{i=0}^{2n-1} V_i$, where $|V_i|=k$ and $V_i \cap V_j = \emptyset$ for $i \neq j$. Let $\lambda$ be an injection, $\lambda : V_i \to \{0_i, 1_i, 2_i, \dots, (k-1)_i\}$, for $i=0,1,\dots,2n-1$. By $H_{ij}$ we denote the bipartite subgraph of $G$ induced on the vertices of the partite sets $V_i$ and $V_j$ with edges of the mixed length $\ell_{ij}$, and by $H_i$ we denote the subgraph of $G$ induced on the vertices of $V_i$ with edges of the pure length $\ell_{ii}$. We say that $G$ has a* fixing blended labeling *(briefly a* fixing labeling*) if there exists an underlying tree $T$ on $2n$ vertices with a $\rho$-symmetric graceful labeling such that the following holds:*

*(1) For some vertex $s \in T$ and its symmetric image $t = s+n \pmod{2n}$ is*
$\{\ell_{ss}(x_s,y_s) | (x_s,y_s) \in E(H_s)\} = \{1,2,\dots,(k-1)/2\}$, *and*
$\{\ell_{tt}(x_t,y_t) | (x_t,y_t) \in E(H_t)\} = \{1,2,\dots,(k-1)/2\}$.

*(2) Let $F = \{i \in T \,|\, i \neq s,\, i \neq s+n\,;\, \deg(x_i) = 1 \text{ for each } x_i \in H_{ij} \text{ and } j \in N(i)\}$, then $F$ is the* set of fixable vertices *in $T$ for given $G$, and each vertex $i$ in $F$ is called* fixable. *Let $V_F$ be any independent set of fixable vertices in $T$ called the* fixed set. *A vertex $i \in V_F$ is called a* fixed vertex.

*Then for every edge $(i,j) \in E(T)$ is one of the endvertices $i$ or $j$ the fixed vertex or $\{\ell_{ij}(x_i,y_j) | (x_i,y_j) \in E(H_{ij})\} = \{0,1,2,\dots,k-1\}$.*

Notice that the fixed vertices are not uniquely determined, since there might be several ways to choose the set $V_F \subseteq F$. The set of fixed vertices $V_F$ might be chosen to be a maximal independent subset of the set $F$ of fixable vertices, but it might be chosen to be also the empty set, depending on the structure of the labeled graph $G$. In the case that $V_F = \emptyset$ a fixing labeling of $G$ is also a $2n$-cyclic labeling.

We will show that also this labeling allows a $G$-decomposition of a complete graph $K_{2nk}$ if $G$ has the labeling.

**Lemma 3.11** *Let a graph $G$ with $2nk-1$ edges, for $k$ odd and $k,n>1$, have a fixing blended labeling. Then there exists a $G$-decomposition of $U(T,s;k)$ into $k$ copies of $G$.*

*Proof* Let a graph $U(T,s;k)$ have the vertex set $V(U(T,s;k)) = \bigcup_{i=0}^{2n-1} V_i$, where $V_i \cap V_j = \emptyset$ for $i \neq j$ and $V_i = \{0_i, 1_i, 2_i, \dots, (k-1)_i\}$, $i = 0,1,2,\dots,2n-1$. Let $V_F$ be any independent set of fixable vertices in $T$ for a given graph $G$, and $\pi = \pi_0\pi_1\pi_2\dots\pi_{2n-1}$, where $\pi_i$ is a cyclic permutation $\alpha_k$ on vertices of $V_i$ if $i \notin V_F$, and $\pi_i$ is an identity permutation $\iota$ on vertices of $V_i$ if $i \in V_F$.

Now suppose that $G_0 = G$ and $G_r = \pi^r[G]$ for $r = 1, 2, \ldots, k-1$, then $G_0, G_1, \ldots, G_{k-1}$ are $k$ isomorphic copies of $G$ on vertices of $U(T, s; k)$. We show that $G_0, G_1, \ldots, G_{k-1}$ is a $G$-decomposition of $U(T, s; k)$.

In $G_0$ the labeling of the subgraph $H_s$ induced by $\lambda$ is a $\rho$-labeling. Since $s \notin V_F$, the copy of $H_s$ in $G_r$ is $\pi_s^r[H_s] = \alpha_k^r[H_s]$. This means that with each permutation $\pi$ of $G$ we obtain a rotation of $H_s$ on the vertices of $V_s$. The $\rho$-labeling of $H_s$ guarantees (Theorem 2.3) that the complete graph $K_{V_s}$ in $U(T, s; k)$ is cyclically decomposed into $k$ copies of $H_s$. Similarly, $K_{V_{s+n}}$ in $U(T, s; k)$ is cyclically decomposed into $k$ copies of $H_{s+n}$, while we make permutations $\pi$ of $G$.

Also $2n - 1$ complete bipartite graphs $K_{V_i,V_j}$ in $U(T, s; k)$ are decomposed while permuting $G$. This is easy to see if neither $i$ nor $j$ is a fixed vertex. Then the labeling of $H_{ij}$ induced by $\lambda$ is a bipartite $\rho$-labeling, and the copy of $H_{ij}$ in $G_r$ is obtained by the permutation $\pi_i^r \pi_j^r[H_{ij}] = \alpha_k^r \alpha_k^r[H_{ij}]$. Thus the vertices of both partite sets $V_i$ and $V_j$ permute separately under the cyclic permutation $\alpha_k$ while $G$ permutes under the permutation $\pi$, and $K_{V_i,V_j}$ is decomposed bi-cyclically into $k$ copies of $H_{ij}$, which is guaranteed by the bipartite $\rho$-labeling of $H_{ij}$ (see page 7).

The remaining case is when one of the endvertices of the edge $(i, j) \in T$ is a fixed vertex. Since the pair $(i, j)$ is unordered we can assume without loss of generality that $i \in V_F$. Notice that if $i \in V_F$, then for any $(i, j) \in T$ is $j \notin V_F$. Because $i$ is fixable, any vertex $x_i \in V_i$ has exactly one neighbor in $V_j$ for any $j$ such that $(i, j) \in T$. So there are exactly $k$ edges in the subgraph $H_{ij}$. In $G_r$ the copy of $H_{ij}$ is $\pi_i^r \pi_j^r[H_{ij}] = \iota \alpha_k^r[H_{ij}]$. The vertices of $V_i$ are fixed under identity permutation $\iota$ and the vertices of $V_j$ are permuted by the cyclic permutation $\alpha_k$. Let $e_0 = (x_i, y_j)$ be the single edge incident with the vertex $x_i$ in $H_{ij}$. The edge $e_0$ has the mixed length $\ell_{ij}(e_0) = y - x \pmod k$. The copy of the edge $e_0$ in $G_r$ is $e_r = (\pi^r(x_i), \pi^r(y_j)) = (\iota(x_i), \alpha_k^r(y_j)) = (x_i, (y + r)_j)$, so it has the mixed length $l_{ij} = r + y - x \pmod k$. For $r = 0, 1, 2, \ldots, k-1$ we obtain $k$ copies of the edge $e_0$, which are all incident with $x_i$ and have all different mixed lengths $l_{ij} = 0, 1, 2, \ldots, k-1$. The same is true for any vertex $x_i \in V_i$, and so while the vertices of $H_{ij}$ are permuted by the permutation $\iota\alpha_k$ we obtain $k$ different edges incident to each vertex $x_i$ in $V_i$, which are all together $k^2$ different edges of $K_{V_i,V_j}$. Thus also in this case $K_{V_i,V_j}$ is decomposed into $k$ isomorphic copies of $H_{ij}$.

This completes the proof that there is a $G$-decomposition of $U(T, s; k)$ when $G$ has a fixing labeling. □

**Theorem 3.12** *Let a graph $G$ with $2nk-1$ edges, for $k$ odd and $k, n > 1$, have a fixing blended labeling. Then there exists a $G$-decomposition of $K_{2nk}$ into $nk$ copies of $G$.*

*Proof.* By Lemma 3.5 the complete graph $K_{2nk}$ can be factorized into $n$ copies of $U(T, s; k)$, and by Lemma 3.11 the graph $U(T, s; k)$ is decomposable into $k$ copies of $G$ if $G$ has a fixing labeling. Therefore $G$ decomposes $K_{2nk}$ into $nk$ isomorphic copies. □

Fixing labelings proved to be useful for classification of caterpillars with diameter 4 on $2nk$ vertices, as is shown further in Chapter 6. Finally we provide an example of a tree $G$ with a fixing labeling, which shall help the reader to understand easier the concept of this new labeling.

**Construction 3.13** Let $G$ be a tree with 29 edges and the vertex set $V(G) = \bigcup_{i=0}^{9} V_i$, where $V_i \cap V_j = \emptyset$ for $i \neq j$ and $V_i = \{0_i, 1_i, 2_i\}$ for any $i = 0, 1, \ldots, 9$. To find a fixing labeling of $G$ we use the symmetric underlying tree $T$ on 10 vertices with the symmetric graceful labeling given in Figure 3.4.

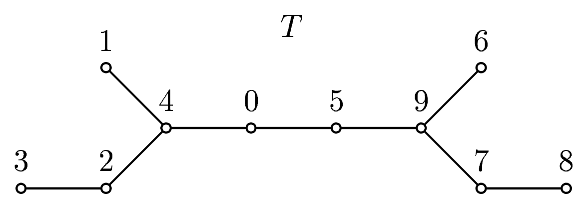



Figure 3.4: *Symmetric graceful labeling of an underlying tree $T$.*

To construct the subgraphs with the pure edges we choose the partite sets $V_0$ and $V_5$ corresponding to the vertex 0 and its symmetric image 5 in $T$. Then the tree $G$ given in figures 3.6 and 3.8 has a fixing labeling, and there are more options to form a $G$-decomposition of $U(T, 0; 3)$, depending on a choice of the fixed set $V_F$. The factors of $U(T, 0; 3)$ are $G_r = \pi^r[G]$ for $r = 0, 1, 2$ and $\pi = \pi_0\pi_1 \ldots \pi_9$, where $\pi_i$ is a permutation on the vertices of $V_i$ such that $\pi_i = \iota$ for $i \in V_F$, otherwise $\pi_i = \alpha_3$. In figures we have omitted the labels of the vertices of $G$, since it should be obvious, also from the previous examples, that they are assigned consecutively in the same way in each partite set.

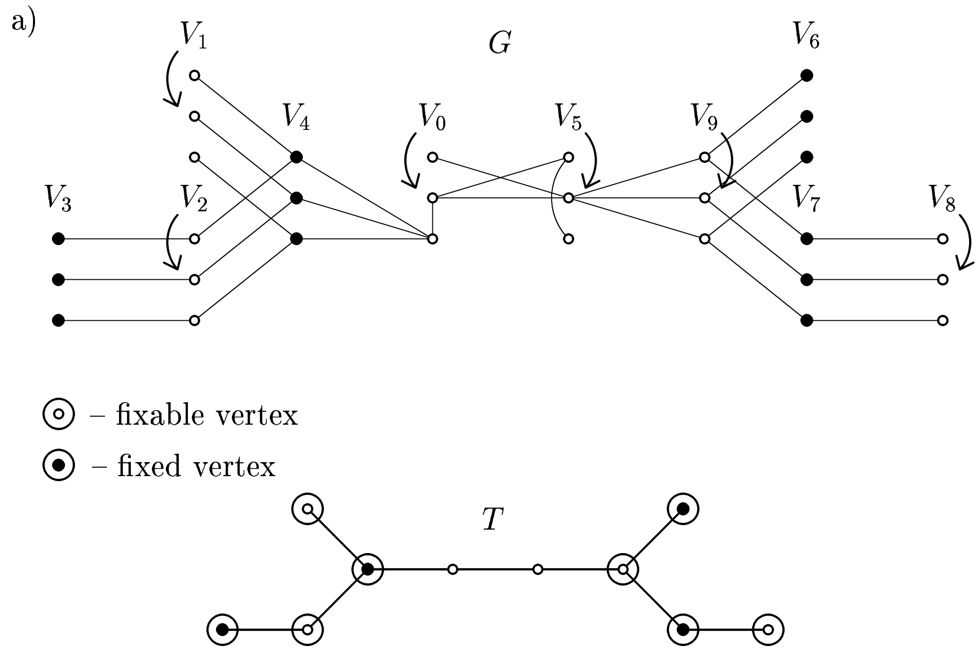



Figure 3.6: *Fixing labeling of G with* $F = \{1, 2, 3, 4, 6, 7, 8, 9\}$ *and* $V_F = \{3, 4, 6, 7\}$.

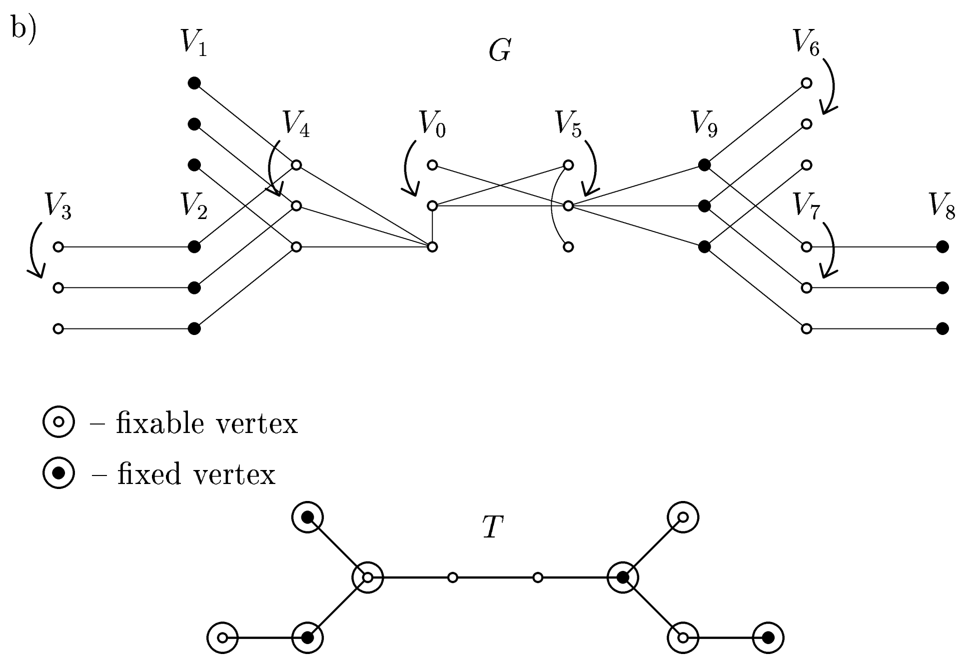



Figure 3.8: *Fixing labeling of G with* $F = \{1, 2, 3, 4, 6, 7, 8, 9\}$ *and* $V_F = \{1, 2, 8, 9\}$.

# Chapter 4

# $\mathbf{K_{4n}}$ decompositions

The case when the number of vertices of the complete graph $K_{4n}$ is a power of two is not covered by the methods for decompositions of $K_{2nk}$ introduced in the previous chapter. It is impossible to split the vertex set of $K_{4n} = K_{2^q}$ into partite sets of the same odd size. If the number of the vertices in a partite set is even the cyclic decomposition based on the $\rho$-labeling within a partite set cannot be used.

If we try to use a graph $G$ with $n$ edges and a $\rho$-labeling (see Definition 2.2) for a cyclic decomposition of $K_{2n}$, after $2n$ rotations of $G$ each edge of $K_{2n}$ is covered exactly once, except of the edges of the maximum length $n$ which are covered twice. Suppose $V(K_{2n}) = Z_{2n}$ and let $G_r = \alpha_{2n}^r[G]$, for $r = 0, 1, \ldots, 2n$. The copy of the edge $e_0 = (a, a+k)$ of the length $k$ in $G_0$ is the edge $e_r = (a+r, a+k+r)$ of the length $k$ in $G_r$, where $a \in \{0, 1, \ldots, 2n-1\}$ and $k \in \{1, 2, \ldots, n\}$. Suppose $e_0 = e_r$ for some $r \neq 0$. This happens in two cases:

(i) If
$$a \equiv a + r \pmod{2n} \quad \text{and} \quad a + k \equiv a + k + r \pmod{2n}.$$
Thus
$$r \equiv 0 \pmod{2n},$$
which contradicts the assumptions $r = 0, 1, \ldots, 2n-1$ and $r \neq 0$.

(ii) If
$$a \equiv a + k + r \pmod{2n} \quad \text{and} \quad a + k \equiv a + r \pmod{2n}.$$
We obtain the set of congruences
$$\begin{aligned} 0 &\equiv k + r \pmod{2n}, \\ k &\equiv r \pmod{2n}, \end{aligned}$$

which simplifies to

$$\begin{aligned} 2k &\equiv 0 \pmod{2n}, \\ 2r &\equiv 0 \pmod{2n}. \end{aligned}$$

The only solution for $r = 1, 2, \ldots, 2n-1$ and $k = 1, 2, \ldots, n$ is $r = n$ and $k = n$.

Thus the only repeated edge is the edge of the maximum length $k = n$ always after $r = n$ rotations. This can be easily observed from Figure 4.1.

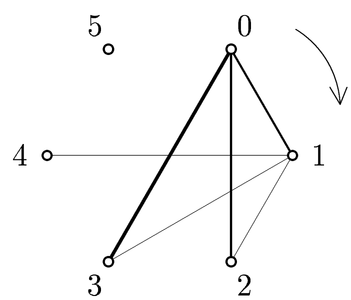



Figure 4.1: *Cyclic covering of $K_6$ by a tree with graceful labeling.*

## 4.1 Switching labelings and diameters of spanning trees

As was already mentioned, possible approach which can be used for decompositions of any $K_{4n}$ and does allow more general constructions of spanning trees than just symmetric ones is based on the switching blended labeling (see Definition 2.9). Nevertheless, we have found that the labeling is not suitable to solve the problems we are interested in, which is implied by the following theorem.

**Theorem 4.1** *If a tree $T$ on $4n$ vertices, where $n \geq 2$, allows a switching blended labeling, then* $\operatorname{diam} T > 4$.

*Proof.* Suppose to the contrary that a tree $T$ with $4n$ vertices has a switching labeling, and $\operatorname{diam} T = d \leq 4$. Then there is the edge $e_0 = (i_0, (i+n)_0)$ of the maximum pure length $\ell_{00}(e_0) = n$ in $T$. Let $e_1$ be the edge of the same pure length $\ell_{11}(e_1) = n$ in $V_1$, such that $e_1 = (j_1, (j+n)_1) \notin T$ and $\varphi(i_0) = j_1$, $\varphi((i+n)_0) = (j+n)_1$.

By $G$ we denote the graph $G = T + e_1$. Then the graph $G - e_0$ is isomorphic to $T$ and in $G$ there is a cycle $C_p$, which contains both edges $e_0, e_1$. Since the endvertices of $e_0$ are both in $V_0$ and the endvertices of $e_1$ are both in $V_1$, the minimum length of the cycle $C_p$ is $p = 4$.

Suppose first that $p = 4$. It means that $C_4 = i_0, (i+n)_0, (j+n)_1, j_1$ or $C_4 = i_0, (i+n)_0, j_1, (j+n)_1$. Notice that these cases are equivalent, since $j = j + n + n \pmod{2n}$. Hence we investigate just the former case. Then the edges $(i_0, j_1)$ and $((i+n)_0, (j+n)_1)$ must be in $T$. But this is not possible, because they are both of the same mixed length $\ell_{01}((i_0, j_1)) = j - i \pmod{2n}$, and $\ell_{01}(((i+n)_0, (j+n)_1)) = j + n - (i+n) = j - i \pmod{2n}$, which contradicts property (3) of the switching labeling. Therefore the length of the cycle $C_p$ is at least $p = 5$ and the diameter $d$ of $T$ is at least 4.

Now suppose that $p = 5$. Then there is a cycle $C_5 = i_0, (i+n)_0, (j+n)_1, j_1, v$ (again the case $C_5 = i_0, (i+n)_0, j_1, (j+n)_1, v$ is equivalent). In order of diameter $d$ of the tree $T$ (or equivalently of $G - e_1$ or $G - e_0$) to be $d = 4$, all other edges in $T$ must be incident to the vertex $v$. This is true because if there is an edge $xi_0$, where $x \neq (i+n)_0, v$, then from (4) it follows that there must be also an edge $yj_1, y \neq (j+n)_1, v$, and vice versa. But then there is the path $x, i_0, v, j_1, (j+n)_1, (i+n)_0$ in $G - e_0$ or $x, j_1, v, i_0, (i+n)_0, (j+n)_1$ in $G - e_1$, both of them of length 5, which contradicts our assumption that $d \leq 4$. Similarly, if there is one of edges $x(i+n)_0, y(j+n)_1$, where $x \neq i_0, (j+n)_1$ and $y \neq j_1, (i+n)_0$, then there must be the other one, too. Then again there is the path $x, (i+n)_0, (j+n)_1, j_1, v, i_0$ in $G - e_0$ or $x, (j+n)_1, j_1, v, i_0, (i+n)_0$ in $G - e_1$, giving the same contradiction.

But now if the vertex $v$ belongs to $V_0$, all its neighbors except for $j_1$ belong to $V_0$ and there is only one pure edge in $V_1$, namely $(j_1, (j+n)_1)$ of length $n \geq 2$. This is impossible, since the tree $T$ must contain edges of all pure lengths $\ell_{11} = 1, 2, \ldots, n$. The same argument holds when $v \in V_1$ and the proof is complete. □

It is obvious now that the method based on switching labelings is not sufficient to answer the questions about diameters of spanning trees or to complete the classification of caterpillars with diameter 4 for factorization of $K_{4n}$.

## 4.2 Swapping labeling

In this section we introduce a new type of the vertex labeling, namely swapping labeling, which allows the decompositions of $K_{4n}$. Similarly as for a graph $G$ with a switching labeling we split the vertex set of $G$ with a swapping labeling into two equal partite sets and we require for $G$ certain type of an isomorphism. Despite of that, in general the labelings seem to be easier to find than switching labelings. Swapping labeling enabled us to complete the solutions of the problems on diameters of the spanning trees and classification of caterpillars in the case when the number of vertices of $K_{4n}$ is a power of two as is shown later.

**Definition 4.2** *Let $G$ be a graph with $4n-1$ edges and the vertex set $V(G) = V_0 \cup V_1$, $V_0 \cap V_1 = \emptyset$, and $|V_0| = |V_1| = 2n$. Let $\lambda$ be an injection, $\lambda : V_i \longrightarrow \{0_i, 1_i, \dots, (2n-1)_i\}$ for $i = 0, 1$. The pure length $\ell_{ii}$, for $i \in \{0, 1\}$ and the mixed length $\ell_{01}$ of an edge are defined as in Definition 2.7 of the blended labeling.*

*Then $G$ has a* swapping blended labeling *(briefly* swapping labeling*) if*

*(1)* $\{\ell_{ii}(x_i, y_i) | (x_i, y_i) \in E(G)\} = \{1, 2, \dots, n\} \text{ for } i = 0, 1,$

*(2) there exists an isomorphism $\varphi$ such that $G$ is isomorphic to $G \setminus \{(k_0, (k+n)_0), (l_1, (l+n)_1)\} \cup \{(k_0, (l+n)_1), ((k+n)_0, l_1)\}$,*

*(3)* $\{\ell_{01}(x_0, y_1) | (x_0, y_1) \in E(G)\} = \{0, 1, \dots, n-1\} \setminus \{\ell_{01}(k_0, (l+n)_1)\}.$

We shall notice again that $G$ with a swapping labeling can be split into subgraphs $H_0$ and $H_1$ on the vertices of $V_0$ and $V_1$ respectively and a bipartite subgraph $H_{01}$ with the partite sets $V_0$ and $V_1$. The labelings of $H_0$ and $H_1$ induced by $\lambda$ are again $\rho$-labelings (condition (1)), and the labeling induced by $\lambda$ on the vertices of $H_{01}$ is an "almost" bipartite $\rho$-labeling. It is not a true bipartite $\rho$-labeling, since one edge of the mixed length $\ell_{01}(k_0, (l+n)_1) = l + n - k \pmod{2n}$ is missing (condition (3)).

It is not difficult to observe what happens if we let $G$ rotate bi-cyclically so that the vertices of $V_0$ or $V_1$ permute separately under cyclic permutation $\alpha_{2n}$. Then $K_{V_0}$ is decomposed into $2n$ copies of $H_0$, but since the number of vertices of $V_0$ is even each edge of the maximum pure length is covered twice. Similarly it holds for $K_{V_1}$. Therefore we keep the edges of the maximum pure length in $H_0$ and $H_1$ only for the first $n$ rotations. In the remaining $n$ rotations they are exchanged (or swapped) for the mixed edges of the missing length in $H_{01}$. The isomorphism $\varphi$ required by the condition (2) guarantees that after swapping the edges isomorphic copies of $G$ are obtained.

**Theorem 4.3** *Let $G$ be a graph on $4n$ vertices with $4n-1$ edges which has a swapping blended labeling. Then there exists a $G$-decomposition of $K_{4n}$ into $2n$ isomorphic copies of $G$.*

*Proof.* Suppose that a graph $G$ on $4n$ vertices has a swapping labeling $\lambda$. By $U$ we denote the graph $G - \{(k_0, (k+n)_0), (l_1, (l+n)_1)\}$, where $k, l \in \{0, 1, \dots, 2n-1\}$. Let $K_{4n}$ has the vertex set $V(K_{4n}) = V_0 \cup V_1 = \{0_0, 1_0, \dots, (2n-1)_0\} \cup \{0_1, 1_1, \dots, (2n-1)_1\}$. It means that we view $K_{4n}$ as a union of the two complete graphs $K_{2n} = K_{V_0} = K_{V_1}$ and the complete bipartite graph $K_{2n,2n} = K_{V_0,V_1}$.

We define $U_0, U_1, \dots, U_{2n-1}$ by $U_r = \pi^r[U]$ for $r = 0, 1, \dots, 2n-1$, where $\pi = \pi_0\pi_1$ and $\pi_i$ is the cyclic permutation $\alpha_{2n}$ on the vertices of $V_i$ for $i \in \{0, 1\}$. Then $U_0, U_1, \dots, U_{2n-1}$ are $2n$ isomorphic copies of the graph $U$ on the vertices of $K_{4n}$.

If the edge $(x_i, (x+a)_i)$ is the unique edge of the pure length $\ell_{ii} = a$, $1 \le a \le n-1$ for $i \in \{0, 1\}$ in $U_0$ then $(\pi^r(x_i), \pi^r((x+a)_i)) = ((x+r)_i, (x+a+r)_i)$ is the unique edge of the same length $\ell_{ii} = a$ in $U_r$. Similarly, if the edge $(x_0, (x+b)_1)$ is the unique edge of the mixed length $\ell_{01} = b$, $0 \le b \le n-1$, and $b \ne \ell_{01}(k_0, (l+n)_1)$ in $U_0$, then $(\pi^r(x_0), \pi^r((x+b)_1)) = ((x+r)_0, (x+b+r)_1)$ is the unique edge of the mixed length $\ell_{01} = b$ in $U_r$. Obviously, for $r = 0, 1, \dots, 2n-1$ there are all $2n$ edges of each pure or mixed length in $K_{4n}$ covered exactly once with two exceptions. In copies of $U$ does not appear any edge of the maximum pure length $\ell_{00} = \ell_{11} = n$ and any edge of the mixed length $\ell_{01}(k_0, (l+n)_1) = l + n - k \pmod{2n}$.

Each copy of the graph $U$ can be completed to a copy of the graph $G$. We let $G_0$ be $U_0 \cup \{(k_0, (k+n)_0), (l_1, (l+n)_1)\}$ and for $r = 1, 2, \dots, n-1$ we define $G_r = U_r \cup \{((k+r)_0, (k+n+r)_0), ((l+r)_1, (l+n+r)_1)\}$. Hence we have used all $n$ edges of the maximum pure length $n$ of $K_{V_0}$ and also of $K_{V_1}$.

Because $G$ has a swapping labeling, $G$ is isomorphic to $U_0 \cup \{(k_0, (l+n)_1), ((k+n)_0, l_1)\}$. Therefore we can set the next copy of $G$ to be $G_n = U_n \cup \{((k+n)_0, (l+n+n)_1), ((k+n+n)_0, (l+n)_1)\} = U_n \cup \{((k+n)_0, l_1), (k_0, (l+n)_1)\}$, and for $r = n+1, n+2, \dots, 2n-1$ we obtain remaining $n-1$ copies as $G_r = U_r \cup \{((k+n+r)_0, (l+r)_1), ((k+r)_0, (l+n+r)_1)\}$. It is easy to check that we have used in remaining $n$ copies of $G$ all $2n$ edges of the mixed length $\ell_{01} = l+n-k$ of $K_{V_0,V_1}$. It is so because the endvertices in $V_0$ of the added edges $((k+n+r)_0, (l+r)_1)$ and $((k+r)_0, (l+n+r)_1)$ are in “distance” $n$. Therefore when $r$ is changed from

$r = n$ to $r = 2n - 1$ the added edges have as endvertices all $2n$ different vertices of $V_0$, and to have the same edge with the different endvertex is absurd. The same is true for the endvertices in the partite set $V_1$.

Thus $G_0, G_1, \ldots, G_{2n-1}$ form the $G$-decomposition of $K_{4n}$ and the proof is complete. □

We again conclude with an example of a graph which has the labeling. Even if the factorizations of any $K_{4n}$ into Hamiltonian paths are known and easily found using any of the previously known methods, we choose the graph to be the path on 8 vertices, $P_8$. The simplicity of the example allows to observe easier the existence of the isomorphism $\varphi$ and how the factors are formed. See Figures 4.2 and 4.3.

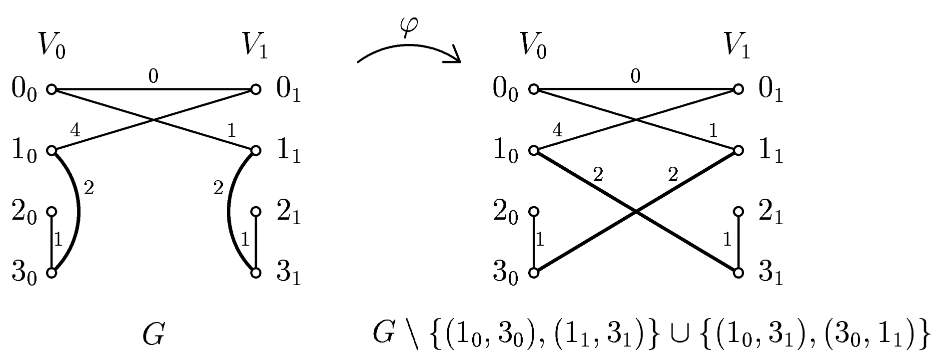



Figure 4.2: *Swapping labeling of $G = P_8$.*

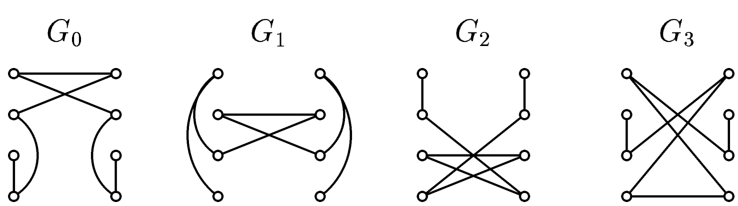



Figure 4.3: *$P_8$-factorization of $K_8$ based on the swapping labeling.*

# Chapter 5

# Spanning trees with given diameter

In this chapter we give an answer to the question if for a given number $d$ there exists a spanning tree factorization of $K_{2n}$ such that the spanning tree has the diameter $d$. The diameter $d$ can have any of the considerable values which are $3 \leq d \leq 2n-1$. The only spanning tree of $K_{2n}$ with the smallest diameter $d = 2$ is the star $K_{1,2n-1}$. But obviously a factorization into stars does not exist, since Degree Condition 2.1 is not satisfied.

As was already mentioned Fronček positively answered the question about diameters of spanning trees for factorizations of $K_{4n+2}$ [6].

**Theorem 5.1** (Fronček) *For every $d$ such that $3 \leq d \leq 4n+1$, where $n \geq 1$, there is a factorization of $K_{4n+2}$ into isomorphic spanning trees with diameter $d$.*

Therefore it remains to solve the problem whenever the number of vertices of a complete graph is a multiple of 4.

While solving the problem we found factorizations based on $2n$-cyclic labelings first [14]. Recall that the method can be used for factorizations of $K_{2nk}$, where $n, k > 1$ and $k$ is odd. Of course the case when the number of the vertices is a power of two cannot be solved by this method. Later the method based on swapping labelings enabled us to answer the question about diameters completely. We introduce here both types of constructions. The reason is that the spanning trees with a diameter $d$ for which we have found $2n$-cyclic labelings have different structure than the spanning trees with a diameter $d$ for which we have found swapping labelings. The problem is quite useful for demonstration of both of the methods. Consequently one will have an opportunity to compare the methods and gain more intuition in deciding which method is more suitable depending on the structure of a spanning tree.

## 5.1 Constructions based on $2n$-cyclic labelings

The method of decomposition based on $2n$-cyclic blended labeling can be used whenever the number of vertices of $K_{4n}$ is not a power of two. By this condition we are left with complete graphs $K_{2^q k}$, where $k$ is odd and $k, q > 1$. Therefore we construct spanning trees of $K_{2^q k}$ with $2^q$-cyclic blended labelings. Because for each such a spanning tree there must be also an underlying tree on $2^q$ vertices with $\rho$-symmetric graceful labeling, we first introduce a class of symmetric graceful trees which are used in constructions.

All symmetric graceful trees we deal with are caterpillars. A caterpillar on $n$ vertices, which is a star $K_{1,h}$, where $1 \leq h \leq n-1$, with a path $P_{n-h}$ attached to its central vertex is called a *broom* and denoted by $B(n,h)$. By $X(2n,h)$ we denote the symmetric caterpillar with banks $H, H'$ both isomorphic to $B(n,h)$ and with the symmetric edge connecting the endvertices of the paths $P_{n-h}$. In other words, the tree $X(2n,h)$ is a union of two stars $K_{1,h}$ and the path $P_{2(n-h)}$ connecting their central vertices. To obtain a symmetric graceful labeling of $X(2n,h)$ it is sufficient to find a graceful labeling of one bank $H = B(n,h)$ since the labels of the other bank $H'$ are induced by the isomorphism $\psi(i) = i + n \pmod{2n}$ (see Definition 2.5).

There are of course more ways how to assign the labels to the vertices of $B(n,h)$ to obtain a graceful labeling. We will consider the following labeling.

**Graceful labeling of a broom $B(n,h)$**

- The label 0 is assigned to the central vertex of $K_{1,h}$, the labels $n-1, n-2, \ldots, n-h$ are assigned to the $h$ attached vertices of degree one. Lengths of the edges are $n-1, n-2, \ldots, n-h$.

- The vertices of the path $P_{n-h}$ receive the labels:

  $(i)$ $0, n-h-1, 1, n-h-2, \ldots, \frac{n-h}{2}-1, \frac{n-h}{2}$ for $n-h$ even,

  $(ii)$ $0, n-h-1, 1, n-h-2, \ldots, \frac{n-h-1}{2}+1, \frac{n-h-1}{2}$ for $n-h$ odd, consecutively.

The edges of the path have remaining lengths $n-h-1, n-h-2, \ldots, 1$. For an example of this labeling see Figure 5.1.

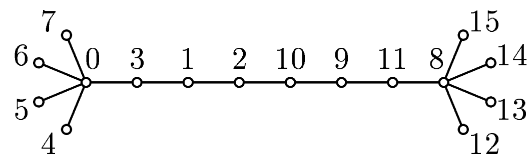



Figure 5.1: *Symmetric graceful labeling of* $X(16,4)$.

Before we state the theorem we define two types of trees with bipartite $\rho$-labelings.

**Construction of $S_I$ and $S_{II}$**

By $S_I$ and $S_{II}$ we denote double stars with bipartite $\rho$-labelings and the vertex set $V(S_I) = V(S_{II}) = V_i \cup V_j$, $V_i = \{0_i, 1_i, 2_i, \ldots, (k-1)_i\}$, $V_j = \{0_j, 1_j, 2_j, \ldots, (k-1)_j\}$, where $k = 2m+1$ for $m \geq 1$.

- The double star $S_I$ is constructed as two stars $K_{1,m-1}$ with the central vertices $m_i$ and $m_j$ connected by the edge $(m_i, m_j)$ of the mixed length $\ell_{ij} = 0$. The endvertices connected to the central vertex $m_i$ are $0_j, 1_j, \ldots, (m-1)_j$. The edges have mixed lengths $\ell_{ij} = m+1, m+2, \ldots, 2m$. The endvertices connected to the central vertex $m_j$ are $0_i, 1_i, \ldots, (m-1)_i$ so that the edges have the missing lengths $\ell_{ij} = 1, 2, \ldots, m$.

- The double star $S_{II}$ is isomorphic to $S_I$ so that there is an isomorphism $f : V(S_I) \longrightarrow V(S_{II})$ defined by $f(x_r) = (2m-x)_r$ for every vertex $x_r \in V(S_I)$ and $r \in \{i, j\}$.

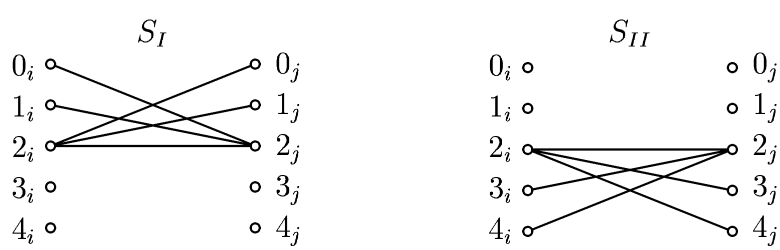



Figure 5.2: *Double stars* $S_I$ *and* $S_{II}$ *for* $k = 5$.

**Construction of $C_I(D)$ and $C_{II}(D)$**

By $C_I(D)$ or $C_{II}(D)$ we denote the tree with a bipartite $\rho$-labeling, diameter $D$, and the vertex set $V_i \cup V_j$, $V_i = \{0_i, 1_i, 2_i, \ldots, (k-1)_i\}$, $V_j = \{0_j, 1_j, 2_j, \ldots, (k-1)_j\}$, where $k = 2m+1$ for $m \geq 1$. The diameter $D$ is odd, ranging from minimum 3 to maximum $k$. Let $D = 2t+1$, where $1 \leq t \leq m$.

- The tree $C_I(D)$, for $t$ odd, has the diametrical bipartite path $P_{D+1} = m_i, 0_j, (m-1)_i, 1_j, \ldots, (m - \frac{t-1}{2})_i, (\frac{t-1}{2})_j, (\frac{t-1}{2})_i, (m - \frac{t-1}{2})_j, \ \ldots, 1_i, (m-1)_j, 0_i, m_j$.

  For $t$ even, $P_{D+1} = m_i, 0_j, (m-1)_i, 1_j, \ldots, (\frac{t}{2}-1)_j, (m-\frac{t}{2})_i, \ (m-\frac{t}{2})_j, (\frac{t}{2}-1)_i, \ldots, 1_i, (m-1)_j, 0_i, m_j$.

  The edges on the path have the mixed lengths $\ell_{ij} = m+1, m+2, \ldots, m+t, 0, m-t+1, m-t+2, \ldots, m-1, m$, and the missing lengths are $\ell_{ij} = 1, 2, \ldots, m-t$ and $m+t+1, m+t+2, \ldots, 2m$.

  We obtain the edges of the missing lengths by adding two stars $K_{1,m-t}$ with the central vertices on the path $P_{D+1}$. When $t$ is odd, the central vertices are $\left(\frac{t-1}{2}\right)_r$, $r \in \{i,j\}$. The vertices of degree one are in the other partite set than the central vertex. They are $(\frac{t-1}{2}+1)_s, (\frac{t-1}{2}+2)_s, \ldots, (m - \frac{t-1}{2} - 1)_s$, where $s = i$ for $r = j$ and $s = j$ for $r = i$. When $t$ is even, the central vertices are $\left(\frac{t}{2}-1\right)_r$. The endvertices in the opposite partite set are $(\frac{t}{2})_s, (\frac{t}{2}+1)_s, \ldots, (m - \frac{t}{2} - 1)_s$.

- The tree $C_{II}(D)$ is isomorphic to $C_I(D)$ by the isomorphism $f : V(C_I(D)) \longrightarrow V(C_{II}(D))$ defined as $f(x_r) = (2m-x)_r$ for every vertex $x_r \in V(C_I(D))$ and $r \in \{i,j\}$.

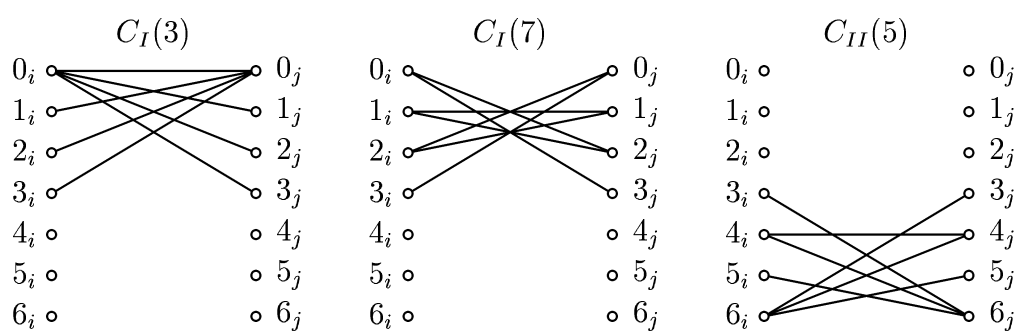



Figure 5.3: *$C_I(D)$ and $C_{II}(D)$ for $k = 7$.*

**Theorem 5.2** *For any $d$, $3 \leq d \leq 2^q k - 1$, there exists a tree $T$ with the diameter $d$ such that there is a $T$-factorization of the complete graph $K_{2^q k}$, where $q, k > 1$ and $k$ is odd.*

*Proof.* To obtain a spanning tree of $K_{2^q k}$ with any odd diameter is easy. We can take for instance $X(2^q k, h)$, which cyclically factorizes $K_{2^q k}$ and has the diameter $d = 2(2^{q-1}k - h) + 1$, where $1 \leq h \leq 2^{q-1}k - 1$. If $h = 2^{q-1}k - 1$, the caterpillar $X(2^q k, h)$ is a double star with the diameter $d = 3$, which is the smallest possible. If $h = 1$, $X(2^q k, h)$ is the path $P_{2^q k}$, and the diameter is the largest possible $d = 2^q k - 1$. Further we will concentrate only on spanning trees with an even diameter.

We will complete the proof in three steps, constructing spanning trees of even diameters with a $2^q$-cyclic blended labeling. We always consider a spanning tree $T$ with the vertex set $V(T) = \bigcup_{i=0}^{2^q-1} V_i$, where $V_i \cap V_j = \emptyset$ for $i \neq j$ and $V_i = \{0_i, 1_i, 2_i, \ldots, (k-1)_i\}$, $i = 0, 1, 2, \ldots, 2^q - 1$. We set $k = 2m + 1$.

(1) Stretching the underlying tree into Hamiltonian path (diameters: $4 \leq d \leq 2^q$).

As the underlying tree we consider $X(2^q, h)$ with the symmetric graceful labeling given above. We will construct subgraphs $H_{ij}$ with mixed edges for each $(i, j) \in E(X(2^q, h))$ and subgraphs $H_0$ and $H_{2^{q-1}}$ with pure edges separately.

We construct each $H_{ij}$ corresponding to an edge $(i, j)$ on the path $P_{2(2^{q-1}-h)}$ as a double star. More precisely, we alternate double stars $S_I$ and $S_{II}$.

When $2^{q-1} - h$ is even, $H_{ij} = S_I$ for

$$(i, j) \in \{(x, 2^{q-1} - h - 1 - x),\ (x + 2^{q-1}, 2^q - h - 1 - x)\},$$

where $0 \leq x \leq \frac{2^{q-1}-h}{2} - 1$, and $H_{ij} = S_{II}$ for

$$(i, j) \in \{(2^{q-1} - h - x, x),\ (2^q - h - x, x + 2^{q-1})\} \cup \{(\tfrac{2^{q-1}-h}{2}, \tfrac{2^{q-1}-h}{2} + 2^{q-1})\},$$

where $1 \leq x \leq \frac{2^{q-1}-h}{2} - 1$.

When $2^{q-1} - h$ is odd, $H_{ij} = S_I$ for

$$(i,j) \in \{(x, 2^{q-1} - h - 1 - x),\ (x + 2^{q-1}, 2^q - h - 1 - x)\} \cup \{(\tfrac{2^{q-1}-h-1}{2}, \tfrac{2^{q-1}-h-1}{2} + 2^{q-1})\},$$

where $0 \leq x \leq \frac{2^{q-1}-h-1}{2} - 1$, and $H_{ij} = S_{II}$ for

$$(i,j) \in \{(2^{q-1} - h - x, x),\ (2^q - h - x, x + 2^{q-1})\},$$

where $1 \leq x \leq \frac{2^{q-1}-h-1}{2}$.

The subgraphs $H_{ij}$ corresponding to the edges connecting $2h$ endvertices in $X(2^q, h)$ are constructed as the stars $K_{1,2m+1}$.

$$\text{For} \quad (i,j) \in \{(0, 2^{q-1} - 1), (0, 2^{q-1} - 2), \ldots, (0, 2^{q-1} - h)\},$$

the star $K_{1,2m+1}$ has the central vertex $(m+1)_0$ and the attached vertices of degree one are all $2m+1$ vertices of $V_j$.

$$\text{For} \quad (i,j) \in \{(2^{q-1}, 2^q - 1), (2^{q-1}, 2^q - 2), \ldots, (2^{q-1}, 2^q - h)\},$$

the star $K_{1,2m+1}$ has the central vertex $m_{2^{q-1}}$ and again $2m+1$ endvertices in $V_j$.

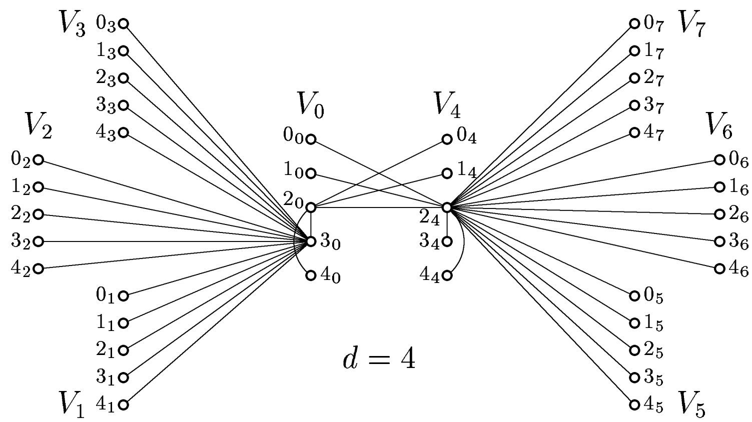
Obviously, in each star $K_{1,2m+1}$ we have $2m+1$ edges, one edge of each mixed length $\ell_{ij} = 0, 1, \ldots, 2m$.

To obtain $H_0$ and $H_{2^{q-1}}$ we add the star $K_{1,m}$ on vertices of $V_i$ for $i \in \{0, 2^{q-1}\}$. The central vertex of $K_{1,m}$ is $m_i$ and the endvertices are $(m+1)_i, (m+2)_i, \ldots, (k-1)_i$, so that we have all required edges of pure lengths $\ell_{ij} = 1, 2, \ldots, m$.

Now if we "glue" all subgraphs $H_{ij}$, $H_0$, and $H_{2^{q-1}}$ together, we obtain the tree $T$ with the $2^q$-cyclic labeling which guarantees the $2^q$-cyclic $T$-factorization of $U(X(2^q, h), 0, k)$ and consequently the $T$-factorization of $K_{2^q k}$.

Our spanning tree $T$ has the diameter $d = 2^q - 2h + 2$. It is so because each of the $2^q - 2h - 1$ double stars, $S_I$ or $S_{II}$, contributes by 1 to the diameter $d$ of $T$, and the stars $K_{1,m}$ and $K_{1,2m+1}$ contribute together by 3. For $h$ ranging from 1 to $2^{q-1} - 1$ we get spanning trees with even diameters from the interval $4 \leq d \leq 2^q$. See examples in Figure 5.4.

The underlying tree is $X(8,3)$.




The underlying tree is $X(8,1)$.

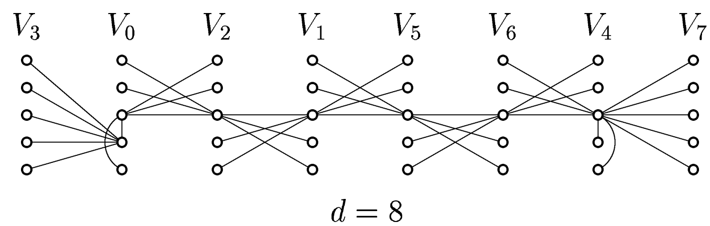



Figure 5.4: *Spanning trees of $K_{40}$ with 8-cyclic blended labelings and diameters $d = 4$ and $d = 8$.*

(2) Stretching the bipartite paths (diameters: $2^q + 2 \leq d \leq 2^q k - k + 1$).

The largest diameter in the previous case was obtained for $h = 1$ when the underlying tree was the path $X(2^q, 1) = P_{2^q}$. The underlying tree cannot be stretched any more, therefore to obtain larger diameter than $d = 2^q$ we have to increase the diameters of the subgraphs $H_{ij}$.

Suppose the underlying tree is $X(2^q, 1) = P_{2^q}$, again with the symmetric graceful labeling given above (see page 34). We start with a spanning tree $T$ of the odd diameter $d = 2^q - 1$. We let each subgraph $H_{ij}$ corresponding to the edge $(i, j) \in E(P_{2^q})$ be a double star $S_I$ or $S_{II}$.

For

$$(i,j) \in \{(2^{q-1}-1-x, x),\ (2^q-1-x, x+2^{q-1})\},$$

where $0 \le x \le 2^{q-2}-1$, the subgraph $H_{ij}$ is constructed as $S_I$.

For

$$(i,j) \in \{(x, 2^{q-1}-2-x,),\ (x+2^{q-1}, 2^q-2-x,)\} \cup \{(2^{q-2}-1, 3\cdot 2^{q-2}-1)\},$$

where $0 \le x \le 2^{q-2}-2$, the subgraph $H_{ij}$ is constructed as $S_{II}$.

We choose the endvertices of $P_{2^q}$, which are $2^{q-1}-1$ and $2^q-1$, to construct two subgraphs $H_{2^{q-1}-1}$, $H_{2^q-1}$ with pure edges. The subgraph $H_{2^{q-1}-1}$ is the star $K_{1,m}$ with the central vertex $0_i$ and $m$ vertices of degree one $(m+1)_i, (m+2)_i, \ldots, (2m)_i$, where $i = 2^{q-1}-1$. The subgraph $H_{2^q-1}$ is also the star $K_{1,m}$ with the central vertex $m_i$ and $m$ vertices of degree one $(m+1)_i, (m+2)_i, \ldots, (2m)_i$, where $i = 2^q-1$.

All subgraphs $H_{2^q-1}$, $H_{2^{q-1}-1}$, and $H_{ij}$ give together the spanning tree $T$ of $U(P_{2^q}, 2^{q-1}-1; k)$ with the $2^q$-cyclic labeling. Diametrical path of $T$ can be chosen so that the subgraphs $H_{ij} = S_I$ corresponding to the first and the last edge on $P_{2^q}$ contribute to the diameter $d$ by 2 and all the other $2^q-3$ subgraphs $H_{ij}$ contribute by 1. Two stars $K_{1,m}$ do not increase diameter and so $d = 2^q+1$.

Now we replace the first double star $S_I$ corresponding to the first edge on $P_{2^q}$ by the tree $C_I(D)$. Diameter $D$ of $C_I(D)$ is odd, ranging from 3 to $k$, which extends the diameter $d$ of the spanning tree always by 2 from $2^q+2$ to $2^q-1+k$. Similarly we replace stepwise all $2^q-1$ double stars $S_I$ and $S_{II}$ by trees $C_I(D)$ and $C_{II}(D)$, respectively. When one of the stars is replaced and $D$ is changed gradually we obtain spanning trees with the next $\frac{k-1}{2}$ even diameters. The largest diameter is $d = 2^q-1+k+(2^q-2)(k-1) = 2^qk-k+1$. Overall we obtain spanning trees with even diameters $2^q+2 \le d \le 2^qk-k+1$. Examples are shown in Figure 5.5.

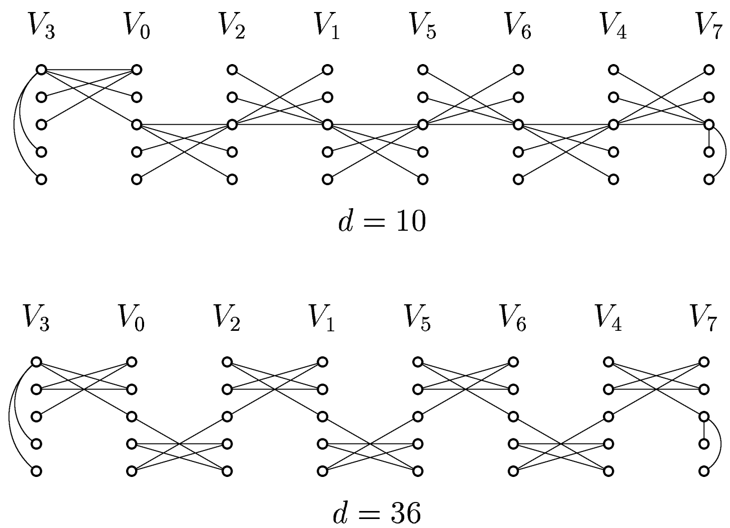



Figure 5.5: *Spanning trees of* $K_{40}$ *with* 8*-cyclic labelings and diameters* $d = 10$ *and* $d = 36$.

(3) Stretching subgraphs with pure edges (diameters: $2^q k - k + 2 \leq d \leq 2^q k - 1$)

In this case the underlying tree is of course again the path $P_{2^q}$. The subgraphs $H_{ij}$, for each edge $(i,j) \in E(P_{2^q})$ are constructed as for the longest diameter in the previous case. It means that they alternate between the graphs $C_I(k)$ and $C_{II}(k)$. The only way how to increase the diameter $d$ of the spanning tree $T$ is to extend the diameter of the subgraphs $H_{2^{q-1}-1}$ and $H_{2^q-1}$ with pure edges.

We start with the odd diameter $d = 2^q k - k + 2$ which is obtained if both subgraphs $H_{2^{q-1}-1}$ and $H_{2^q-1}$ are the stars $K_{1,m}$ with the central vertices $m_i$, where $i \in \{2^{q-1} - 1, 2^q - 1\}$. Then we convert one of the stars, say in partite set $V_i$ for $i = 2^{q-1} - 1$, to a broom $B(m+1, s)$, where $1 \leq s \leq m - 1$. If $m + 1 - s = 2r$, the vertices of the path $P_{m+1-s}$ are $m_i, 2m_i, (m+1)_i, (2m-1)_i, \ldots, (m+r-1)_i, (2m+1-r)_i$, and the star $K_{1,s}$ has the central vertex $(2m+1-r)_i$ with attached vertices of degree one, $(m+r)_i, (m+r+1)_i, \ldots, (2m-r)_i$. If $m + 1 - s = 2r + 1$, the path $P_{m+1-s}$ has the vertices $m_i, 2m_i, (m+1)_i, (2m-1)_i, \ldots, (2m+1-r)_i, (m+r)_i$, and the star $K_{1,s}$ has the central vertex $(m+r)_i$. The attached endvertices are $(m+r+1)_i, (m+r+2)_i, \ldots, (2m-r)_i$. The edges have in both cases pure lengths $\ell_{ii} = m, m-1, \ldots, 1$. Each broom $B(m+1, s)$ contributes by the diameter $m + 1 - s$.

When $s$ is changing from $m-1$ to 1, we obtain the spanning trees with even and odd diameters $2^qk-k+3, 2^qk-k+4, \ldots, 2^qk-k+m+1 = 2^qk-\frac{k+1}{2}+1$. We can repeat the procedure with the brooms in the partite set $V_{2^q-1}$ to obtain the spanning trees with the missing diameters $2^qk - \frac{k+1}{2} + 2, 2^qk - \frac{k+1}{2} + 3, \ldots, 2^qk - \frac{k+1}{2} + m = 2^qk - 1$. See example in Figure 5.6.

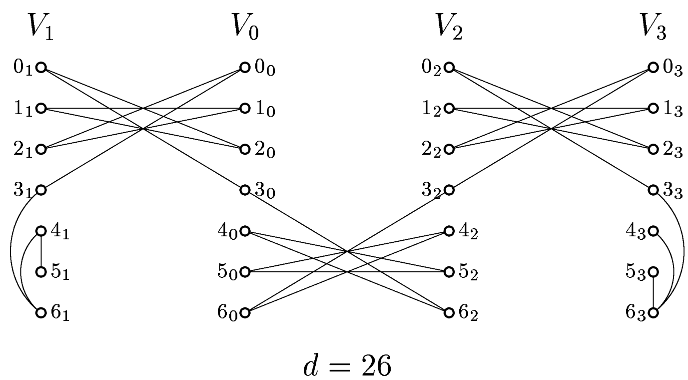



Figure 5.6: *Spanning tree of $K_{28}$ with 4-cyclic blended labeling and diameter $d = 26$.*

Now we have constructed spanning trees of all possible diameters $3 \leq d \leq 2^qk-1$ and so the proof is complete. □

It remains to solve the problem for $K_{2^q}$. This case is covered by the result introduced in the following section, where we use swapping labelings.

## 5.2 Constructions based on swapping labelings

With the swapping labeling available we prove the following theorem.

**Theorem 5.3** *For any $d$, $3 \leq d \leq 4n-1$, there exists a tree $T$ with the diameter $d$ such that there is a $T$-factorization of the complete graph $K_{4n}$, where $n$ is a positive integer.*

*Proof.* In constructions we consider a spanning tree $T$ with the vertex set $V(T) = \bigcup_{i=0}^{1} V_i$, where $V_0 \cap V_1 = \emptyset$ and $V_i = \{0_i, 1_i, 2_i, \ldots, (2n-1)_i\}$, for $i = 0, 1$. We will view each spanning tree with a swapping labeling as a union of the subgraphs $H_0, H_1$ and $H_{01}$. Subgraphs $H_0, H_1$ and $H_{01}$ will contribute by $d_0$, $d_1$ and $d_{01}$ respectively to the diameter $d = \text{diam}(T)$ so that $d = d_0 + d_1 + d_{01}$.

(1) Stretching the subgraphs $H_0$ and $H_1$ (diameters: $3 \le d \le 2n-1$).

$H_{01}$ is constructed as the double star (similarly to the construction of $S_I$ in page 35). Two stars $K_{1,n-1}$ have the central vertices $(n-1)_0, (n-1)_1$ connected by an edge of the mixed length $\ell_{01} = 0$. The endvertices attached to $(n-1)_i$ for $i \in \{0,1\}$ are $0_j, 1_j, \ldots, (n-2)_j$, where $j = 0$ for $i = 1$ and $j = 1$ for $i = 0$. The edges have the mixed lengths $\ell_{01} = 1, 2, \ldots, n-1, n+1, n+2, \ldots, 2n-1$. There is no edge of the length $\ell_{01} = n$ in $H_{01}$.

The smallest diameter is obtained if $H_0$ and $H_1$ are the stars $K_{1,n}$ with the central vertices $(n-1)_i$, $i \in \{0,1\}$, and endvertices $n_i, (n+1)_i, \ldots, (2n-1)_i$. Clearly, edges have all required pure lengths in both subgraphs. In this case $d_0 = d_1 = d_{01} = 1$ which gives $d = 3$.

Further we increase the diameter $d$ by converting one of the subgraphs $H_0$ or $H_1$ or finally both of them to a broom $B(n+1, t)$, where $1 \le t \le n-2$. We choose to start with the subgraph $H_0$.

- If $n+1-t = 2r$ the vertices of the path $P_{n+1-t}$ are $(2n-1)_0, (n-1)_0, (2n-2)_0, n_0, \ldots, (2n-r)_0, (n-2+r)_0$, and the star $K_{1,t}$ has the central vertex $(n-2+r)_0$ with attached endvertices $(n-1+r)_0, (n+r)_0, \ldots, (2n-r-1)_0$.
- If $n+1-t = 2r+1$ the vertices of the path $P_{n+1-t}$ are $(2n-1)_0, (n-1)_0, (2n-2)_0, n_0, \ldots, (n-2+r)_0, (2n-r-1)_0$, and the star $K_{1,t}$ has the central vertex $(2n-r-1)_0$ with attached endvertices $(n-1+r)_0, (n+r)_0, \ldots, (2n-r-2)_0$.

In both cases the edges have all required pure lengths $\ell_{00} = 1, 2, \ldots, n$. Each broom contributes by diameter $d_0 = n-t$. Because $d_1 = d_{01} = 1$ we obtain $d = n-t+2$, and for $1 \le t \le n-2$ is $4 \le d \le n+1$.

Further we apply the same procedure to the graph $H_1$ which so far remained to be the star $K_{1,n}$. Then $H_0$ contributes by the maximum value $d_0 = n-1$, $d_{01} = 1$, and $d_1 = n-t$. This yields $d = 2n-t$, and for $1 \le t \le n-2$ is $n+2 \le d \le 2n-1$.

By that we completed the construction of spanning trees with diameters from the interval $3 \le d \le 2n-1$. For an examples see Figure 5.7.

To show that our spanning trees have swapping labelings it remains to find an isomorphisms required by condition (2) of Definition 4.2. For the case

(1) each spanning tree $T$ is isomorphic to $G = T\setminus\{((n-1)_0,(2n-1)_0),((n-1)_1,(2n-1)_1)\}\cup\{((n-1)_0,(2n-1)_1),((n-1)_1,(2n-1)_0)\}$ by an isomorphism $\varphi : T \longrightarrow G$ such that $\varphi((2n-1)_0) = (2n-1)_1$, $\varphi((2n-1)_1) = (2n-1)_0$, and $\varphi(x_i) = x_i$ for any vertex $x_i \in V(T)$ different from $(2n-1)_0$ or $(2n-1)_1$.

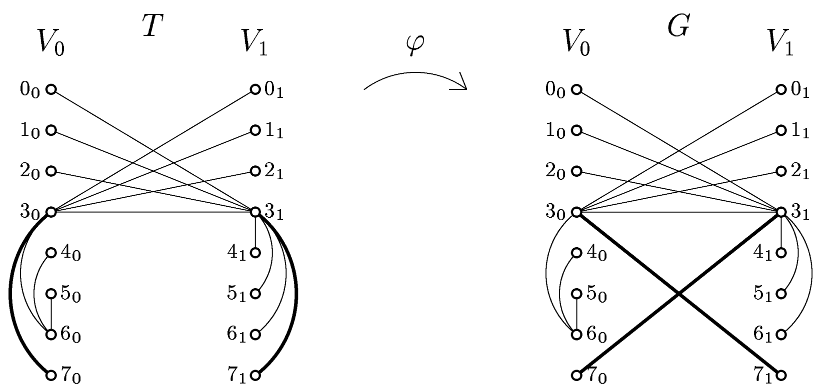



Figure 5.7: *Spanning tree of $K_{16}$ with the swapping labeling and diameter $d = 4$.*

(2) Stretching the subgraph $H_{01}$ (diameters: $2n \le d \le 4n-1$).

We start with the construction of the subgraphs $H_0, H_1$. $H_0$ is constructed as a broom $B(n+1,2)$.

- If $n-1 = 2r$ the vertices of the path $P_{n-1}$ are $(n-1)_0, (2n-1)_0, n_0, (2n-2)_0, \ldots, (n-2+r)_0, (2n-r)_0$, and the star $K_{1,2}$ has the central vertex $(2n-r)_0$ with attached endvertices $(2n-1-r)_0, (2n-2-r)_0$.
- If $n-1 = 2r+1$ the vertices of the path $P_{n-1}$ are $(n-1)_0, (2n-1)_0, n_0, (2n-2)_0, \ldots, (2n-r)_0, (n-1+r)_0$, and the star $K_{1,2}$ has the central vertex $(n-1+r)_0$ with attached endvertices $(n+r)_0, (n+r+1)_0$.

Again in both cases edges have the pure lengths $\ell_{00} = 1, 2, \ldots, n$. The subgraph $H_0$ contributes to the whole diameter $d$ of the spanning tree by $d_0 = \operatorname{diam}(P_{n-1}) + 1 = n-1$.

$H_1$ is the path $P_{n+1}$. For $n+1 = 2s$ the vertices of the path are $(n-1)_1, (2n-1)_1, n_1, (2n-2)_1, \ldots, (2n-s)_1$. For $n+1 = 2s+1$ the vertices of the path are $(n-1)_1, (2n-1)_1, n_1, (2n-2)_1, \ldots, (n-s+1)_1$. It is easy to check that the edges have the pure lengths $\ell_{11} = 1, 2, \ldots, n$, and the path contributes by the diameter $d_1 = n$.

In the first step we let the subgraph $H_{01}$ be the double star as given by the construction in case (1). Then $d_{01} = 1$, and the whole diameter of the spanning tree is $d = n - 1 + n + 1 = 2n$.

Further we increase the diameter $d_{01}$ by replacing bipartite double star by a graph similar to $C_I(D)$ (see page 36). It means that we will obtain only odd values of $d_{01}$ from the interval $3 \leq d_{01} \leq 2n - 1$. Let $d_{01} = 2r + 1$, where $1 \leq r \leq n - 1$.

For $r$ odd, $H_{01}$ has the diametrical bipartite path $P_{d_{01}+1} = (n-1)_0, 0_1, (n-2)_0, 1_1, \ldots, (n-1-\frac{r-1}{2})_0, (\frac{r-1}{2})_1, (\frac{r-1}{2})_0, (n-1-\frac{r-1}{2})_1, \ldots, 1_0, (n-2)_1, 0_0, (n-1)_1$,

for $r$ even, $P_{d_{01}+1} = (n-1)_0, 0_1, (n-2)_0, 1_1, \ldots, (\frac{r}{2}-1)_1, (n-1-\frac{r}{2})_0, (n-1-\frac{r}{2})_1, (\frac{r}{2}-1)_0, \ldots, 1_0, (n-2)_1, 0_0, (n-1)_1$. The edges of the path have the mixed lengths $\ell_{01} = n+1, n+2, \ldots, n+r, 0, n-r+1, n-r+2, \ldots, n-2, n-1$.

To obtain the edges of the missing lengths we add two stars $K_{1,n-1-r}$ with the central vertices on the path $P_{d_{01}+1}$. For $r$ odd, the central vertices are $\left(\frac{r-1}{2}\right)_i$, $i \in \{0,1\}$. The attached endvertices are $(\frac{r-1}{2}+1)_j, (\frac{r-1}{2}+2)_j, \ldots, (n-2-\frac{r-1}{2})_j$, where $j = 1$ if $i = 0$ and $j = 0$ if $i = 1$. For $r$ even, the central vertices are $\left(\frac{r}{2}-1\right)_i$, $i \in \{0,1\}$ with the endvertices in the opposite partite set $(\frac{r}{2})_j, (\frac{r}{2}+1)_j, \ldots, (n-2-\frac{r}{2})_j$, where $j = 1$ if $i = 0$ and $j = 0$ if $i = 1$. The edges have in both cases the missing mixed lengths $\ell_{01} = 1, 2, \ldots, n-r, n+r+1, n+r+2, \ldots, 2n-1$. There is no edge of the mixed length $\ell_{01} = n$.

By this construction we obtained the spanning trees with diameters $d = n - 1 + n + 2r + 1 = 2n + 2r$. Since $1 \leq r \leq n-1$ we have all even values of $d$ from the interval $2n + 2 \leq d \leq 4n - 2$. The odd values of $d$ are obtained when the subgraph $H_0$ is replaced by the path $P_{n+1}$ constructed in the same way as for the subgraph $H_1$. An example is shown in Figure 5.8.

Each spanning tree $T$ constructed in the case (2) is isomorphic to $G = T \setminus \{((n-1)_0, (2n-1)_0), ((n-1)_1, (2n-1)_1)\} \cup \{((n-1)_0, (2n-1)_1), ((n-1)_1, (2n-1)_0)\}$ by the isomorphism $\varphi : T \longrightarrow G$, such that $\varphi(x_i) = x_i$ if $x \in \{n, n+1, \ldots, 2n-1\}$ and $i \in \{0,1\}$, $\varphi(x_0) = x_1$ and $\varphi(x_1) = x_0$ if $x \in \{0, 1, \ldots, n-1\}$.

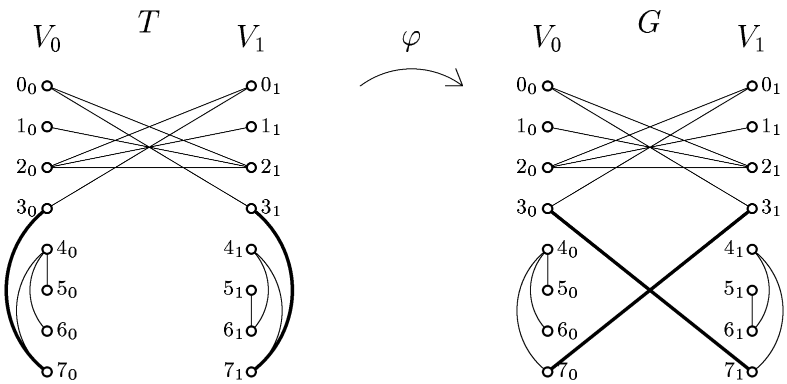



Figure 5.8: *Spanning tree of $K_{16}$ with the swapping labeling and diameter $d = 12$.*

We have found swapping labelings of spanning trees on $4n$ vertices with diameters $3 \leq d \leq 4n-1$. Since a swapping labeling of a $T$ guarantees the existence of a $T$-factorization of $K_{4n}$ (Theorem 4.3), our proof is complete. □

Finally we can conclude this section by the following statement, which is just the direct consequence of Theorem 5.1 and Theorem 5.3.

**Theorem 5.4** *For any integer $d$, such that $3 \leq d \leq 2n-1$ the complete graph $K_{2n}$ can be factorized into $n$ isomorphic copies of a spanning tree with diameter $d$.*

# Chapter 6

# Caterpillars

This chapter is devoted to the problem of isomorphic factorizations of $K_{2n}$ into caterpillars with diameter 4. By a *caterpillar* we mean a tree such that by deleting of all vertices of degree one we obtain a path $P$. (We consider one isolated vertex to be a path $P_1$ of length 0.) The path $P$ is called the *spine* of the caterpillar.

A caterpillar with the smallest diameter is a star, and we know by now that a factorization of any complete graph with more than 2 vertices into stars does not exist. If the diameter of a caterpillar on $2n$ vertices is 3, then the caterpillar is a double star. As was already mentioned, it is a well known fact that each complete graph $K_{2n}$ can be factorized into symmetric double stars [4]. If a double star is not symmetric, one of the central vertices of the double star has a degree larger than $n$, thus by Degree Condition 2.1 a factorization does not exist. Therefore the first interesting case is when the diameter of the caterpillar is 4.

Each caterpillar can be characterized by the degree sequence of the vertices of the spine. The spine of the caterpillar with $d = 4$ consists of three vertices and two edges. Further we will use the same notation as in [7] or [16]. We denote the endvertices of the spine by $A$ and $C$ and the central vertex by $b$. The two edges of the spine are then $(A, b)$ and $(b, C)$. By a $(d_1, d_2, d_3)$-caterpillar we denote the caterpillar of diameter 4, such that $\deg(A) = d_1$, $\deg(b) = d_2$, and $\deg(C) = d_3$. By a $[t_1, t_2, t_3]$-caterpillar where $t_1 \geq t_2 \geq t_3$ we specify the degrees of the vertices of the spine without determining their exact order on the spine.

Known necessary conditions for a $[t_1, t_2, t_3]$-caterpillar on $2n$ vertices to factorize $K_{2n}$ are the following. By Degree Condition the largest degree of a caterpillar is at most $n$, which implies $t_1 \leq n$. Moreover D. Fronček showed in [6] that the largest degree must be $n$, thus $t_1 = n$. Obviously the sum of the degrees of the vertices on the spine is always $t_1 + t_2 + t_3 = 2n + 1$. In combination with the previous condition we obtain $t_2+t_3 = n+1$. It was proved by P. Eldergill [4] that a $(d_1, 2, d_3)$-caterpillar does not factorize $K_{2n}$ for any $n$. He also showed that the $(2, 3, 2)$-caterpillar does not factorize $K_6$. This, together with the following theorem by M. Kubesa [17] is the complete classification of caterpillars with diameter 4 for factorization of $K_{2n}$ when $n$ is odd.

**Theorem 6.1** (Kubesa) *Let $n$ be an odd integer, $n \geq 5$. Let $R_{2n}$ be a caterpillar on $2n$ vertices with diameter 4. For any $R_{2n}$ which is a $(n, d_2, d_3)$-caterpillar with $3 \leq d_2 \leq n-1$, $d_2+d_3 = n+1$ or a $(d_1, n, d_3)$-caterpillar with $2 \leq d_1 \leq n-1$, $d_1 + d_3 = n + 1$ there is an $R_{2n}$-factorization of $K_{2n}$.*

In following two sections we complement this results in order to obtain the classification of caterpillars with diameter 4 for factorization of $K_{2n}$ when $n$ is even. We split the problem into two cases according to the number of vertices of the complete graph. Firstly for $K_{2n}$, where $n$ is even but not a power of two, we use the method of factorization based on fixing labelings. Secondly for $K_{2n}$, where $n$ is a power of two, we use swapping labelings.

## 6.1 Caterpillars on $2^q k$ vertices

In the following four lemmas we give constructions of fixing labelings for caterpillars $R_{2nk}$ on $2nk$ vertices with $d = 4$, such that $2nk = 2^q k$, where $q, k > 1$. Then the number of vertices $2^q k$ is a multiple of 4, but different from a power of 2. For each graph with a fixing labeling there must exist an underlying tree $T$ with a $\rho$-symmetric graceful labeling. Further in our constructions we use a symmetric double star $S_{2n}$ on $2n$ vertices, where $n > 1$.

### Symmetric graceful labeling of a double star $S_{2n}$

To obtain a symmetric graceful labeling of $S_{2n}$, we assign labels 0 and $n$ to the central vertices of the stars $K_{1,n-1}$, which are then connected by the symmetric edge $(0, n)$ of the maximum length $n$. The labels of the vertices of degree 1 joined to the central vertex 0 are $1, 2, \ldots, n-1$. Thus the edges have the lengths $1, 2, \ldots, n-1$. The labels of the vertices of degree 1 joined to the central vertex $n$ are $n+1, n+2, \ldots, 2n-1$, and the edges again have the lengths $1, 2, \ldots, n-1$.

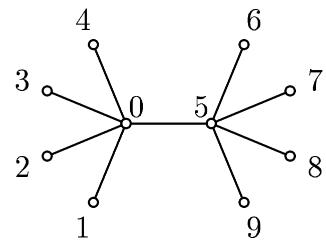



Figure 6.1: *Symmetric graceful labeling of* $S_{10}$.

The constructions differ when one of the endvertices of the spine has the largest possible degree, we can assume $\deg(A) = \Delta(R_{2nk}) = nk$, from the constructions when the central vertex of the spine has the largest possible degree $\deg(b) = \Delta(R_{2nk}) = nk$. We start with the case when $\deg(A) = nk$.

Every $(nk, m, nk+1-m)$-caterpillar, where $3 \leq m \leq k-1$ can be reduced by "cutting off" $2(n-1)k$ vertices of degree one to a $(k, m, k+1-m)$-caterpillar on $2k$ vertices. Kubesa in [17] gives constructions of blended $\rho$-labelings for $(k, m, k+1-m)$-caterpillars, where $k > 1$ and odd, and $3 \leq m \leq k-1$. Kubesa's constructions can be easily extended to the constructions of $2n$-cyclic labelings of $(nk, m, nk+1-m)$-caterpillars. Just to recall, a $2n$-cyclic labeling is also a fixing labeling with empty fixed set $V_F = \emptyset$. Based on Kubesa's results we prove the following lemma.

**Lemma 6.2** *Let* $2nk = 2^q k$*, where* $q, k > 1$ *and* $k$ *is odd. Then every* $(nk, m, nk+1-m)$*-caterpillar, where* $3 \leq m \leq k-1$*, has a* $2n$*-cyclic blended labeling.*

*Proof.* As the underlying tree we consider $S_{2n}$ with the symmetric graceful labeling given above. Let $R_{2nk}$ be an $(nk, m, nk+1-m)$-caterpillar, such that $n = 2^{q-1}$, where $q, k > 1$, and $k$ is odd, with the vertex set $V_i = \{0_i, 1_i, 2_i, \ldots, (k-1)_i\}$, for $i = 0, 1, \ldots, 2n-1$. Let $R_{2k}$ be a $(k, m, k+1-m)$-caterpillar, where $k > 1$ and odd, and $3 \leq m \leq k-1$, with the vertex set $V_i = \{0_i, 1_i, 2_i, \ldots, (k-1)_i\}$, for $i = 0$ and $n$.

We construct a $(k, m, k+1-m)$-caterpillar with a blended $\rho$-labeling on the vertices of the partite sets $V_0$ and $V_n$ according to the construction given by Kubesa. (In Kubesa's construction the partite sets are denoted by $V_0$ and $V_1$.)

To have a $2n$-cyclic labeling of $R_{2nk}$ we need to add bipartite subgraphs $H_{i0}$ and $H_{jn}$ for $i = 1, 2, \ldots, n-1$ and $j = n+1, n+2, \ldots, 2n-1$ with bipartite $\rho$-labelings. We construct each subgraph $H_{i0}$ as the star $K_{1,k}$. The central vertex is the vertex $A$ on the spine of $R_{2k}$. Each of the subgraphs $H_{jn}$ is again the star $K_{1,k}$ with the central vertex $C$ on the spine of $R_{2k}$. This is possible since in Kubesa's constructions the vertices $A$ and $C$ of the spine of a caterpillar $R_{2k}$ are always in different partite sets. Then $H_{i0}$ and $H_{jn}$ have exactly $k$ mixed edges of all different mixed lengths $\ell_{0i}$ and $\ell_{nj}$, respectively. In this way $(n-1)k$ vertices are connected to the vertex $A$ and another $(n-1)k$ vertices are connected to the vertex $C$ of $R_{2k}$, thus we have obtained a caterpillar $R_{2nk}$ with $2n$-cyclic labeling. Examples are shown in Figure 6.3. □

**Lemma 6.3** *Let $2nk = 2^q k$, where $q, k > 1$ and $k$ is odd. Then every $(nk, m, nk+1-m)$-caterpillar, where $k \leq m \leq nk-1$, has a fixing blended labeling.*

*Proof.* Again the underlying tree is $S_{2n}$ with given symmetric graceful labeling. Let $R_{2nk}$ be an $(nk, m, nk+1-m)$-caterpillar, such that $n = 2^{q-1}$, $q, k > 1$ and $k$ is odd, with the vertex set $V(R_{2nk}) = \bigcup_{i=0}^{n} V_i$, where $V_i = \{0_i, 1_i, 2_i, \ldots, (k-1)_i\}$, for $i = 0, 1, \ldots, 2n-1$. We set $k = 2h+1$. Let the vertices of the spine have the labels:

$$A = 0_n,\ b = 0_0,\ \text{and}\ C = (2h)_0.$$

We start with the construction of a $2n$-cyclic labeling for the case when $m = k$. The labeling will then be easily transformed to fixing labelings for all remaining cases $k < m \leq nk-1$.

Then the $(nk, k, nk+1-k)$-caterpillar consists of

(i) Subgraphs $H_0$ and $H_n$ with $\rho$-labelings. There are pure edges $(0_0, (2h)_0)$, $(0_0, (2h-1)_0), \ldots, (0_0, (h+1)_0)$ of all the lengths $\ell_{00} = 1, 2, \ldots, h$ and pure edges $(0_n, (2h)_n), (0_n, (2h-1)_n), \ldots, (0_n, (h+1)_n)$ of all the lengths $\ell_{nn} = 1, 2, \ldots, h$.

(ii) Bipartite subgraph $H_{0n}$ with a bipartite $\rho$-labeling. $H_{0n}$ contains mixed edges $(0_0, 0_n), (0_0, 1_n), \ldots, (0_0, h_n)$ of the lengths $\ell_{0n} = 0, 1, 2, \ldots, h$, and mixed edges $(h_0, 0_n), ((h-1)_0, 0_n), \ldots, (1_0, 0_n)$ of the remaining lengths $\ell_{0n} = h+1, h+2, \ldots, 2h$.

(iii) Bipartite subgraphs $H_{i0}$ for $i = 1, 2, \ldots, n-1$, and $H_{jn}$ for $j = n+1, n+2, \ldots, 2n$, which again have bipartite $\rho$-labelings. Each of the subgraphs $H_{i0}$ is the star $K_{1,k}$ with the central vertex $(2h)_0$. Obviously in each of them there are exactly $k$ mixed edges of all different mixed lengths $\ell_{0i}$. Similarly, each of the subgraphs $H_{jn}$ is the star $K_{1,k}$ with the central vertex $0_n$ and $k$ mixed edges of different mixed lengths $\ell_{nj}$.

Now it suffices to change slightly the previous construction to obtain fixing labelings when $k < m \leq nk - 1$.

Let $i \in V_F$ for $i = 1, 2, \ldots, n-1$. Then an $(nk, m, nk+1-m)$-caterpillar consists of the same subgraphs $H_0$ and $H_n$ with $\rho$-labelings and the same bipartite subgraph $H_{0n}$ with a bipartite $\rho$-labeling as given in steps $(i)$ and $(ii)$ of the previous construction. Also each $H_{jn}$ for $j = n+1, n+2, \ldots, 2n$ is again the star $K_{1,k}$ with the central vertex $0_n$ as in step $(iii)$.

Each vertex of the $n$ vertices in the fixed partite sets $V_i$ can be connected arbitrarily to the vertex $b = 0_0$ or $C = 2h_0$ so that we obtain required degrees.

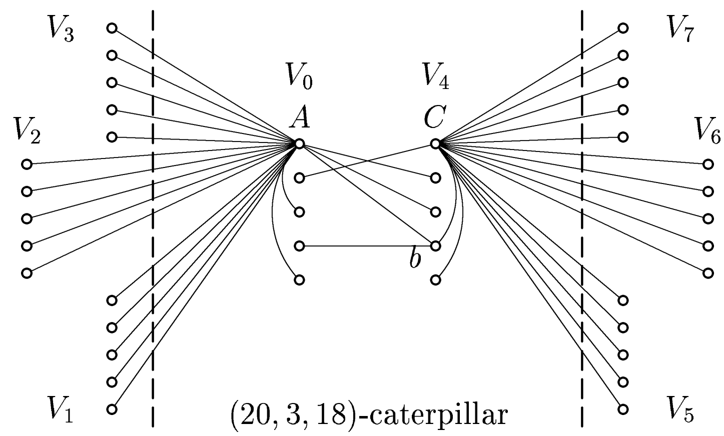
For example let $m - k = pk + r$, where $0 \leq r < k$. Since $k < m \leq nk - 1$, it holds that $0 \leq p \leq n - 2$. We connect all the vertices of the fixed partite sets $V_i$ for $i = 1, 2, \ldots, p$, and the vertices $0_{p+1}, 1_{p+1}, 2_{p+1}, \ldots, r_{p+1}$ of the fixed partite set $V_{p+1}$ to the vertex $0_0$. The remaining $nk - m$ vertices in the fixed partite sets are connected to the vertex $(2h)_0$. □

For an easier verification of the proofs we will demonstrate our constructions on an example. Particularly we will illustrate by figures labelings of all caterpillars with 40 vertices and $d = 4$, which are considered in Lemmas 6.2 and 6.3. For $R_{2nk} = R_{40}$ is $k = 5$ and $n = 4$. We deal now with the cases when $\deg(A) = \Delta(R_{40}) = 20$. An overview is given in Table 6.1.

| $R_{40}$ | type of the labeling | Figure | Lemma |
|---|---|---|---|
| (20,3,8) | 2n-cyclic labeling | 6.3 | 6.2 |
| (20,4,17) | 2n-cyclic labeling | 6.3 | 6.2 |
| (20,5,16) | 2n-cyclic labeling | 6.4 | 6.3 |
| (20,6,15) | fixing labeling | 6.4 | 6.3 |
| ⋮ | ⋮ | ⋮ | ⋮ |
| (20,19,2) | fixing labeling | 6.4 | 6.3 |

Table 6.1: *Caterpillars on* 40 *vertices with* $d = 4$ *and* $\deg(A) = 20$ *that factorize* $K_{40}$.

*By "cutting of" 30 vertices* $(5,3,3)$*-caterpillar is obtained with the blended* $\rho$*-labeling given by Kubesa.*




*By "cutting of" 30 vertices* $(5,4,2)$*-caterpillar is obtained with the blended* $\rho$*-labeling given by Kubesa.*

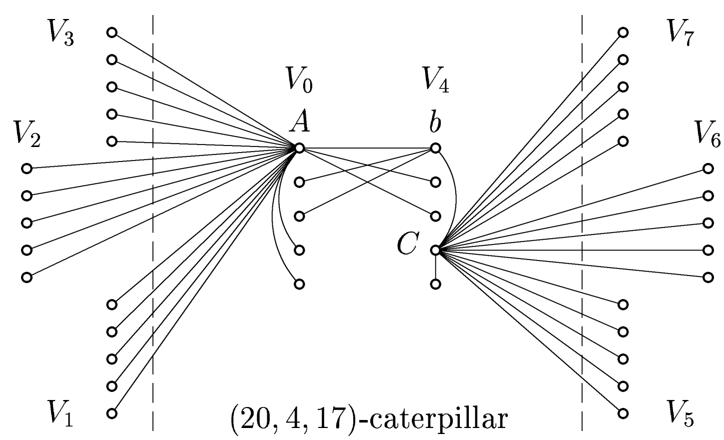



Figure 6.3: 8-*cyclic labelings of caterpillars on* 40 *vertices with* $\deg(A) = 20$ *and diameter* $d = 4$.

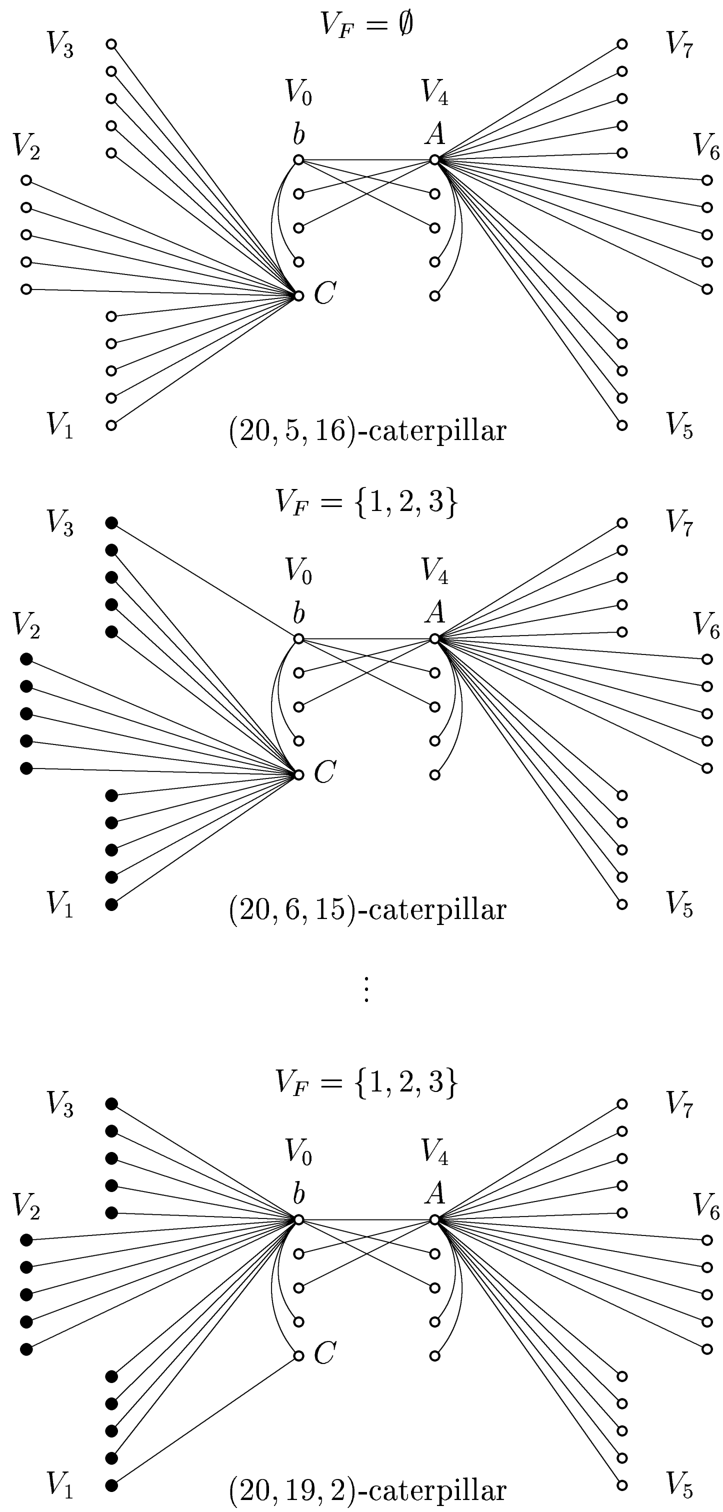



Figure 6.4: *Fixing labelings of caterpillars on* 40 *vertices with* $\deg(A) = 20$ *and diameter* $d = 4$.

Now we continue with the constructions of labelings for caterpillars $R_{2nk}$ where $\deg(b) = nk$. Without loss of generality we can assume that $\deg(A) \leq \deg(C)$. Also each $(m, nk, nk+1-m)$-caterpillar, where $2 \leq m < \frac{k+1}{2}$, can be reduced by "cutting off" $2(n-1)k$ vertices of degree one to an $(m, k, k+1-m)$-caterpillar on $2k$ vertices which allows a blended $\rho$-labeling as was proved by Kubesa [17]. Again we use Kubesa's constructions to find $2n$-cyclic labelings of $(m, nk, nk+1-m)$-caterpillars.

**Lemma 6.4** *Let $2nk = 2^q k$, where $q, k > 1$ and $k$ is odd. Then every $(m, nk, nk+1-m)$-caterpillar, where $2 \leq m < \frac{k+1}{2}$, has a $2n$-cyclic blended labeling.*

*Proof.* The underlying tree is $S_{2n}$ with the graceful symmetric labeling given on page 48. Let $R_{2nk}$ be an $(m, nk, nk+1-m)$-caterpillar, such that $n = 2^{q-1}$, where $q, k > 1$ and $k$ is odd. The vertex set is $V_i = \{0_i, 1_i, 2_i, \ldots, (k-1)_i\}$, for $i = 0, 1, \ldots, 2n-1$. Let $R_{2k}$ be an $(m, k, k+1-m)$-caterpillar, where $k > 1$ and odd, and $2 \leq m < \frac{k+1}{2}$, with the vertex set $V_i = \{0_i, 1_i, 2_i, \ldots, (k-1)_i\}$, for $i = 0$ and $n$.

We take a caterpillar $R_{2k}$ on the vertices of the partite sets $V_0$ and $V_n$ with a blended $\rho$-labeling given by Kubesa. (In Kubesa's notation partite sets are denoted by $V_0$ and $V_1$.)

To obtain a $2n$-cyclic labeling of $R_{2nk}$ we add bipartite subgraphs $H_{i0}$ and $H_{jn}$ for $i = 1, 2, \ldots, n-1$ and for $j = n+1, n+2, \ldots, 2n-1$ with bipartite $\rho$-labelings. Since in Kubesa's constructions the vertices $b$ and $C$ of the spine of the caterpillar $R_{2k}$ are in different partite sets, the following construction is possible. Each subgraph $H_{i0}$ is the star $K_{1,k}$ with the central vertex $C$ on the spine of $R_{2k}$. Each subgraph $H_{jn}$ is also the star $K_{1,k}$ with the central vertex $b$ of the spine of $R_{2k}$. Then each $H_{i0}$ and $H_{jn}$ has exactly $k$ mixed edges of all different mixed lengths $\ell_{0i}$ and $\ell_{nj}$, respectively.

In this way $(n-1)k$ vertices are connected to the vertex $b$ and another $(n-1)k$ vertices are connected to the vertex $C$ of $R_{2k}$, thus we obtained a caterpillar $R_{2nk}$ with a $2n$-cyclic labeling. Examples are shown in Figure 6.5. □

**Lemma 6.5** *Let $2nk = 2^q k$, where $q, k > 1$ and $k$ is odd. Then every $(m, nk, nk+1-m)$-caterpillar, where $\frac{k+1}{2} \leq m \leq \frac{nk}{2}$, has a fixing blended labeling.*

*Proof.* As the underlying tree we again consider $S_{2n}$ with the given symmetric graceful labeling (see page 48). Let $R_{2nk}$ be an $(m, nk, nk+1-m)$-caterpillar, such that $n = 2^{q-1}$, $q, k > 1$ and $k$ is odd, with the vertex set $V(R_{2nk}) = \bigcup_{i=0}^{n} V_i$, where $V_i = \{0_i, 1_i, 2_i, \ldots, (k-1)_i\}$, for $i = 0, 1, \ldots, 2n-1$. We set $k = 2h+1$. Suppose the vertices of the spine have the labels:

$$A = h_0,\ b = 0_n \text{ and } C = 0_0.$$

First we construct a $2n$-cyclic labeling for the case when $m = \frac{k+1}{2}$. Similarly to the proof of Lemma 6.3 we transform this labeling to fixing labelings for all remaining cases $\frac{k+1}{2} < m \leq \frac{nk}{2}$.

Then the $(\frac{k+1}{2}, nk, nk+1-\frac{k+1}{2})$-caterpillar consists of

(i) Subgraphs $H_0$ and $H_n$ with $\rho$-labelings. There are pure edges $(h_0, (h+1)_0)$, $(h_0, (h+2)_0), \ldots, (h_0, (2h)_0)$ of all the lengths $\ell_{00} = 1, 2, \ldots, h$ and pure edges $(0_n, (2h)_n), (0_n, (2h-1)_n), \ldots, (0_n, (h+1)_n)$ of all the lengths $\ell_{nn} = 1, 2, \ldots, h$.

(ii) Bipartite subgraph $H_{0n}$ with a bipartite $\rho$-labeling. $H_{0n}$ contains mixed edges $(0_0, 0_n), (0_0, 1_n), \ldots, (0_0, h_n)$ of the lengths $\ell_{0n} = 0, 1, 2, \ldots, h$, and mixed edges $(h_0, 0_n), ((h-1)_0, 0_n), \ldots, (1_0, 0_n)$ of the remaining lengths $\ell_{0n} = h+1, h+2, \ldots, 2h$.

(iii) Bipartite subgraphs $H_{i0}$ for $i = 1, 2, \ldots, n-1$, and $H_{jn}$ for $j = n+1, n+2, \ldots, 2n$, with bipartite $\rho$-labelings. Each of the subgraphs $H_{i0}$ is the star $K_{1,k}$ with the central vertex $0_0$. Also each of the subgraphs $H_{jn}$ is the star $K_{1,k}$ with the central vertex $0_n$. It is obvious that in each of them are $k$ mixed edges of all different mixed lengths $\ell_{0i}$ or $\ell_{nj}$, respectively.

Now we change slightly the previous construction to obtain fixing labelings when $\frac{k+1}{2} < m \leq \frac{nk}{2}$.

We choose the fixed set $V_F$ as follows. Let $i \in V_F$ for $i = 1, 2, \ldots, n-1$. Then an $(m, nk, nk+1-m)$-caterpillar consists of the same subgraphs $H_0$ and $H_n$ with $\rho$-labelings and the same bipartite subgraph $H_{0n}$ with a bipartite $\rho$-labeling as given in steps $(i)$ and $(ii)$ of the previous construction. Also, each $H_{jn}$ for $j = n+1, n+2, \ldots, 2n$ is again the star $K_{1,k}$ with the central vertex $0_n$ as in step $(iii)$.

We reconnect some of the vertices in the fixed partite sets $V_i$ to the vertex $A = h_0$ so that the required degrees of the vertices $A$ and $C$ are obtained. One of many possibilities how to do that is the following.

Let $m - \frac{k+1}{2} = pk + r$, where $0 \le r < k$. Since $\frac{k+1}{2} \le m \le \frac{nk}{2}$ is $0 \le p \le \frac{n}{2} - 1$. We connect to the vertex $A = h_0$ all the vertices of the fixed partite sets $V_i$ for $i = 1, 2, \ldots, p$ and also the vertices $0_{p+1}, 1_{p+1}, 2_{p+1}, \ldots, r_{p+1}$ of the fixed partite set $V_{p+1}$. Remaining $nk - \frac{k-1}{2} - m$ vertices in fixed partite sets are connected to the vertex $C = 0_0$. □

In Table 6.2 we consider caterpillars $R_{40}$ with $\deg(b) = 20$. Their labelings are given in proofs of Lemmas 6.4 and 6.5.

| $R_{40}$ | type of the labeling | Figure | Lemma |
|---|---|---|---|
| (2,20,19) | 2n-cyclic labeling | 6.5 | 6.4 |
| (3,20,18) | 2n-cyclic labeling | 6.6 | 6.5 |
| (4,20,17) | fixing labeling | 6.6 | 6.5 |
| ⋮ | ⋮ | ⋮ | ⋮ |
| (10,20,11) | fixing labeling | 6.6 | 6.5 |

Table 6.2: *Caterpillars on* 40 *vertices with* $d = 4$ *and* $\deg(b) = 20$ *that factorize* $K_{40}$.

*By "cutting of" 30 vertices is obtained* $(2,5,4)$*-caterpillar with the blended* $\rho$*-labeling given by Kubesa.*

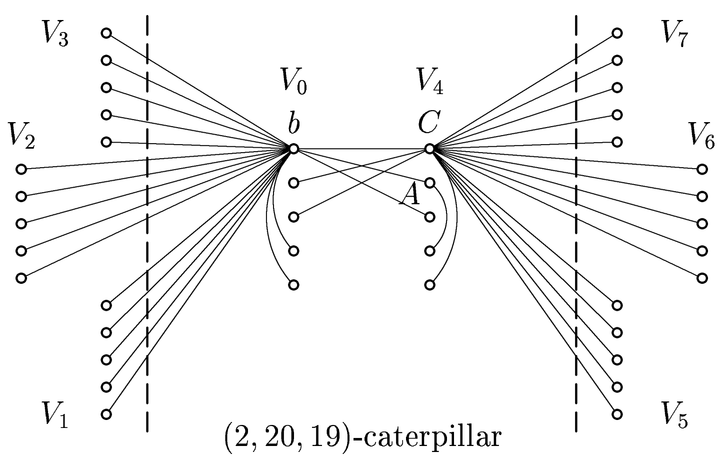



Figure 6.5: 8*-cyclic labeling of caterpillar on* 40 *vertices with* $\deg(b) = 20$ *and diameter* $d = 4$.

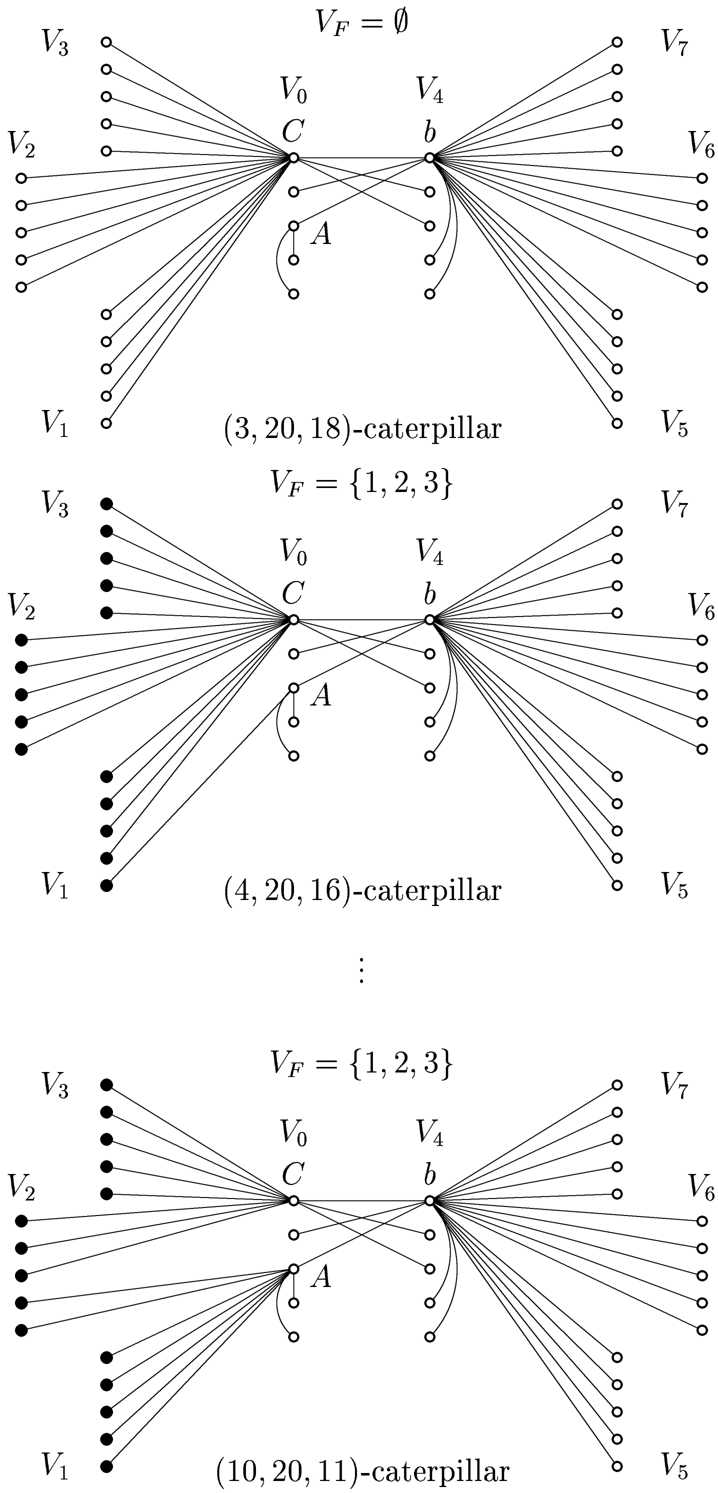



Figure 6.6: *Fixing labelings of caterpillars on* 40 *vertices with* $\deg(b) = 20$ *and diameter* $d = 4$.

Based on previous four lemmas we can state the following theorem.

**Theorem 6.6** *Let $2nk = 2^q k$, where $q, k > 1$ and $k$ is odd. Let $R_{2nk}$ be a caterpillar with diameter 4. There is an $R_{2nk}$-factorization of $K_{2nk}$ if and only if $R_{2nk}$ is an $(nk, d_2, d_3)$-caterpillar with $3 \leq d_2 \leq nk - 1$, $d_2 + d_3 = nk + 1$ or a $(d_1, nk, d_3)$-caterpillar with $2 \leq d_1 \leq nk - 1$, $d_1 + d_3 = nk + 1$.*

*Proof.* The theorem gives the necessary and sufficient condition for the existence of an $R_{2nk}$-factorization of $K_{2nk}$ when $n = 2^q$, $q, k > 1$ and $k$ is odd. The sufficiency is a direct consequence of Theorem 3.12 and Lemmas 6.2, 6.3, 6.4, and 6.5, where fixing labelings of $R_{2nk}$ are given. The necessity follows from already mentioned results by Eldergill [4] and Frônček [6]. □

## 6.2 Caterpillars on $2^q$ vertices

In this section we focus on the case of factorization of $K_{2n}$ into caterpillars with diameter four which is not covered by any of the previously mentioned results. It is the case when the number of vertices of $K_{2n}$ is a power of two, thus $2n = 2^q$ for $q > 2$. We do not consider $K_4$ since the diameter of a spanning tree on 4 vertices is at most 3.

We set $2n = 4k$ and show that there exist factorizations of $K_{4k}$, where $4k = 2^q$ using the method based on swapping labelings. The constructions of labelings will again differ depending on which of the vertices on the spine of $R_{4k}$ has the largest degree $\Delta(R_{4k}) = 2k$. In the proofs we always consider a caterpillar $R_{4k}$, where $4k = 2^q$, $q > 2$, with the vertex $V(R_{4k}) = V_0 \cup V_1$, $V_0 \cup V_1 = \emptyset$, and $V_i = \{0_i, 1_i, \ldots, (2k-1)_i\}$ for $i \in \{0, 1\}$. To find a swapping labeling of $R_{4k}$, we give labelings of the subgraphs $H_0$, $H_1$ and the bipartite subgraph $H_{01}$ separately and then we show that there exists also the required isomorphism $\varphi$ (see Definition 4.2).

**Lemma 6.7** *Let $4k = 2^q$, where $q > 2$. Then every $(2k, m, 2k - m + 1)$-caterpillar, for $3 \leq m \leq 2k - 1$, has a swapping blended labeling.*

*Proof.* Let $R_{4k}$ be a $(2k, m, 2k - m + 1)$-caterpillar, where $k = 2^{q-2}$ and $q > 2$. We split the proof of the lemma into two cases.

*Case 1* For $3 \leq m \leq k + 1$.
We let the vertices of the spine of $R_{4k}$ have the labels:

$$A = 0_0,\ b = (k + m - 2)_1,\ \text{and}\ C = (k)_1.$$

$H_0$ – has the pure edges $(0_0,(2k-1)_0),(0_0,(2k-2)_0),\ldots,(0_0,(2k-m+2)_0)$ of the pure lengths $\ell_{00}=1,2,\ldots,m-2$ and the pure edges $(0_0,(m-1)_0),(0_0,m_0),\ldots,(0_0,k_0)$ of the pure lengths $\ell_{00}=m-1,m,\ldots,k$.

$H_1$ – has the pure edges $(k_1,(k+1)_1),(k_1,(k+2)_1,\ldots,(k_1,(2k-1)_1))$ and the pure edge $(0_1,k_1)$ of all the required pure lengths $\ell_{11}=1,2,\ldots,k$.

$H_{01}$ – has the mixed edges $(0_0,1_1),(0_0,2_1),\ldots,(0_0,(k-1)_1)$ of the mixed lengths $\ell_{01}=1,2,\ldots,k-1$ and the mixed edges $((m-2)_0,(k+m-2)_1),((m-3)_0,(k+m-2)_1),\ldots,(0_0,(k+m-2)_1)$ of the mixed lengths $\ell_{01}=k,k+1,\ldots,k+m-2$. Further, for $m\neq k+1$, there are the mixed edges $((2k-m+1)_0,k_1),((2k-m)_0,k_1),\ldots,((k+1)_0,k_1)$ of the mixed lengths $\ell_{01}=k+m-1,k+m,\ldots,2k-1$. There is no edge of the mixed length $\ell_{01}=0$.

*Case 2* For $k+2\leq m\leq 2k-1$.
The vertices of the spine of $R_{4k}$ are assigned the labels:

$$A=0_0,\ b=(k)_1,\ \text{and}\ C=(m-k-1)_1.$$

$H_0$ – has the pure edges $(0_0,k_0),(0_0,(k+1)_0),\ldots,(0_0,(2k-1)_0)$ of all the required pure lengths $\ell_{00}=k,k-1,\ldots,1$.

$H_1$ – has the pure edges $(k_1,0_1),(k_1,1_1),\ldots,(k_1,(m-k-1)_1)$ of the pure lengths $\ell_{11}=k,k-1,\ldots,2k-m+1$ and the pure edges $((k-1)_1,(m-k-1)_1),((k-2)_1,(m-k-1)_1),\ldots,((m-k)_1,(m-k-1)_1$ of the pure lengths $\ell_{11}=2k-m,2k-m-1,\ldots,1$.

$H_{01}$ – has the mixed edges $(0_0,k_1),(0_0,(k+1)_1),\ldots,(0_0,(2k-1)_1)$ of the mixed lengths $\ell_{01}=n,n+1,\ldots,2n-1$ and the mixed edges $(1_0,k_1),(2_0,k_1),\ldots,((k-1)_0,k_1)$ with the remaining mixed lengths $\ell_{01}=k-1,k-2,\ldots,1$. There is again no edge of the mixed length $\ell_{01}=0$.

Each caterpillar $R_{4k}$ constructed in *Case 1* or *Case 2* is isomorphic to the caterpillar $G=R_{4k}\setminus\{(0_0,k_0),(0_1,k_1)\}\cup\{(0_0,0_1),(k_0,k_1)\}$ by the isomorphism $\varphi:R_{4k}\longrightarrow G$, such that $\varphi(k_0)=0_1,\varphi(0_1)=k_0$, and $\varphi(x_r)=x_r$ for any $x_r\in V(R_{4k})-\{k_0,0_1\}$, $r\in\{0,1\}$. It means that we are adding two mixed edges, $(0_0,0_1),(k_0,k_1)$, of the missing mixed length $\ell_{01}=0$ to the last $k$ factors $G_k,G_{k+1},\ldots,G_{2k-1}$, while the pure edges $(0_0,k_0),(0_1,k_1)$ of the maximum pure length $\ell_{00}=\ell_{11}=k$ are omitted. □

**Lemma 6.8** *Let $4k = 2^q$, where $q > 2$. Then every $(m, 2k, 2k - m + 1)$-caterpillar, for $2 \leq m \leq k$, has a swapping blended labeling.*

*Proof.* Let $R_{4k}$ be an $(m, 2k, 2k - m + 1)$-caterpillar, where $k = 2^{q-2}$ and $q > 2$. We again split constructions of the labelings into two cases.

*Case 1* For $2 \leq m \leq k - 1$.
To the vertices of the spine of $R_{4k}$ we assign the labels:

$$A = (k - m)_1,\ b = 0_0,\ \text{and } C = (k)_1.$$

$H_0$ – has the pure edges $(0_0, 1_0), (0_0, 2_0), \dots, (0_0, k_0)$ of the pure lengths $\ell_{00} = 1, 2, \dots, k$.

$H_1$ – has the pure edges $(k_1, (k+1)_1), (k_1, (k+2)_1), \dots, (k_1, (2k - m)_1)$ of the pure lengths $\ell_{11} = 1, 2, \dots, k - m$, the pure edges $((k-m)_1, (2k-1)_1), ((k-m)_1, (2k-2)_1), \dots, (k-m)_1, (2k-m+1)_1$ of the pure lengths $\ell_{11} = k - m + 1, k - m + 2, \dots, k - 1$, and finally the pure edge $(0_1, k_1)$ of the remaining length $\ell_{11} = k$.

$H_{01}$ – has the mixed edges $(0_0, 1_1), (0_0, 2_1), \dots, (0_0, k_1)$ of the mixed lengths $\ell_{01} = 1, 2, \dots, k$ and the mixed edges $((2k-1)_0, k_1), ((2k-2)_0, k_1), \dots, ((k+1)_0, k_1)$ of the mixed lengths $\ell_{01} = k + 1, k + 2, \dots, 2k - 1$. The edge of the mixed length $\ell_{01} = 0$ is missing.

*Case 2* For $m = k$.
The vertices of the spine of $R_{4k}$ have the labels:

$$A = (k - m)_1,\ b = 0_0,\ \text{and } C = (k)_1.$$

$H_0$ – has the pure edges $(0_0, (2k-1)_0), (0_0, 2_0), (0_0, 3_0), \dots, (0_0, k_0)$ of the pure lengths $\ell_{00} = 1, 2, 3, \dots, k$.

$H_1$ – has the pure edges $(k_1, (k-1)_1), ((2k-1)_1, 1_1), ((2k-1)_1, 2_1), \dots, ((2k-1)_1, (k-2)_1)$ of the lengths $\ell_{11} = 1, 2, 3, \dots, k - 1$ and the pure edge $(0_1, k_1)$ of the remaining length $\ell_{11} = k$.

$H_{01}$ – has the mixed edges $(1_0, k_1), (0_0, k_1), (0_0, (k+1)_1) \dots, (0_0, (2k-1)_1)$ of the lengths $\ell_{01} = k - 1, k, k + 1, \dots, 2k - 1$ and the mixed edges $((k+1)_0, (2k-1)_1), ((k+2)_0, (2k-1)_1), \dots, ((2k-2)_0, (2k-1)_1)$ of the lengths $\ell_{01} = k - 2, k - 3, \dots, 1$. Again the edge of the mixed length $\ell_{01} = 0$ is missing.

Similarly to the proof of the previous lemma each caterpillar $R_{4k}$ constructed in *Case 1* or *Case 2* is isomorphic to the caterpillar $G = R_{4k} \setminus \{(0_0, k_0), (0_1, k_1)\} \cup \{(0_0, 0_1), (k_0, k_1)\}$ by the isomorphism $\varphi : R_{4k} \longrightarrow G$, such that $\varphi(k_0) = 0_1$,$\varphi(0_1) = k_0$, and $\varphi(x_r) = x_r$ for any $x_r \in V(R_{4k}) - \{k_0, 0_1\}$, $r \in \{0, 1\}$.
□

As an example of the constructions given in the proofs of Lemmas 6.7 and 6.8 we show the swapping labelings of all caterpillars $R_{16}$ with diameter four for which there exist factorizations of $K_{16}$. An overview is given by Table 6.3.

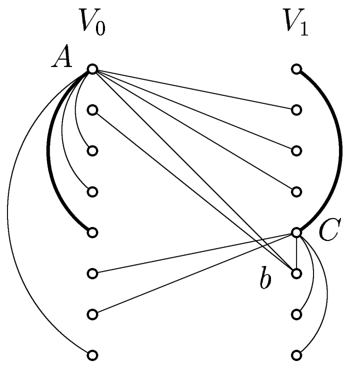
| $R_{16}$ | Figure | Lemma | Case |
|---|---|---|---|
| (8,3,6) | 6.7 | 6.7 | 1 |
| (8,4,5) | 6.7 | 6.7 | 1 |
| (8,5,4) | 6.7 | 6.7 | 1 |
| (8,6,3) | 6.7 | 6.7 | 2 |
| (8,7,2) | 6.7 | 6.7 | 2 |
| (2,8,7) | 6.8 | 6.8 | 1 |
| (3,8,6) | 6.8 | 6.8 | 1 |
| (4,8,5) | 6.8 | 6.8 | 2 |

Table 6.3: *Caterpillars on* 16 *vertices with* $d = 4$ *that factorize* $K_{16}$.

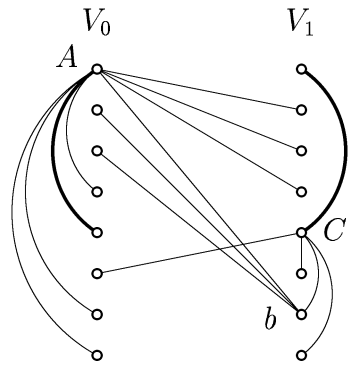
Previous two lemmas allow us to state the following theorem.

**Theorem 6.9** *Let $R_{4k}$, where $4k = 2^q$, $q > 2$, be a caterpillar with diameter 4. There is an $R_{4k}$-factorization of $K_{4k}$ if and only if $R_{4k}$ is an $(2k, d_2, d_3)$-caterpillar with $3 \leq d_2 \leq 2k-1$, $d_2 + d_3 = 2k+1$, or a $(d_1, 2k, d_3)$-caterpillar with $2 \leq d_1 \leq 2k-1$, $d_1 + d_3 = 2k+1$.*

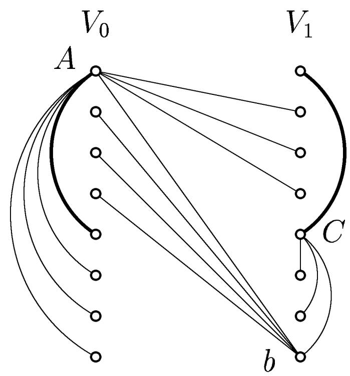
*Proof.* The sufficiency of the conditions for the existence of an $R_{4k}$-factorization of $K_{4k}$ when $4k = 2^q$, $q > 2$ is a direct consequence of Theorem 4.3 and Lemmas 6.7 and 6.8 where swapping labelings for $R_{4k}$ are given. The necessity follows from the results by Eldergil [4] and Frončo [6]. □



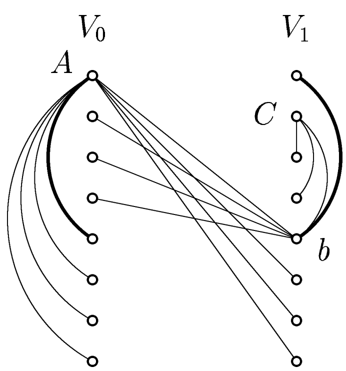

(8, 3, 6)-caterpillar



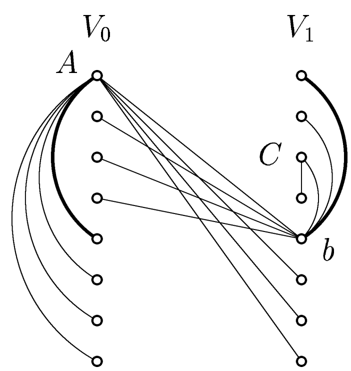

(8, 4, 5)-caterpillar




(8, 5, 4)-caterpillar

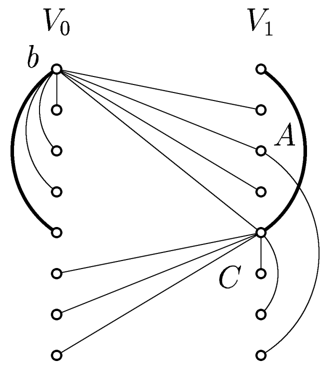



(8, 6, 3)-caterpillar

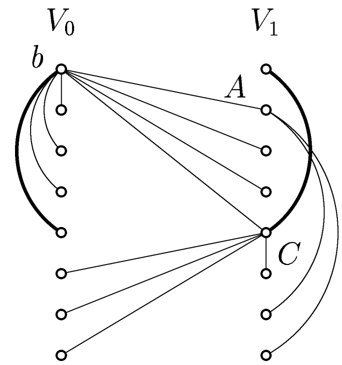



(8, 7, 2)-caterpillar

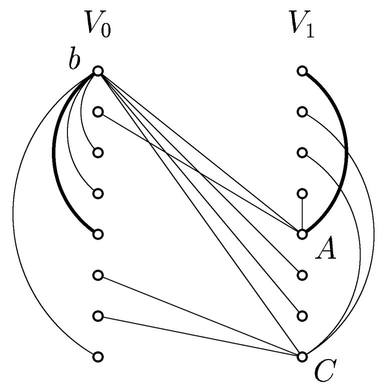
Figure 6.7: *Swapping labelings of caterpillars on* 16 *vertices with* $\deg(A) = 8$ *and diameter* $d = 4$.




(2, 8, 7)-caterpillar




(3, 8, 6)-caterpillar




(4, 8, 5)-caterpillar

Figure 6.8: *Swapping labelings of caterpillars on* 16 *vertices with* $\deg(b) = 8$ *and diameter* $d = 4$.

Finally we summarize the results on factorizations of $K_{2n}$ into caterpillars with diameter 4.

**Theorem 6.10** *Let $R_{2n}$ be a caterpillar on $2n$ vertices with diameter 4, where $n$ is an integer, $n > 2$. There exists an $R_{2n}$-factorization of $K_{2n}$ if and only if $\Delta(R_{2n}) = n$ and $R_{2n}$ is not a $(d_1, 2, d_3)$-caterpillar or the $(2, 3, 2)$-caterpillar.*

*Proof.* Follows from Theorems 6.1, 6.6, and 6.9, and the results obtained by Eldergil [4] and Fronček [6] for $R_{2n}$-factorizations of $K_{2n}$. □

# Chapter 7

# Conclusion

The objective of this thesis was to obtain results on isomorphic factorizations of $K_{4n}$ into spanning trees complementing the results known for spanning tree factorizations of $K_{4n+2}$. Particularly, our goal was to decide if for any given $d$, such that $3 \leq d \leq 4n-1$, there exist a factorization of $K_{4n}$ into a spanning tree with the diameter $d$ and to complete the classification of caterpillars with diameter 4 that factorize $K_{4n}$. We have shown that to solve these problems, it was necessary to find new methods for isomorphic decompositions of $K_{4n}$.

We have introduced two new methods, which together allowed us to solve the problems mentioned above. The first method can be used to find $G$-decompositions of a complete graph $K_{2nk}$, where $n, k > 1$ and $k$ is odd, into $nk$ copies of a graph $G$ with $2nk-1$ edges. The method is based on a new type of vertex labeling which we call the fixing blended labeling. As a special case of the fixing labeling we distinguish the $2n$-cyclic blended labeling. The fixing labeling is a further generalization of the blended $\rho$-labeling introduced by D. Fronček as a tool for spanning tree factorizations of $K_{4n+2}$. We have used $2n$-cyclic labelings to show that there are factorizations of $K_{2nk}$ for $2n = 2^q$, where $q > 1$, into spanning trees with given diameter. Further we gave fixing labelings of all caterpillars on $2^q k$ vertices with diameter 4, which were admissible for factorization of $K_{2^q k}$. By admissible we mean caterpillars that are not excluded by the results of P. Eldergill [4] and D. Fronček [7].

The second method is suitable for a $G$-decomposition of $K_{4n}$ into $2n$ copies of a given graph $G$ with $4n-1$ edges. Again the method is based on a vertex labeling, which we call the swapping blended labeling. This method enabled us to find the factorizations also when the number of vertices of a complete graph is a power of 2, which is the only case not covered by the method based on the fixing labeling. We have answered the question about factorizations of $K_{4n}$ into spanning trees with a given diameter $d$ completely by giving the swapping labelings of trees on $4n$ vertices for $d$, where $3 \leq d \leq 4n-1$. We have completed the classification of caterpillars with diameter 4 by giving swapping labelings of all admissible caterpillars on $2^q$ vertices, where $q > 2$. Our results together with the results due to P. Eldergill [4], D. Fronček [7], and M. Kubesa [17] give the complete answer about factorizations of $K_{2n}$ into caterpillars with diameter 4.

As was already mentioned D. Fronček [7] and M. Kubesa [16] attempted to give similar classification of caterpillars on $4n+2$ vertices with diameter 5, but for certain subcases the problem remains unsolved. This naturally suggests the direction of further investigations on spanning tree factorizations of $K_{4n}$. The methods introduced in this thesis seem to be promising to obtain complete results also for this problem, in fact we have already found fixing labelings of certain subclass of caterpillars on $2^q k$ vertices with diameter 5.

In [8] we have shown that there are factorizations of $K_{2nk}$ into trees which belong to a special subclass of lobsters with diameter $d = 4$ by constructing $2n$-cyclic labelings of these graphs. With the fixing labeling available it is likely to extend the result for a wider subclass of lobsters with $d = 4$. In [18] M. Kubesa gave blended labelings of another special class of lobsters with $d = 4$. Any tree with diameter 4 is either a lobster or a caterpillar or the path $P_4$. Therefore, if we decide the existence of the factorization of $K_{2n}$ into copies of any lobster with $d = 4$, we obtain the complete classification of all trees with $d = 4$ for factorization of $K_{2n}$. Unfortunately, it seems that solving such a problem even for factorizations of $K_{4n+2}$ is rather difficult.

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