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Scalable Total BETI for 2D and 3D Contact Problems

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Abstract

The thesis focuses on the solution of both coercive and semi–coercive contact problems by using the Boundary Element Tearing and Interconnecting (BETI) method, which represents a boundary element counterpart of the Finite Element Tearing and Interconnecting (FETI) method. The BETI approach, which uses "tearing" the domain into non–overlapping subdomains and subsequent "gluing" along the artificial interfaces by Lagrange multipliers, is based on the symmetric discretization of the local Steklov–Poincaré operator and its suitable boundary element approximation. We combine BETI with the preconditioning by the projectors to the so–called natural coarse grid and apply recently proposed optimal algorithms for the solution of bound and equality constrained quadratic programming problems in order to develop a theoretically supported scalable solver for elliptic boundary variational inequalities.

In the following text, we cover particularly the application of BETI to two chosen model contact problems. The first one is a 2D semi–coercive multibody contact problem described by the Laplace operator and the second one is a 3D coercive contact problem of linear homogeneous isotropic elastostatics. The theoretical results are validated by the numerical experiments which demonstrate the scalability of the presented method. Finally, we discuss our first results obtained for 3D Hertz problem.

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Chapter 1

Introduction

The Boundary Element Tearing and Interconnecting (BETI) method was originally introduced by Langer and Steinbach [32] as a boundary element counterpart of the Finite Element Tearing and Interconnecting (FETI) domain decomposition method proposed by Farhat and Roux [28, 29] for parallel solution of linear problems described by elliptic partial differential equations. Both methods are based on decomposition of the original domain into non–overlapping subdomains and application of the duality. The continuity of the solution across the subdomain interfaces is then enforced by Lagrange multipliers and the primal problem is reduced to a small, relatively well–conditioned, typically equality constrained quadratic programming problem. An important feature of this approach is that the solution of the system of such subproblems may be efficiently parallelized.

Generally, the main idea behind non–overlapping domain decomposition methods is splitting the spatial domain into smaller ones, overlapping only on their interfaces, and then, instead of the large problem formulated on the original domain, we solve many smaller problems formulated on subdomains. These subproblems are linked together by suitable conditions. The idea of domain decomposition is quite natural, for instance, when different physical models are needed to be used in different parts of the domain.

Boundary Element Method (BEM) has certainly its significance between modern numerical methods of mathematical modelling. The main benefit of this method, comparing to the well–established and widely used Finite Element Method (FEM), is that the formulation of the problem is reduced to the boundary which yields an important dimension reduction. Due to the fact that we handle no mesh inside the body, BEM is very well applicable to exterior or shape optimization problems. Usage of BEM is quite natural when dealing with problems formulated on unbounded domains. On the other hand, application of BEM leads in general to densely populated stiff-

ness matrices. This represents a certain drawback in comparison with using FEM which results in stiffness matrices with the vast majority of entries vanished. This disadvantage causing large storage requirements and unwelcome limitations on the fineness of discretization may be effectively overcome by the so–called Fast BEMs that have been studied recently. Methods such as Fast Multipole [39, 40] or hierarchical matrices [3] reduce the solution time and memory storage requirement significantly.

In this work, we mainly focus on the application of the BETI method to elliptic variational inequalities, where the resulting quadratic programming problem is constrained not only by equalities, but also by bound constraints that are associated with the related contact conditions. We exploit here the idea of "All Floating" or "Total" variant of the BETI/FETI method introduced independently by Of [38] and Dostál et al. [20], respectively. This approach enforces the fixation along the Dirichlet part of the boundary by additional Lagrange multipliers. Although our method is based on that introduced by Langer and Steinbach [32], we cannot use their preconditioning strategy, since their preconditioner transforms the bound constraints into more general inequalities, which prevents usage of our in a sense optimal algorithms. For this reason we employ, instead, preconditioning by the projectors to the so–called natural coarse grid and develop a scalable algorithm for the solution of both coercive and semi–coercive contact problems [7]. The key tool is an observation of Langer and Steinbach [32] that yields a spectral equivalence of the discrete approximate Steklov–Poincaré operators generated by the FETI and BETI methods. This way we can use the analysis of Farhat et al. [27], which gives the upper bound on the spectrum of the preconditioned dual stiffness matrix as CH/h, where C is a constant independent of discretization and decomposition parameters h and H, respectively.

The thesis is structured as follows. In Chapter 2, we intend to recall the definition of Sobolev spaces and some basic results about existence and uniqueness of the solution of abstract variational inequality.

Let us state beforehand that Chapters 3, 4, and 5 are split into two main parts, where the first one is concerned with the analysis corresponding to the Laplace operator and the second one focuses on the analysis corresponding to linear homogeneous isotropic elastostatics. In Chapter 3, we introduce the representation formula and by application of the interior trace operator and the associated conormal derivative operator, we obtain a system of integral equations valid on the boundary. Then we continue with the definitions and properties of the well–known single and double layer potential operators, adjoint double layer potential operator, hypersingular integral operator, and Newton (volume) potential operators. Finally, we introduce the Dirichlet–Neumann map and define the Steklov–Poincaré operator by the

corresponding boundary integral operators.

In Chapter 4, we formulate our model contact problems. As indicated before, the first one is a 2D semi–coercive multibody problem described by the Laplace operator and the second one is a 3D coercive contact problem of linear homogeneous isotropic elastostatics. Then we apply the non–overlapping domain decomposition and discuss existence and uniqueness of the solutions of the resulting boundary variational inequalities.

Chapter 5 focuses firstly on the appropriate boundary element approximations of the Steklov–Poincaré and Newton operators, which are needed due to implicit definitions of both operators. Then we continue with the boundary element discretization by using the Ritz method. At the end, we sketch a possible evaluation of the Newton potential.

Chapter 6 introduces the primal and dual formulations of the discretized model problems. As discussed briefly above, we employ the natural coarse grid to obtain a preconditioned bound and equality constrained quadratic programming problem. An important result is that if we refine the mesh and increase the number of subdomains so that the ratio H/h is kept fixed, we have still the same bounds on the spectrum of the preconditioned dual stiffness matrix. Finally, a few paragraphs are devoted to the description of stable evaluation of the left generalized inverse of the corresponding local stiffness matrix.

In Chapter 7, we introduce a pair of algorithms SMALBE and MPRGP proposed by Dostál [13] and Dostál and Schöberl [25], respectively. The algorithm SMALBE is based on semi–monotonic augmented Lagrangians and its unique feature is a bound on number of iterations in bounds on the spectrum of the preconditioned dual stiffness matrix. The algorithm MPRGP is a quadratic programming algorithm using classical conjugate gradient steps, gradient projections, and proportioning. The rate of convergence of this algorithm is also given in bounds on the spectrum of the preconditioned dual stiffness matrix.

Chapter 8 contains results of our numerical experiments, in which we employed both algorithms described in Chapter 7. First of all, we demonstrate the numerical scalability of our method on the solutions of both model contact problems. Finally, we show the numerical results for 3D Hertz problem.

In Conclusion, we summarize achieved results of the thesis and outline future work connected to this topic.

Notations

$\equiv$	identical equality
$\approx$	approximation
$\mathbb{R}$	real axis
$\mathbb{N}$	set of all natural numbers
$\mathbb{R}_+$	interval $(0, \infty)$
$\overline{\Omega}$	closure of a set Ω
$\operatorname{diam}\Omega$	diameter of a set Ω, i.e. $\operatorname{diam}\Omega := \sup\{\|x-y\| : \ x, y \in \Omega\}$
$\partial\Omega$	boundary of a set Ω
$\operatorname{meas}\Gamma$	Lebesgue measure of a set Γ
$\operatorname{Span} S$	linear hull of a set S
$C^\infty(\Omega)$	space of all real functions with continuous derivatives of all orders on a set Ω
Dv	domain of a function v
$\operatorname{supp} v$	support of a function v, i.e. $\operatorname{supp} v := \overline{\{x \in Dv : \ v(x) \neq 0\}}$
$\liminf v_n$	limit inferior of a sequence $\{v_n\}$, i.e. $\liminf v_n := \sup\limits_{n\geq 0} \inf\limits_{k\geq n} v_k$
$\rightharpoonup$	weak convergence symbol
$v\|_\Gamma$	restriction of a function v to a set Γ
I	identity mapping
$\operatorname{Ker} v$	kernel (null space) of a function v
$\|\cdot\|$	Euclidean norm
$(\cdot,\cdot)$	Euclidean scalar product
$\|\cdot\|_{\mathcal{V}}$	norm defined in a vector space $\mathcal{V}$
$\langle\cdot,\cdot\rangle_{\mathcal{V}}$	scalar product defined in a vector space $\mathcal{V}$
$\triangle$	Laplace's operator
∇u	gradient of a function u
$\operatorname{div}\underline{u}$	divergence of a vector field $\underline{u}$
δ_{ij}	Kronecker's delta
δ_y	Dirac's δ–distribution at a point y
$\mathsf{A}\,\|\,M$	restriction of a matrix A to a set M
I	identity matrix
O	zero matrix
$\operatorname{Im}\mathsf{A}$	range of a matrix A

$\sigma(\mathsf{A})$	spectrum of a matrix A
$\lambda_{\min}(\mathsf{A})$	minimum eigenvalue of a real symmetric matrix A
$\lambda_{\max}(\mathsf{A})$	maximum eigenvalue of a real symmetric matrix A
$\|\mathsf{A}\|$	spectral norm of a real symmetric positive semi–definite matrix A, i.e. $\|\mathsf{A}\| := \lambda_{\max}(\mathsf{A})$
$\kappa(\mathsf{A})$	spectral condition number of a real symmetric positive definite matrix A, i.e. $\kappa(\mathsf{A}) := \lambda_{\max}(\mathsf{A})/\lambda_{\min}(\mathsf{A})$
$\square$	end of proof

Chapter 2

Preliminaries

2.1 Sobolev spaces

In this work, we shall deal only with the boundary value problems of the second order and the introduction of the Sobolev space of the first order shall be sufficient. In what follows, we shall assume that Ω denotes a non–empty bounded Lipschitz domain in $\mathbb{R}^d$, $d = 2, 3$, with the boundary $\Gamma := \partial\Omega$.

Let us define the Sobolev space

$$H^1(\Omega)$$

as a completion

$$(C^\infty(\overline{\Omega}),\, \|\cdot\|_{H^1(\Omega)}),$$

where for all $u \in C^\infty(\overline{\Omega})$ we define

$$\|u\|_{H^1(\Omega)} := \sqrt{\|u\|^2_{L^2(\Omega)} + |u|^2_{H^1(\Omega)}}$$

with

$$|u|^2_{H^1(\Omega)} := \int_\Omega \|\nabla u(x)\|^2 \,\mathrm{d}x.$$

It can be shown that

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) :\ \frac{\partial u}{\partial x_i} \in L^2(\Omega) \ \text{ for } i = 1, \ldots, d \right\},$$

where the derivatives are considered in the distributional sense. It holds that $H^1(\Omega)$ is the Hilbert space with the scalar product

$$\langle u,\, v\rangle_{H^1(\Omega)} := \langle u,\, v\rangle_{L^2(\Omega)} + \langle \nabla u,\, \nabla v\rangle_{L^2(\Omega)}\,. \tag{2.1}$$

Let $\Gamma_u \subset \Gamma$ satisfy $\operatorname{meas}\Gamma_u > 0$. A completion

$$(C_0^\infty(\overline{\Omega}, \Gamma_u),\ \|\cdot\|_{H^1(\Omega)}),$$

where $C_0^\infty(\overline{\Omega}, \Gamma_u)$ contains functions from $C^\infty(\overline{\Omega})$ that are zero on Γ_u, shall be denoted by

$$H_0^1(\Omega, \Gamma_u).$$

$H_0^1(\Omega, \Gamma_u)$ is the Hilbert space with the scalar product given by (2.1).

Theorem 2.1 *(Friedrichs) Functional $|\cdot|_{H^1(\Omega)}$ is on $H_0^1(\Omega, \Gamma_u)$ an equivalent norm to $\|\cdot\|_{H^1(\Omega)}$.*

Theorem 2.2 *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded Lipschitz domain. Then there is a unique linear continuous mapping*

$$\gamma_0 :\ H^1(\Omega) \mapsto L^2(\Gamma)$$

satisfying

$$\gamma_0 u = u|_\Gamma \qquad \text{for all } u \in C^\infty(\overline{\Omega}).$$

We call $\gamma_0 u \in L^2(\Gamma)$ a trace of a function $u \in H^1(\Omega)$.

We shall denote the trace space of $H^1(\Omega)$ by $H^{1/2}(\Gamma)$, i.e.

$$H^{1/2}(\Gamma) := \gamma_0(H^1(\Omega)).$$

In $H^{1/2}(\Gamma)$, we introduce a norm

$$\|v\|_{H^{1/2}(\Gamma)} := \sqrt{\|v\|^2_{L^2(\Gamma)} + |v|^2_{H^{1/2}(\Gamma)}},$$

where

$$|v|^2_{H^{1/2}(\Gamma)} := \int_\Gamma \int_\Gamma \frac{(v(x) - v(y))^2}{\|x - y\|^d}\, \mathrm{d}s_x\, \mathrm{d}s_y.$$

Let us note that $H^{1/2}(\Gamma)$ is the Hilbert space. Furthermore, it can be observed that there is a $k > 0$ such that

$$\|\gamma_0 u\|_{H^{1/2}(\Gamma)} \le k\|u\|_{H^1(\Omega)} \qquad \text{for all } u \in H^1(\Omega). \tag{2.2}$$

The dual space to $H^{1/2}(\Gamma)$ with respect to the $L^2(\Gamma)$ scalar product shall be denoted by

$$H^{-1/2}(\Gamma)$$

and the norm in $H^{-1/2}(\Gamma)$ is given by

$$\|w\|_{H^{-1/2}(\Gamma)} := \sup_{0 \ne v \in H^{1/2}(\Gamma)} \frac{|\langle w, v\rangle_{L^2(\Gamma)}|}{\|v\|_{H^{1/2}(\Gamma)}}.$$

Theorem 2.3 *(Green) Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded Lipschitz domain, $u, v \in H^1(\Omega)$, and $i \in \{1, \ldots, d\}$. Then*

$$\left\langle \frac{\partial u}{\partial x_i}, v \right\rangle_{L^2(\Omega)} = \int_\Gamma \gamma_0 u(x) \gamma_0 v(x) n_i(x) \, \mathrm{d}s_x - \left\langle u, \frac{\partial v}{\partial x_i} \right\rangle_{L^2(\Omega)},$$

where $n_i(x)$ is the ith component of the exterior unit normal vector defined for almost all $x \in \Gamma$.

2.2 Variational inequalities

Here we intend to recall some well–known results of the analysis of variational inequalities which we shall use later. First of all, let us introduce notations and definitions needed in the following theorems.

Let

i) $\mathcal{V}$ be a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ and norm $\|\cdot\|_{\mathcal{V}}$;

ii) $\mathcal{K} \subset \mathcal{V}$ be a closed, convex, nonempty subset that needs not be a subspace of $\mathcal{V}$;

iii) $\mathcal{F}$ be a continuous linear functional on $\mathcal{V}$;

iv) $\mathcal{A}$ be a bilinear form on $\mathcal{V}$.

Definition 2.1

- *$\mathcal{A}$ is bounded on $\mathcal{K}$ if there is an $M > 0$ such that*

$$|\mathcal{A}(u, v)| \le M \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}} \qquad \text{for all } u, v \in \mathcal{K};$$

- *$\mathcal{A}$ is symmetric on $\mathcal{K}$ if*

$$\mathcal{A}(u, v) = \mathcal{A}(v, u) \qquad \text{for all } u, v \in \mathcal{K};$$

- *$\mathcal{A}$ is semi–elliptic on $\mathcal{K}$ if*

$$\mathcal{A}(u, u) \ge 0 \qquad \text{for all } u \in \mathcal{K};$$

- *$\mathcal{A}$ is elliptic on $\mathcal{K}$ if there is an $\alpha > 0$ such that*

$$\mathcal{A}(u, u) \ge \alpha \|u\|_{\mathcal{V}}^2 \qquad \text{for all } u \in \mathcal{K}.$$

Definition 2.2 *Functional $\mathcal{G} : \mathcal{V} \mapsto \mathbb{R}$ is*

- *coercive on $\mathcal{K}$ if*

$$\left.\begin{array}{r} v \in \mathcal{K} \\ \|v\|_{\mathcal{V}} \to \infty \end{array}\right\} \Rightarrow \ \mathcal{G}(v) \to \infty.$$

- *convex on $\mathcal{K}$ if*

$$\mathcal{G}(tu + (1-t)v) \leq t\,\mathcal{G}(u) + (1-t)\,\mathcal{G}(v)$$

for all $u, v \in \mathcal{K}$ and all $t \in (0,1)$.

Theorem 2.4 *Let $\mathcal{G} : \ \mathcal{V} \mapsto \mathbb{R}$ be continuous, coercive, and convex on $\mathcal{K}$. Then there exists at least one solution of the minimization problem: find $u \in \mathcal{K}$ such that*

$$\mathcal{G}(u) = \min\{\mathcal{G}(v) : \ v \in \mathcal{K}\}. \tag{2.3}$$

`Proof.` Let $\mathcal{B}_R$ be a closed–origin–centered ball of radius $R \in \mathbb{R}_+$, i.e.

$$\mathcal{B}_R := \{v \in \mathcal{V} : \ \|v\|_{\mathcal{V}} \leq R\}.$$

Then the coercivity of $\mathcal{G}$ on $\mathcal{K}$ yields existence of a large enough $R \in \mathbb{R}_+$ such that

$$\inf_{v \in \mathcal{K}} \mathcal{G}(v) = \inf_{v \in \mathcal{K} \cap \mathcal{B}_R} \mathcal{G}(v) =: q.$$

Let us now consider a sequence $\{u_n\}$, $u_n \in \mathcal{K} \cap \mathcal{B}_R$, such that

$$\mathcal{G}(u_n) \to q.$$

Since $\{u_n\}$ is bounded and $\mathcal{V}$ is the Hilbert space, there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u \in \mathcal{V}$ satisfying

$$u_{n_k} \rightharpoonup u.$$

Let us recall that every closed convex set is weakly closed, i.e. $u \in \mathcal{K}$, and since every continuous convex functional on a closed convex set is weakly lower semi–continuous, we get

$$\mathcal{G}(u) \leq \liminf \mathcal{G}(u_{n_k}) = \lim \mathcal{G}(u_{n_k}) = q \leq \mathcal{G}(u).$$

□

Now we shall be concerned with the `variational inequality`: find $u \in \mathcal{K}$ such that

$$\mathcal{A}(u,\, v-u) \geq \mathcal{F}(v-u) \qquad \text{for all } v \in \mathcal{K}. \tag{2.4}$$

Theorem 2.5 *(analogue to the Lax–Milgram theorem for variational inequalities) Let $\mathcal{A}$ be bounded and elliptic on $\mathcal{V}$. Then there exists a unique solution $u \in \mathcal{K}$ of problem (2.4).*

`Proof.` See [30].

□

Moreover, let us define the so–called `energy functional` $\mathcal{J}$ on $\mathcal{V}$ by

$$\mathcal{J}(v) := \frac{1}{2}\,\mathcal{A}(v,\, v) - \mathcal{F}(v). \tag{2.5}$$

Theorem 2.6 *If $\mathcal{A}$ is bounded, symmetric, and semi–elliptic on $\mathcal{V}$, then the energy functional $\mathcal{J}$ defined by (2.5) is convex on $\mathcal{V}$.*

`Proof.` The aim is to show that

$$\mathcal{J}(tu + (1-t)v) \le t\,\mathcal{J}(u) + (1-t)\mathcal{J}(v).$$

for all $u,\, v \in \mathcal{V}$ and all $t \in (0,\, 1)$. Since $\mathcal{F}$ is linear on $\mathcal{V}$, it suffices to prove that

$$v \mapsto \mathcal{A}(v,\, v)$$

is convex on $\mathcal{V}$. Let $u,\, v \in \mathcal{V}$ and $t \in (0,\, 1)$ be arbitrary. From the semi–ellipticity and symmetry of $\mathcal{A}$ on $\mathcal{V}$ we obtain

$$0 \le \mathcal{A}(u - v,\, u - v) = \mathcal{A}(u,\, u) - 2\,\mathcal{A}(u,\, v) + \mathcal{A}(v,\, v). \tag{2.6}$$

Then, by (2.6), we get

$$\begin{aligned}
&\mathcal{A}(tu + (1-t)v,\, tu + (1-t)v) = \\
&\qquad = t^2\mathcal{A}(u,\, u) + 2\,t(1-t)\,\mathcal{A}(u,\, v) + (1-t)^2\mathcal{A}(v,\, v) \\
&\qquad \le t^2\mathcal{A}(u,\, u) + t(1-t)\,[\mathcal{A}(u,\, u) + \mathcal{A}(v,\, v)] + (1-t)^2\mathcal{A}(v,\, v) \\
&\qquad = t\,\mathcal{A}(u,\, u) + (1-t)\,\mathcal{A}(v,\, v),
\end{aligned}$$

which completes the proof.

□

Theorem 2.7 *Let $\mathcal{A}$ be bounded, symmetric, and semi–elliptic on $\mathcal{V}$. Then problem (2.4) is equivalent to the minimization problem: find $u \in \mathcal{K}$ such that*

$$\mathcal{J}(u) = \min\left\{\mathcal{J}(v) :\ v \in \mathcal{K}\right\}.$$

`Proof.` Suppose $u \in \mathcal{K}$ is a solution of variational inequality (2.4). Let us pick any $v \in \mathcal{K}$ and put $z := v - u \in \mathcal{V}$. Thus we have

$$\mathcal{A}(u, z) \geq \mathcal{F}(z).$$

Then due to the assumed symmetry and semi–ellipticity of $\mathcal{A}$ on $\mathcal{V}$, we obtain

$$\begin{aligned} \mathcal{J}(v) = \mathcal{J}(z+u) &= \frac{1}{2}\mathcal{A}(z+u, z+u) - \mathcal{F}(z+u) \\ &= \frac{1}{2}\mathcal{A}(z,z) + \mathcal{A}(u,z) + \frac{1}{2}\mathcal{A}(u,u) - \mathcal{F}(z) - \mathcal{F}(u) \\ &\geq \mathcal{J}(u). \end{aligned}$$

Conversely, assume $u \in \mathcal{K}$ minimizes the energy functional $\mathcal{J}$ on $\mathcal{K}$. Let us choose a $v \in \mathcal{K}$. Then

$$\phi(t) := \mathcal{J}((1-t)u + tv) \geq \mathcal{J}(u) = \phi(0) \qquad \text{for all } t \in [0,1].$$

Now let us take a look at the function ϕ. The symmetry of $\mathcal{A}$ on $\mathcal{V}$ yields

$$\phi(t) = \frac{1}{2}(1-t)^2\mathcal{A}(u,u) + (1-t)\,t\mathcal{A}(u,v) + \frac{1}{2}t^2\mathcal{A}(v,v) - (1-t)\mathcal{F}(u) - t\mathcal{F}(v)$$

for all $t \in [0,1]$, and therefore

$$\phi'_+(0) = -\mathcal{A}(u,u) + \mathcal{A}(u,v) + \mathcal{F}(u) - \mathcal{F}(v) = \mathcal{A}(u, v-u) - \mathcal{F}(v-u).$$

Since $\phi'_+(0) \geq 0$, the proof is finished.

□

Chapter 3

The Dirichlet–Neumann map for elliptic partial differential operators

3.1 Laplace operator

Let us begin with some notations that we shall use through the following text.

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded Lipschitz domain with the boundary Γ. We shall consider the interior trace operator

$$\gamma_0 : \ H^1(\Omega) \mapsto H^{1/2}(\Gamma)$$

and the interior conormal derivative operator

$$\gamma_1 : \ H^1_{\triangle}(\Omega) \mapsto H^{-1/2}(\Gamma),$$

where

$$H^1_{\triangle}(\Omega) := \left\{ v \in H^1(\Omega) : \ -\triangle v \in L^2(\Omega) \right\},$$

satisfying

$$\gamma_1 u(x) = \frac{\partial u}{\partial \underline{n}}(x) \qquad \text{for all } u \in C^\infty(\overline{\Omega}),\ x \in \Gamma,$$

with $\underline{n}(x)$ denoting the exterior unit normal vector defined for almost all $x \in \Gamma$. Let us note that the operator γ_1 is a linear and bounded mapping on $H^1_{\triangle}(\Omega)$.

3.1.1 Fundamental solution, Green's representation formula

Definition 3.1 *A function* $U: \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R}$ *defined by*

$$U(x,y) := -\frac{1}{2\pi} \log \|x-y\| \tag{3.1}$$

is called the ***fundamental solution of the Laplace operator*** *in* $\mathbb{R}^2$.

Definition 3.2 *A function* $U: \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}$ *defined by*

$$U(x,y) := \frac{1}{4\pi} \frac{1}{\|x-y\|} \tag{3.2}$$

is called the ***fundamental solution of the Laplace operator*** *in* $\mathbb{R}^3$.

Indeed, it can be shown [36, 45] that U satisfies (in the distributional sense)

$$-\triangle_x U(x,y) = \delta_y(x) \quad \text{for } x, y \in \mathbb{R}^d,\ d = 2, 3.$$

Here and in what follows, we consider the function U defined by (3.1) or (3.2) with regard to the dimension d.

Now we shall introduce theorems which are discussed in more detail in [8].

Theorem 3.1 *(First Green's formula) For any* $u \in H^1_{\triangle}(\Omega)$ *and* $v \in H^1(\Omega)$ *there holds*

$$-\int_\Omega \triangle u(x) v(x)\,\mathrm{d}x = \int_\Omega \nabla u(x) \nabla v(x)\,\mathrm{d}x - \int_\Gamma \gamma_1 u(x) \gamma_0 v(x)\,\mathrm{d}s_x. \tag{3.3}$$

Theorem 3.2 *(Second Green's formula) For any* $u, v \in H^1_{\triangle}(\Omega)$ *there holds*

$$\int_\Omega \triangle u(x) v(x) - u(x) \triangle v(x)\,\mathrm{d}x = \int_\Gamma \gamma_1 u(x) \gamma_0 v(x) - \gamma_0 u(x) \gamma_1 v(x)\,\mathrm{d}s_x.$$

Theorem 3.3 *(Green's representation formula) Let* $f \in L^2(\Omega)$ *and* $u \in H^1(\Omega)$ *be a distributional solution of the Poisson equation*

$$-\triangle u(x) = f(x) \quad \text{for } x \in \Omega.$$

Then

$$u(x) = \int_\Omega f(y)\, U(x,y)\,\mathrm{d}y + \int_\Gamma \gamma_1 u(y)\, U(x,y)\,\mathrm{d}s_y - \int_\Gamma \gamma_0 u(y)\, \gamma_{1,y} U(x,y)\,\mathrm{d}s_y \tag{3.4}$$

for $x \in \Omega$.

3.1.2 Boundary integral operators

By applying the operators γ_0 and γ_1 to the representation formula (3.4) [41], we get the system of integral equations valid on Γ

$$\begin{pmatrix} \gamma_0 u \\ \gamma_1 u \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K' \end{pmatrix} \begin{pmatrix} \gamma_0 u \\ \gamma_1 u \end{pmatrix} + \begin{pmatrix} N_0 f \\ N_1 f \end{pmatrix}. \tag{3.5}$$

First, let us focus on the well–known boundary integral operators that appear in (3.5). We define for $x \in \Gamma$ the `single layer potential operator` by

$$(Vt)(x) := \int_\Gamma t(y)\, U(x,y)\, \mathrm{d}s_y, \tag{3.6}$$

the `double layer potential operator` by

$$(Ku)(x) := \int_\Gamma u(y)\, \gamma_{1,y} U(x,y)\, \mathrm{d}s_y, \tag{3.7}$$

the `adjoint double layer potential operator` by

$$(K't)(x) := \int_\Gamma t(y)\, \gamma_{1,x} U(x,y)\, \mathrm{d}s_y, \tag{3.8}$$

and finally the `hypersingular integral operator` by

$$(Du)(x) := -\gamma_{1,x} \int_\Gamma u(y)\, \gamma_{1,y} U(x,y)\, \mathrm{d}s_y. \tag{3.9}$$

The `Newton` or `volume potential operators` from (3.5) are given for $x \in \Gamma$ by

$$(N_0 f)(x) := \int_\Omega f(y)\, U(x,y)\, \mathrm{d}y \tag{3.10}$$

and

$$(N_1 f)(x) := \int_\Omega f(y)\, \gamma_{1,x} U(x,y)\, \mathrm{d}y.$$

The boundary integral operators have the following properties [8, 41, 45]:

Theorem 3.4 *The single layer potential operator*

$$V : \; H^{-1/2}(\Gamma) \mapsto H^{1/2}(\Gamma)$$

given by (3.6) is linear, bounded, symmetric, and, for the case of the dimension $d = 2$ *provided*

$$\operatorname{diam}\Omega < 1, \tag{3.11}$$

$H^{-1/2}(\Gamma)$*–elliptic satisfying*

$$\langle Vt,\, t\rangle_{L^2(\Gamma)} \geq \alpha^V \left\|t\right\|^2_{H^{-1/2}(\Gamma)} \qquad \textit{for all } t \in H^{-1/2}(\Gamma).$$

Theorem 3.5 *The double layer potential operator*

$$K:\ H^{1/2}(\Gamma) \mapsto H^{1/2}(\Gamma)$$

given by (3.7) is linear and bounded. Moreover, there hold the representations

$$(Kv)(x) = \frac{1}{2\pi}\int\limits_{\Gamma} \frac{(x-y, \underline{n}(y))}{\|x-y\|^2} v(y)\,\mathrm{d}s_y$$

for $x \in \Gamma$ *and* $d = 2$ *and*

$$(Kv)(x) = \frac{1}{4\pi}\int\limits_{\Gamma} \frac{(x-y, \underline{n}(y))}{\|x-y\|^3} v(y)\,\mathrm{d}s_y$$

for $x \in \Gamma$ *and* $d = 3$.

Theorem 3.6 *The adjoint double layer potential operator*

$$K':\ H^{-1/2}(\Gamma) \mapsto H^{-1/2}(\Gamma)$$

given by (3.8) is linear and bounded.

Theorem 3.7 *The hypersingular integral operator*

$$D:\ H^{1/2}(\Gamma) \mapsto H^{-1/2}(\Gamma)$$

given by (3.9) is linear, bounded, symmetric, and $H^{1/2}(\Gamma)$*–semi–elliptic satisfying*

$$\langle Dv,\, v\rangle_{L^2(\Gamma)} \geq \overline{\alpha}^D \left|v\right|^2_{H^{1/2}(\Gamma)} \qquad \textit{for all } v \in H^{1/2}(\Gamma) \tag{3.12}$$

and

$$\langle Dv,\, v\rangle_{L^2(\Gamma)} \geq \alpha^D \left\|v\right\|^2_{H^{1/2}(\Gamma)} \qquad \textit{for all } v \in H^{1/2}(\Gamma)_{/\operatorname{Ker} D}, \tag{3.13}$$

where $H^{1/2}(\Gamma)_{/\operatorname{Ker} D}$ *is the space of all functions from* $H^{1/2}(\Gamma)$ *that are orthogonal to* $\operatorname{Ker} D$. *Moreover, for* $d = 2$ *and* $u, v \in H^{1/2}(\Gamma) \cap C(\Gamma)$ *there holds the representation*

$$\langle Du,\, v\rangle_{L^2(\Gamma)} = -\frac{1}{2\pi}\int\limits_{\Gamma} \operatorname{curl}_\Gamma v(x) \int\limits_{\Gamma} \log\|x-y\| \operatorname{curl}_\Gamma u(y)\,\mathrm{d}s_y\,\mathrm{d}s_x,$$

where

$$\operatorname{curl}_\Gamma v(x) := (\underline{n}(x),\, \underline{\operatorname{curl}}\, \widetilde{v}(x)) \qquad \text{for } x \in \Gamma$$

with the rotation

$$\underline{\operatorname{curl}}\, \widetilde{v}(x) := \begin{pmatrix} \frac{\partial}{\partial x_2} \widetilde{v}(x) \\ -\frac{\partial}{\partial x_1} \widetilde{v}(x) \end{pmatrix}.$$

For $d = 3$ and $u, v \in H^{1/2}(\Gamma) \cap C(\Gamma)$ we have the representation

$$\langle Du,\, v \rangle_{L^2(\Gamma)} = \frac{1}{4\pi} \int_\Gamma \int_\Gamma \frac{(\underline{\operatorname{curl}}_\Gamma u(y),\, \underline{\operatorname{curl}}_\Gamma v(x))}{\|x - y\|} \, \mathrm{d}s_y \, \mathrm{d}s_x$$

with the surface curl operator

$$\underline{\operatorname{curl}}_\Gamma v(x) := \underline{n}(x) \times \nabla \widetilde{v}(x) \qquad \text{for } x \in \Gamma.$$

Here, $\widetilde{v}$ is some locally defined extension of v into a d–dimensional neighbourhood of Γ.

Theorem 3.8 *Let $u \equiv 1$. Then*

$$u \in \operatorname{Ker} \left(\frac{1}{2} I + K \right) \quad \text{and} \quad u \in \operatorname{Ker} D.$$

Moreover,

$$\operatorname{Ker} D = \operatorname{Span} \{1\}.$$

`Proof. i)` Function $u \equiv 1$ satisfies the Laplace equation

$$-\triangle u(x) = 0 \qquad \text{for } x \in \Omega,$$

and therefore it also has to satisfy the system (3.5) with $f = 0$. Since

$$\gamma_1 u(x) = \frac{\partial u}{\partial \underline{n}}(x) = 0 \qquad \text{for } x \in \Gamma,$$

the first equation gives

$$\left(\frac{1}{2} I + K \right) u(x) = 0 \qquad \text{for } x \in \Gamma$$

and the second one yields

$$(Du)(x) = 0 \qquad \text{for } x \in \Gamma.$$

`ii)` Let $v \in \operatorname{Ker} D$. Relation (3.12) yields

$$0 = \langle Dv,\, v\rangle_{L^2(\Gamma)} \geq \overline{\alpha}^D\, |v|^2_{H^{1/2}(\Gamma)} = \overline{\alpha}^D \int\limits_\Gamma \int\limits_\Gamma \frac{(v(x)-v(y))^2}{\|x-y\|^d}\, \mathrm{d}s_x\, \mathrm{d}s_y \geq 0,$$

where $\overline{\alpha}^D$ is a positive constant. Thus

$$v(x) = v(y) \qquad \text{for almost all } x, y \in \Gamma.$$

□

3.1.3 The Steklov–Poincaré operator

Let us assume that the operator V is elliptic on $H^{-1/2}(\Gamma)$. Then from the first equation of (3.5) we get the Dirichlet–Neumann map

$$\gamma_1 u(x) = V^{-1}(\frac{1}{2}I + K)\gamma_0 u(x) - V^{-1}(N_0 f)(x) \tag{3.14}$$

for $x \in \Gamma$. Insertion of (3.14) into the second equation of (3.5) yields another representation of the Dirichlet–Neumann map

$$\begin{aligned} \gamma_1 u(x) \;&=\; \left[(\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K) + D\right]\gamma_0 u(x) + \\ &\quad \left[N_1 - (\frac{1}{2}I + K')V^{-1}N_0\right] f(x) \end{aligned} \tag{3.15}$$

for $x \in \Gamma$. Now let us define the `Steklov-Poincaré operator` by the equivalent representations

$$S \;:=\; V^{-1}(\frac{1}{2}I + K) \tag{3.16}$$

$$\;=\; (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K) + D \tag{3.17}$$

and the `Newton operator` by the equivalent representations

$$\begin{aligned} -N \;&:=\; -V^{-1}N_0 \\ &=\; N_1 - (\frac{1}{2}I + K')V^{-1}N_0, \end{aligned} \tag{3.18}$$

so that we can rewrite the equations (3.14) and (3.15) as

$$\gamma_1 u(x) = (S\gamma_0 u)(x) - (Nf)(x) \quad \text{ for } x \in \Gamma. \tag{3.19}$$

At this moment, let us note that the representation (3.16) together with the Galerkin discretization leads typically to a non–symmetric stiffness matrix. We premise that symmetry of stiffness matrix shall be essential in our analysis, so that we shall consider only the formally symmetric representation (3.17).

Let us introduce a mapping $\mathcal{P}: \ H^{1/2}(\Gamma) \mapsto H^1(\Omega)$ so that for a $u \in H^{1/2}(\Gamma)$ we define $\mathcal{P}u \in H^1(\Omega)$ as a weak solution of the Dirichlet problem

$$\begin{aligned} -\triangle v(x) &= 0 \qquad \text{for } x \in \Omega, \\ v(x) &= u(x) \quad \text{for } x \in \Gamma. \end{aligned}$$

Let us note that $\mathcal{P}$ is linear and bounded on $H^{1/2}(\Gamma)$. It can be further shown that for all $u, v \in H^{1/2}(\Gamma)$ we have

$$\langle Su,\, v\rangle_{L^2(\Gamma)} = \int_\Omega \nabla(\mathcal{P}u)(x)\nabla(\mathcal{P}v)(x)\,\mathrm{d}x. \tag{3.20}$$

Let us define the function space

$$H_0^{1/2}(\Gamma, \Gamma_u) := \{v \in H^{1/2}(\Gamma): \ v(x) = 0 \ \text{ for } x \in \Gamma_u\},$$

where $\Gamma_u \subset \Gamma$ and $\operatorname{meas}\Gamma_u > 0$. Let us note that $H_0^{1/2}(\Gamma, \Gamma_u)$ is a closed subspace of $H^{1/2}(\Gamma)$.

Theorem 3.9 *Let the condition (3.11) be satisfied for the case $d = 2$. Then the Steklov–Poincaré operator*

$$S: \ H^{1/2}(\Gamma) \mapsto H^{-1/2}(\Gamma)$$

is a linear, bounded, symmetric, and semi–elliptic mapping satisfying

$$\langle Sv,\, v\rangle_{L^2(\Gamma)} \geq \alpha^D \|v\|^2_{H^{1/2}(\Gamma)} \qquad \text{for all } v \in H^{1/2}(\Gamma)_{/\operatorname{Ker} D}, \tag{3.21}$$

where $H^{1/2}(\Gamma)_{/\operatorname{Ker} D}$ is the space of all functions from $H^{1/2}(\Gamma)$ that are orthogonal to $\operatorname{Ker} D$. *Note that the positive constant α^D is the same as that in Theorem 3.7.*

Moreover, S is elliptic on $H_0^{1/2}(\Gamma, \Gamma_u)$.

`Proof. i)` The linearity and boundedness of the operators V, K, K', and D imply the linearity and boundedness of the operator S.

`ii)` The symmetry of S follows particularly from the symmetry of V^{-1} and D on $H^{1/2}(\Gamma)$ and from the fact that K' is adjoint to K. Alternatively,

we can proceed as follows. Let $u, v \in H^{1/2}(\Gamma)$ be arbitrary. Then, by (3.20), we get

$$\langle Su,\, v\rangle_{L^2(\Gamma)} = \int_\Omega \nabla(\mathcal{P}u)(x)\nabla(\mathcal{P}v)(x)\,\mathrm{d}x = \langle Sv,\, u\rangle_{L^2(\Gamma)}\,.$$

`iii)` Let $u \in H^{1/2}(\Gamma)$ be arbitrary. Since the operator $\frac{1}{2}I + K'$ is adjoint to the operator $\frac{1}{2}I + K$ and

$$\left\langle V^{-1}v,\, v\right\rangle_{L^2(\Gamma)} \geq 0 \qquad \text{for all } v \in H^{1/2}(\Gamma),$$

we obtain

$$\begin{aligned}\left\langle (\tfrac{1}{2}I + K')V^{-1}(\tfrac{1}{2}I + K)u,\, u\right\rangle_{L^2(\Gamma)} &= \left\langle V^{-1}(\tfrac{1}{2}I + K)u,\, (\tfrac{1}{2}I + K)u\right\rangle_{L^2(\Gamma)} \\ &\geq 0.\end{aligned}$$

Hence and by (3.13), we immediately get

$$\begin{aligned}\langle Su,\, u\rangle_{L^2(\Gamma)} &= \left\langle (\tfrac{1}{2}I + K')V^{-1}(\tfrac{1}{2}I + K)u,\, u\right\rangle_{L^2(\Gamma)} + \langle Du,\, u\rangle_{L^2(\Gamma)} \\ &\geq \langle Du,\, u\rangle_{L^2(\Gamma)} \\ &\geq \alpha^D \left\|u\right\|^2_{H^{1/2}(\Gamma)}\end{aligned}$$

for all $u \in H^{1/2}(\Gamma)_{/\operatorname{Ker} D}$.

`iv)` Let $u \in H_0^{1/2}(\Gamma, \Gamma_u)$ be arbitrary. Then $\mathcal{P}u \in H_0^1(\Omega, \Gamma_u)$ and by using (3.20), Friedrichs' theorem 2.1, and inequality (2.2) we get

$$\begin{aligned}\langle Su,\, u\rangle_{L^2(\Gamma)} = \int_\Omega \left\|\nabla(\mathcal{P}u)(x)\right\|^2 \mathrm{d}x &\geq c_1 \|\mathcal{P}u\|^2_{H^1(\Omega)} \\ &\geq c_2 \|\gamma_0 \mathcal{P}u\|^2_{H^{1/2}(\Gamma)} = c_2 \|u\|^2_{H^{1/2}(\Gamma)}.\end{aligned}$$

□

Let us assume that the condition (3.11) is satisfied for the case $d = 2$ and let us for a while put $u \equiv 1$. Then u satisfies

$$\begin{aligned}-\triangle u(x) &= 0 \quad \text{for } x \in \Omega \\ \gamma_1 u(x) = \frac{\partial u}{\partial \underline{n}}(x) &= 0 \quad \text{for } x \in \Gamma\end{aligned}$$

and, by (3.19), we get

$$(Su)(x) = 0 \qquad \text{for all } x \in \Gamma. \tag{3.22}$$

3.1.4 The Newton operator

We shall be concerned only with the (more simple) representation (3.18) of the Newton operator.

Lemma 3.1 *The Newton potential operator*

$$N_0 : \; L^2(\Omega) \mapsto H^{1/2}(\Gamma)$$

given by (3.10) is a linear and bounded mapping.

`Proof.` See [45].

□

Now the linearity and boundedness of N_0 and V imply the following theorem.

Theorem 3.10 *Let the condition (3.11) be satisfied for the case $d = 2$. Then the Newton operator*

$$N : \; L^2(\Omega) \mapsto H^{-1/2}(\Gamma)$$

is a linear and bounded mapping.

For future purposes, let us state the following lemma.

Lemma 3.2 *Let the condition (3.11) be satisfied for the case $d = 2$ and assume $f \in L^2(\Omega)$. Then*

$$\langle Nf,\, 1\rangle_{L^2(\Gamma)} = \langle f,\, 1\rangle_{L^2(\Omega)}\,.$$

`Proof.` Let $u \in H^1(\Omega)$ be a weak solution of the Dirichlet boundary value problem

$$\begin{aligned} -\triangle u(x) &= f(x) \quad \text{for } x \in \Omega, \\ \gamma_0 u(x) &= 0 \qquad \text{for } x \in \Gamma. \end{aligned}$$

Then

$$\langle -\triangle u,\, 1\rangle_{L^2(\Omega)} = \langle f,\, 1\rangle_{L^2(\Omega)}$$

and also, by (3.19),

$$\gamma_1 u(x) = -(Nf)(x) \quad \text{for } x \in \Gamma. \tag{3.23}$$

From the first Green formula (3.3), we obtain

$$\langle -\triangle u,\, 1\rangle_{L^2(\Omega)} = \langle \nabla u,\, \nabla 1\rangle_{L^2(\Omega)} - \langle \gamma_1 u,\, 1\rangle_{L^2(\Gamma)} = -\langle \gamma_1 u,\, 1\rangle_{L^2(\Gamma)}$$

and by combining the latter relations with (3.23), we get

$$\langle f,\, 1\rangle_{L^2(\Omega)} = -\langle \gamma_1 u,\, 1\rangle_{L^2(\Gamma)} = \langle Nf,\, 1\rangle_{L^2(\Gamma)}\,.$$

□

3.2 Linear homogeneous isotropic elastostatics

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain with a boundary Γ. We shall focus on the elliptic partial differential operator $\mathcal{L}$ defined by

$$(\mathcal{L}\underline{u})_i(x) := -\sum_{j=1}^{3} \frac{\partial}{\partial x_j}\sigma_{ij}(\underline{u}, x) \quad \text{for } x \in \Omega,\ i = 1, 2, 3, \tag{3.24}$$

where the matrix operator σ is given by

$$\sigma_{ij}(\underline{u}, x) := \frac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\sum_{k=1}^{3} e_{kk}(\underline{u}, x) + \frac{E}{1+\nu}e_{ij}(\underline{u}, x) \quad \text{for } i, j = 1, 2, 3 \tag{3.25}$$

with the matrix operator e defined by

$$e_{ij}(\underline{u}, x) := \frac{1}{2}\left(\frac{\partial}{\partial x_i}u_j(x) + \frac{\partial}{\partial x_j}u_i(x)\right) \quad \text{for } i, j = 1, 2, 3 \tag{3.26}$$

and constants $E > 0$, $\nu \in (0, 1/2)$.

Let us note that in the context of linear elastostatics, σ denotes the stress tensor, e is the strain tensor, and E and ν refer to Young's modulus and Poisson's ratio, respectively. The stress–strain relation (3.25) is known as Hook's law.

We shall consider the interior trace operator

$$\gamma_0 :\ [H^1(\Omega)]^3 \mapsto [H^{1/2}(\Gamma)]^3$$

and the interior boundary stress operator

$$\gamma_1 :\ [H^1_{\text{Lamé}}(\Omega)]^3 \mapsto [H^{-1/2}(\Gamma)]^3,$$

where

$$[H^1_{\text{Lamé}}(\Omega)]^3 := \left\{\underline{v} \in [H^1(\Omega)]^3 :\ \mathcal{L}\underline{v} \in [L^2(\Omega)]^3\right\},$$

satisfying

$$\left(\gamma_1\underline{u}\right)_i(x) := \sum_{j=1}^{3} \sigma_{ij}(\underline{u}, x)n_j(x) \qquad \text{for all } \underline{u} \in [C^\infty(\overline{\Omega})]^3,\ x \in \Gamma,$$

$i = 1, 2, 3$, with $n_j(x)$ denoting jth component of the exterior unit normal vector $\underline{n}(x)$ that is defined for almost all $x \in \Gamma$. Let us note that γ_1 is a linear and bounded mapping on $[H^1_{\text{Lamé}}(\Omega)]^3$.

Theorem 3.11 *(Korn's inequality) There is a $k > 0$ such that*

$$\int_\Omega \sum_{i,j=1}^{3} \left(\frac{\partial v_i}{\partial x_j}(x) \right)^2 \mathrm{d}x \leq k \int_\Omega \sum_{i,j=1}^{3} e_{ij}^2(\underline{v}, x)\, \mathrm{d}x \qquad \text{for all } \underline{v} \in [H_0^1(\Omega,\, \Gamma_u)]^3. \tag{3.27}$$

`Proof.` See [36].

□

3.2.1 Fundamental solution, Somigliana's identity

Definition 3.3 *The matrix function $U:\ \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}^{3\times 3}$ defined by*

$$U_{kl}(x,\, y) := \frac{1+\nu}{8\pi E(1-\nu)} \left((3-4\nu) \frac{\delta_{kl}}{\|x-y\|} + \frac{(x_k - y_k)(x_l - y_l)}{\|x-y\|^3} \right)$$

for $k, l = 1, 2, 3$ is called the ***fundamental solution of linear homogeneous isotropic elastostatics****. U is also known as Kelvin's tensor.*

It can be shown that by insertion of σ given by (3.25) into the definition (3.24), we obtain

$$(\mathcal{L}\underline{u})_i(x) = -\mu \triangle \underline{u}(x) - (\lambda + \mu) \nabla \mathrm{div}\, \underline{u}(x) \quad \text{ for } x \in \Omega$$

with the so–called Lamé constants

$$\lambda := \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu := \frac{E}{2(1+\nu)}.$$

And indeed, it can be checked [36, 45] that U is a distributional solution of

$$-\mu \triangle_x \underline{U}_l(x,y) - (\lambda+\mu) \nabla_x \mathrm{div}_x\, \underline{U}_l(x,y) = \delta_y(x) \underline{e}_l \quad \text{ for } x, y \in \mathbb{R}^3$$

and $l = 1, 2, 3$, where $\underline{e}_l$ denotes a standard $\mathbb{R}^3$ basis vector.

Now we shall state the following theorems which are commented in more detail in [36, 41].

Theorem 3.12 *(First Betti's formula) For any $\underline{u} \in [H^1_{Lamé}(\Omega)]^3$ and $\underline{v} \in [H^1(\Omega)]^3$ there holds*

$$\int_\Omega (\mathcal{L}\underline{u}(x),\, \underline{v}(x))\, \mathrm{d}x = \int_\Omega W(\underline{u}(x), \underline{v}(x))\, \mathrm{d}x - \int_\Gamma (\gamma_1 \underline{u}(x),\, \gamma_0 \underline{v}(x))\ \mathrm{d}s_x$$

with

$$\begin{aligned} W(\underline{u}(x),\underline{v}(x)) &:= \sum_{i,j=1}^{3}\sigma_{ij}(\underline{u},x)e_{ij}(\underline{v},x) \\ &= \lambda \operatorname{div}\underline{u}(x)\operatorname{div}\underline{v}(x)+2\mu\sum_{i,j=1}^{3}e_{ij}(\underline{u},x)e_{ij}(\underline{v},x). \quad (3.28) \end{aligned}$$

Theorem 3.13 *(Second Betti's formula) For any* $\underline{u},\underline{v}\in[H^1_{Lamé}(\Omega)]^3$ *there holds*

$$\int_\Omega (\mathcal{L}\underline{v}(x),\,\underline{u}(x))-(\underline{v}(x),\,\mathcal{L}\underline{u}(x))\,\mathrm{d}x=\int_\Gamma(\gamma_1\underline{u}(x),\,\gamma_0\underline{v}(x))-(\gamma_0\underline{u}(x),\,\gamma_1\underline{v}(x))\,\mathrm{d}s_x.$$

Theorem 3.14 *(Somigliana's identity) Let* $\underline{f}\in[L^2(\Omega)]^3$ *and* $\underline{u}\in[H^1(\Omega)]^3$ *be a distributional solution of the equation*

$$\mathcal{L}\underline{u}(x)=\underline{f}(x)\quad \text{for } x\in\Omega.$$

Then

$$\begin{aligned} u_l(x) &= \int_\Omega(\underline{f}(y),\,\underline{U}_l(x,y))\,\mathrm{d}y+\int_\Gamma(\gamma_1\underline{u}(y),\,\underline{U}_l(x,y))\,\mathrm{d}s_y \\ &\quad -\int_\Gamma(\gamma_0\underline{u}(y),\,\gamma_{1,y}\underline{U}_l(x,y))\,\mathrm{d}s_y \end{aligned} \quad (3.29)$$

for $x\in\Omega$ *and* $l=1,2,3$.

3.2.2 Boundary integral operators

By applying the interior trace and boundary stress operators to the Somigliana identity [41], we can get the following system of integral equations

$$\begin{pmatrix}\gamma_0\underline{u}\\ \gamma_1\underline{u}\end{pmatrix}=\begin{pmatrix}\frac{1}{2}I-K^{\text{Lamé}} & V^{\text{Lamé}}\\ D^{\text{Lamé}} & \frac{1}{2}I+(K^{\text{Lamé}})'\end{pmatrix}\begin{pmatrix}\gamma_0\underline{u}\\ \gamma_1\underline{u}\end{pmatrix}+\begin{pmatrix}N_0^{\text{Lamé}}\underline{f}\\ N_1^{\text{Lamé}}\underline{f}\end{pmatrix} \quad (3.30)$$

valid on Γ, where we define for $x\in\Gamma$ and $i=1,2,3$ the `single layer potential operator` by

$$(V^{\text{Lamé}}\underline{t})_i(x):=\int_\Gamma(\underline{t}(y),\,\underline{U}_i(x,y))\,\mathrm{d}s_y, \quad (3.31)$$

the `double layer potential operator` by

$$(K^{\text{Lamé}}\underline{u})_i(x) := \int_\Gamma (\underline{u}(y),\, \gamma_{1,y}\underline{U}_i(x,y))\, \mathrm{d}s_y, \tag{3.32}$$

the `adjoint double layer potential operator` by

$$((K^{\text{Lamé}})'\underline{t})_i(x) := \int_\Gamma (\underline{t}(y),\, \gamma_{1,x}\underline{U}_i(x,y))\, \mathrm{d}s_y, \tag{3.33}$$

and finally the `hypersingular integral operator` by

$$(D^{\text{Lamé}}\underline{u})_i(x) := -\gamma_{1,x} \int_\Gamma (\underline{u}(y),\, \gamma_{1,y}\underline{U}_i(x,y))\, \mathrm{d}s_y. \tag{3.34}$$

The `Newton` or `volume potential operators` used in (3.30) are given for $x \in \Gamma$ and $i = 1, 2, 3$ by

$$(N_0^{\text{Lamé}}\underline{f})_i(x) := \int_\Omega (\underline{f}(y),\, \underline{U}_i(x,y))\, \mathrm{d}y \tag{3.35}$$

and

$$(N_1^{\text{Lamé}}\underline{f})_i(x) := \int_\Omega (\underline{f}(y),\, \gamma_{1,x}\underline{U}_i(x,y))\, \mathrm{d}y.$$

Let us now define the duality coupling of $\underline{u} \in [H^{-1/2}(\Gamma)]^3$ and $\underline{v} \in [H^{1/2}(\Gamma)]^3$ as

$$\langle \underline{u},\, \underline{v} \rangle_\Gamma := \int_\Gamma (\underline{u}(y),\, \underline{v}(y))\, \mathrm{d}s_y.$$

The boundary integral operators have the following properties [41, 43]:

Theorem 3.15 *The single layer potential operator*

$$V^{Lamé} :\ [H^{-1/2}(\Gamma)]^3 \mapsto [H^{1/2}(\Gamma)]^3$$

given by (3.31) is linear, bounded, symmetric, and $[H^{-1/2}(\Gamma)]^3$*–elliptic satisfying*

$$\left\langle V^{Lamé}\underline{w},\, \underline{w} \right\rangle_\Gamma \geq \alpha^{V^{Lamé}} \|\underline{w}\|^2_{[H^{-1/2}(\Gamma)]^3} \quad \textit{for all } \underline{w} \in [H^{-1/2}(\Gamma)]^3.$$

Moreover, for $\underline{w} \in [L^\infty(\Gamma)]^3$ *the operator* $V^{Lamé}$ *can be represented by*

$$(V^{Lamé}\underline{w})_k(x) = \frac{1+\nu}{2E(1-\nu)} \left((3-4\nu)(Vw_k)(x) + \sum_{l=1}^{3} (V_{kl}w_l)(x) \right),$$

where

$$(Vw_k)(x) = \frac{1}{4\pi} \int_\Gamma \frac{w_k(y)}{\|x-y\|} \, \mathrm{d}s_y$$

is the single layer potential operator of the Laplace operator in $\mathbb{R}^3$ *and*

$$(V_{kl} w_l)(x) = \frac{1}{4\pi} \int_\Gamma \frac{(x_k - y_k)(x_l - y_l)}{\|x-y\|^3} w_l(y) \, \mathrm{d}s_y$$

for $k, l = 1, 2, 3$ *and* $x \in \Gamma$.

Let

$$\mathcal{R} := \mathrm{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -x_3 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ 0 \\ -x_1 \end{pmatrix} \right\}$$

be the solution space of the homogeneous Neumann boundary value problem

$$\begin{aligned} \mathcal{L}\underline{u}(x) &= \underline{0} \quad \text{for } x \in \Omega, \\ \gamma_1 \underline{u}(x) &= \underline{0} \quad \text{for } x \in \Gamma. \end{aligned} \tag{3.36}$$

We call $\mathcal{R}$ the space of the rigid body motions: translation and rotation.

Theorem 3.16 *The double layer potential operator*

$$K^{Lamé} : \ [H^{1/2}(\Gamma)]^3 \mapsto [H^{1/2}(\Gamma)]^3$$

given by (3.32) is linear and bounded and for continuous $\underline{v}$ *there holds the representation*

$$(K^{Lamé} \underline{v})_l(x) = (Kv_l)(x) - (V(M(\partial, \underline{n})\underline{v})_l)(x) + 2\mu(V^{Lamé} M(\partial, \underline{n})\underline{v})_l(x), \tag{3.37}$$

for $x \in \Gamma$ *and* $l = 1, 2, 3$, *where* V *and* K *are the single and double layer potential operators for the Laplace operator in* $\mathbb{R}^3$, *in particular,*

$$(Kv_l)(x) = \frac{1}{4\pi} \int_\Gamma \frac{(x - y, \underline{n}(y))}{\|x-y\|^3} v_l(y) \, \mathrm{d}s_y,$$

and $V^{Lamé}$ *is the single layer potential operator of linear elastostatics. The operator* $M(\partial, \underline{n})$ *appearing in (3.37) is the matrix surface curl operator defined by*

$$M_{ij}(\partial_y, \underline{n}_y) := n_j(y) \frac{\partial}{\partial y_i} - n_i(y) \frac{\partial}{\partial y_j} \quad \text{for } i, j = 1, 2, 3. \tag{3.38}$$

Moreover, it holds

$$\left(\frac{1}{2}I + K^{Lamé}\right)\underline{v}(x) = \underline{0} \qquad \text{for all } \underline{v} \in \mathcal{R},\ x \in \Gamma.$$

Theorem 3.17 *The adjoint double layer potential operator*

$$(K^{Lamé})' :\ [H^{-1/2}(\Gamma)]^3 \mapsto [H^{-1/2}(\Gamma)]^3$$

given by (3.33) is linear and bounded.

Theorem 3.18 *The hypersingular integral operator*

$$D^{Lamé} :\ [H^{1/2}(\Gamma)]^3 \mapsto [H^{-1/2}(\Gamma)]^3$$

given by (3.34) is linear, bounded, symmetric, and $[H^{1/2}(\Gamma)]^3$–semi–elliptic with

$$\left\langle D^{Lamé}\underline{w},\, \underline{w}\right\rangle_\Gamma \geq \alpha^{D^{Lamé}} \|\underline{w}\|^2_{[H^{1/2}(\Gamma)]^3} \qquad \text{for all } \underline{w} \in [H^{1/2}(\Gamma)]^3_{/\operatorname{Ker} D^{Lamé}},$$

where $[H^{1/2}(\Gamma)]^3_{/\operatorname{Ker} D^{Lamé}}$ is the space of all functions from $[H^{1/2}(\Gamma)]^3$ that are orthogonal to $\operatorname{Ker} D^{Lamé}$ and $\operatorname{Ker} D^{Lamé}$ is equal to the trace space of $\mathcal{R}$, i.e.

$$(D^{Lamé}\underline{v})(x) = \underline{0} \qquad \text{for all } \underline{v} \in \mathcal{R},\ x \in \Gamma.$$

Moreover, for $\underline{u}$, $\underline{v} \in [H^{1/2}(\Gamma) \cap C(\Gamma)]^3$ we have the representation

$$\begin{aligned}
&\left\langle D^{Lamé}\underline{u},\, \underline{v}\right\rangle_\Gamma = \\
&\quad \frac{\mu}{4\pi}\int_\Gamma\int_\Gamma \frac{1}{\|x-y\|}\sum_{k=1}^{3}\left(\frac{\partial}{\partial S_k(y)}\underline{u}(y),\ \frac{\partial}{\partial S_k(y)}\underline{v}(x)\right)\mathrm{d}s_y\,\mathrm{d}s_x\ + \\
&\quad \frac{\mu}{2\pi}\int_\Gamma\int_\Gamma \frac{1}{\|x-y\|}\Big(M(\partial_x,\underline{n}(x))\underline{v}(x),\ M(\partial_y,\underline{n}(y))\underline{u}(y)\Big)\,\mathrm{d}s_y\,\mathrm{d}s_x\ - \\
&\quad 4\mu^2\int_\Gamma\int_\Gamma \Big(M(\partial_x,\underline{n}(x))\underline{v}(x),\, U(x,y)M(\partial_y,\underline{n}(y))\underline{u}(y)\Big)\,\mathrm{d}s_y\,\mathrm{d}s_x\ + \\
&\quad \frac{\mu}{4\pi}\int_\Gamma\int_\Gamma \sum_{i,j,k=1}^{3} M_{kj}(\partial_x,\underline{n}(x))\underline{v}_i(x)\frac{1}{\|x-y\|}M_{ki}(\partial_y,\underline{n}(y))\underline{u}_j(y)\,\mathrm{d}s_y\,\mathrm{d}s_x,
\end{aligned}$$

where the matrix surface curl operator $M(\partial,\underline{n})$ is defined by (3.38) and

$$\frac{\partial}{\partial S_1(x)} := M_{32}(\partial_x,\underline{n}(x)),$$

$$\frac{\partial}{\partial S_2(x)} := M_{13}(\partial_x, \underline{n}(x)),$$
$$\frac{\partial}{\partial S_3(x)} := M_{21}(\partial_x, \underline{n}(x)).$$

3.2.3 The Steklov–Poincaré Operator

Since the single layer potential operator is $[H^{-1/2}(\Gamma)]^3$– elliptic, its inversion exists. From the first equation of (3.30) we get the Dirichlet–Neumann map

$$\gamma_1 \underline{u}(x) = (V^{\text{Lamé}})^{-1} \left(\frac{1}{2} I + K^{\text{Lamé}} \right) \gamma_0 \underline{u}(x) - (V^{\text{Lamé}})^{-1} (N_0^{\text{Lamé}} \underline{f})(x) \quad (3.39)$$

for $x \in \Gamma$. By insertion of (3.39) into the second equation of (3.30) we get another representation of the Dirichlet–Neumann map

$$\begin{aligned} \gamma_1 \underline{u}(x) \;=\; & \left[\left(\frac{1}{2} I + (K^{\text{Lamé}})' \right) (V^{\text{Lamé}})^{-1} \left(\frac{1}{2} I + K^{\text{Lamé}} \right) + D^{\text{Lamé}} \right] \gamma_0 \underline{u}(x) + \\ & \left[N_1^{\text{Lamé}} - \left(\frac{1}{2} I + (K^{\text{Lamé}})' \right) (V^{\text{Lamé}})^{-1} N_0^{\text{Lamé}} \right] \underline{f}(x) \end{aligned} \quad (3.40)$$

for $x \in \Gamma$. Now let us define the `Steklov-Poincaré operator` by the equivalent representations

$$\begin{aligned} S^{\text{Lamé}} &:= (V^{\text{Lamé}})^{-1} \left(\frac{1}{2} I + K^{\text{Lamé}} \right) && (3.41) \\ &= \left(\frac{1}{2} I + (K^{\text{Lamé}})' \right) (V^{\text{Lamé}})^{-1} \left(\frac{1}{2} I + K^{\text{Lamé}} \right) + D^{\text{Lamé}} && (3.42) \end{aligned}$$

and the `Newton operator` by the equivalent representations

$$\begin{aligned} -N^{\text{Lamé}} &:= -(V^{\text{Lamé}})^{-1} N_0^{\text{Lamé}} && (3.43) \\ &= N_1^{\text{Lamé}} - \left(\frac{1}{2} I + (K^{\text{Lamé}})' \right) (V^{\text{Lamé}})^{-1} N_0^{\text{Lamé}}, \end{aligned}$$

so that we can rewrite equations (3.39) and (3.40) as

$$\gamma_1 \underline{u}(x) = (S^{\text{Lamé}} \gamma_0 \underline{u})(x) - (N^{\text{Lamé}} \underline{f})(x) \quad \text{for } x \in \Gamma. \quad (3.44)$$

Similarly to the case of the Laplace operator, the representation (3.41) together with the Galerkin discretization leads typically to a non–symmetric

stiffness matrix. As stated before, the symmetry of stiffness matrix shall be essential in our analysis, so that we shall consider only the formally symmetric representation (3.42).

Let us introduce a mapping $\mathcal{P}^{\text{Lamé}} : \ [H^{1/2}(\Gamma)]^3 \mapsto [H^1(\Omega)]^3$ so that for a $\underline{u} \in [H^{1/2}(\Gamma)]^3$ we define $\mathcal{P}^{\text{Lamé}}\underline{u} \in [H^1(\Omega)]^3$ as a weak solution of the Dirichlet problem

$$\begin{aligned} \mathcal{L}\underline{v}(x) &= \underline{0} \qquad \text{for } x \in \Omega, \\ \underline{v}(x) &= \underline{u}(x) \quad \text{for } x \in \Gamma. \end{aligned}$$

Let us note that $\mathcal{P}^{\text{Lamé}}$ is linear and bounded on $[H^{1/2}(\Gamma)]^3$. It can be further shown that for all $\underline{u}, \underline{v} \in [H^{1/2}(\Gamma)]^3$ we have

$$\left\langle S^{\text{Lamé}}\underline{u},\, \underline{v} \right\rangle_\Gamma = \int\limits_\Omega W(\mathcal{P}^{\text{Lamé}}\underline{u}(x), \mathcal{P}^{\text{Lamé}}\underline{v}(x))\,\mathrm{d}x \tag{3.45}$$

with W given by (3.28).

Furthermore, let us recall the definition of the function space

$$H_0^{1/2}(\Gamma, \Gamma_u) := \{v \in H^{1/2}(\Gamma) : \ v(x) = 0 \ \text{ for } x \in \Gamma_u\},$$

where $\Gamma_u \subset \Gamma$ and $\operatorname{meas}\Gamma_u > 0$.

Theorem 3.19 *The Steklov–Poincaré operator*

$$S^{Lamé} : \ [H^{1/2}(\Gamma)]^3 \mapsto [H^{-1/2}(\Gamma)]^3$$

is a linear, bounded, symmetric, and semi–elliptic mapping satisfying

$$\left\langle S^{Lamé}\underline{w},\, \underline{w} \right\rangle_\Gamma \geq \alpha^{D^{Lamé}} \|\underline{w}\|^2_{[H^{1/2}(\Gamma)]^3} \qquad \textit{for all } \underline{w} \in [H^{1/2}(\Gamma)]^3_{/\operatorname{Ker} D^{Lamé}},$$

where $[H^{1/2}(\Gamma)]^3_{/\operatorname{Ker} D^{Lamé}}$ *is the space of all functions from* $[H^{1/2}(\Gamma)]^3$ *that are orthogonal to* $\operatorname{Ker} D^{Lamé}$. *Note that the positive constant* $\alpha^{D^{Lamé}}$ *is the same as that in Theorem 3.18.*

Moreover, $S^{Lamé}$ *is elliptic on* $[H_0^{1/2}(\Gamma, \Gamma_u)]^3$.

`Proof.` We proceed analogously to the proof of Theorem (3.9). In particular, relation (3.45) is useful here and to prove the $[H_0^{1/2}(\Gamma, \Gamma_u)]^3$–ellipticity, we also have to use the Korn inequality (3.27).

□

Let us for a while put $\underline{u} \in \mathcal{R}$. Then $\underline{u}$ solves (3.36) and, by (3.44), we immediately get

$$(S^{\text{Lamé}}\underline{u})(x) = \underline{0} \qquad \text{for all } x \in \Gamma. \tag{3.46}$$

3.2.4 The Newton operator

Analogously to the case of the Laplace operator, in what follows, we shall consider only the alternative representation (3.43) of the operator $N^{\text{Lamé}}$.

Lemma 3.3 *The Newton potential operator*

$$N_0^{Lamé} : \ [L^2(\Omega)]^3 \mapsto [H^{1/2}(\Gamma)]^3$$

given by (3.35) is a linear and bounded mapping.

`Proof.` See [45].

□

Now from the linearity and boundedness of $N_0^{\text{Lamé}}$ and $V^{\text{Lamé}}$ we can easily conclude the following theorem.

Theorem 3.20 *The Newton operator*

$$N^{Lamé} : \ [L^2(\Omega)]^3 \mapsto [H^{-1/2}(\Gamma)]^3$$

is a linear and bounded mapping.

Chapter 4

Domain decomposition, Boundary variational inequality

4.1 Laplace operator

4.1.1 Model contact problem

Let us consider the domains

$$\Omega^1 := (0,\,1)^2 \quad \text{and} \quad \Omega^2 := (1,\,2) \times (0,\,1)$$

with boundaries Γ^1 and Γ^2, respectively. Moreover, we denote

$$\begin{aligned}
\Gamma_u^1 &:= \{(0,\,x_2) :\ x_2 \in [0,\,1]\}\,,\\
\Gamma_c &:= \{(1,\,x_2) :\ x_2 \in [0,\,1]\}\,,\\
\Gamma_f^1 &:= \Gamma^1 \setminus \{\Gamma_u^1 \cup \Gamma_c\}\,,\\
\Gamma_f^2 &:= \Gamma^2 \setminus \Gamma_c.
\end{aligned}$$

For better comprehension, see the whole situation depicted in Figure 4.1. Furthermore, let a function $f \in L^2(\Omega^1 \cup \Omega^2)$ satisfy

$$\int_{\Omega^2} f(x)\,\mathrm{d}x < 0. \tag{4.1}$$

We shall now be focused on finding a sufficiently smooth $(u^1,\,u^2)$ satisfying

$$\begin{aligned}
-\triangle u^m(x) &= f(x) \quad && \text{for } x \in \Omega^m,\ m = 1,\,2,\\
u^1(x) &= 0 && \text{for } x \in \Gamma_u^1,\\
t^m(x) := \frac{\partial u^m}{\partial \underline{n}^m}(x) &= 0 && \text{for } x \in \Gamma_f^m,\ m = 1,\,2,
\end{aligned} \tag{4.2}$$

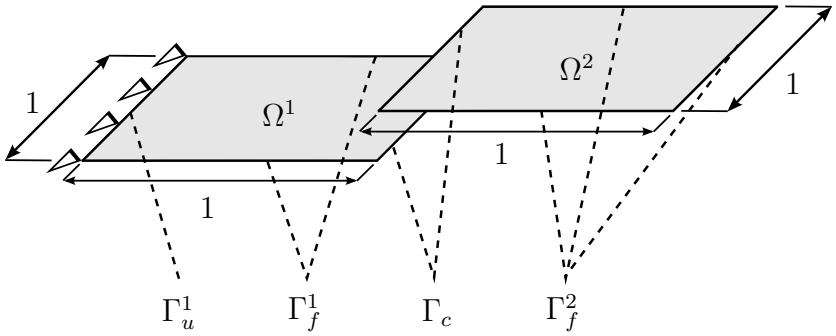

Figure 4.1: Geometry of the 2D model contact problem

together with the contact conditions

$$\left.\begin{aligned} u^2(x) - u^1(x) &\geq 0 \\ t^2(x) &\geq 0 \\ t^2(x)(u^2(x) - u^1(x)) &= 0 \\ t^1(x) + t^2(x) &= 0 \end{aligned}\right\} \text{ for } x \in \Gamma_c \tag{4.3}$$

with $\underline{n}^m(x)$ denoting the exterior unit normal vector of Ω^m defined for almost all $x \in \Gamma^m$, $m = 1, 2$.

Let us further describe the interpretation of the problem (4.2), (4.3). The functions u^1 and u^2 represent vertical displacements of two thin membranes of the shapes Ω^1 and Ω^2, respectively. Both membranes are stretched by horizontal forces with the unit density and they are pushed down by a force with the density f. The left membrane is fixed on the part Γ_u^1; both membranes are free on the parts Γ_f^m. The right membrane floats due to the lack of the Dirichlet condition. The membranes can eventually be in a contact on Γ_c and we assume that the right membrane cannot penetrate the left one. At contact points, which are a priori unknown, the right membrane can press the left membrane down. The third contact condition is the so–called complementarity condition. It says that at points where the membranes do not touch, the right membrane stays free. "Smoothness of the contact" is ensured by the last condition.

4.1.2 Domain decomposition

We now decompose each domain Ω^m into $p^m \in \mathbb{N}$ non–overlapping subdomains, i.e.

$$\overline{\Omega^m} = \bigcup_{i=1}^{p^m} \overline{\Omega_i^m}, \quad \Omega_i^m \cap \Omega_j^m = \emptyset \quad \text{for } i \neq j.$$

Moreover, let us assume that every subdomain Ω_i^m is Lipschitz and fulfils the condition

$$\operatorname{diam}\Omega_i^m < 1. \tag{4.4}$$

Let us denote

$$\Gamma_i^m := \partial\Omega_i^m, \quad \Gamma_{ij}^m := \Gamma_i^m \cap \Gamma_j^m \quad \text{for } i \neq j, \quad \Gamma_s^m := \bigcup_{i=1}^{p^m} \Gamma_i^m$$

and call Γ_s^m the skeleton of Ω^m. Furthermore, let $\underline{n}_i^m(x)$ denote the exterior unit normal vector of Ω_i^m at the point $x \in \Gamma_i^m$.

After we provided both domains with the decompositions, we consider – instead of the original problem (4.2), (4.3) – the system of local boundary value problems

$$\begin{aligned} -\triangle u_i^m(x) &= f(x) && \text{for } x \in \Omega_i^m, \\ u_i^1(x) &= 0 && \text{for } x \in \Gamma_i^m \cap \Gamma_u^1, \\ t_i^m(x) := \frac{\partial u_i^m}{\partial \underline{n}_i^m}(x) &= 0 && \text{for } x \in \Gamma_i^m \cap \Gamma_f^m \end{aligned} \tag{4.5}$$

together with the local contact conditions

$$\left.\begin{aligned} u_j^2(x) - u_i^1(x) &\geq 0 \\ t_j^2(x) &\geq 0 \\ t_j^2(x)(u_j^2(x) - u_i^1(x)) &= 0 \\ t_j^2(x) + t_i^1(x) &= 0 \end{aligned}\right\} \text{ for } x \in \Gamma_i^1 \cap \Gamma_j^2 \tag{4.6}$$

and with the so–called transmission conditions

$$\left.\begin{aligned} u_i^m(x) &= u_j^m(x) \\ t_i^m(x) + t_j^m(x) &= 0 \end{aligned}\right\} \text{ for } x \in \Gamma_{ij}^m. \tag{4.7}$$

The latter conditions link the local problems together. They ensure "smoothness" across the artificial interfaces Γ_{ij}^m.

4.1.3 Boundary weak formulation

First, let us define $H^{1/2}(\Gamma_s^m)$ as the trace space of $H^1(\Omega^m)$ restricted to the skeleton Γ_s^m equipped with the norm

$$\|v\|_{H^{1/2}(\Gamma_s^m)} := \sqrt{\sum_{i=1}^{p^m} \left\| v|_{\Gamma_i^m} \right\|^2_{H^{1/2}(\Gamma_i^m)}}.$$

Moreover,

$$H_0^{1/2}(\Gamma_s^1,\, \Gamma_u^1) := \left\{ v \in H^{1/2}(\Gamma_s^1) :\ v(x) = 0 \quad \text{for } x \in \Gamma_u^1 \right\}.$$

Now we introduce the space

$$\mathcal{V} := H_0^{1/2}(\Gamma_s^1,\, \Gamma_u^1) \times H^{1/2}(\Gamma_s^2) \tag{4.8}$$

equipped with the norm

$$\left\| (v^1,\, v^2) \right\|_{\mathcal{V}} := \sqrt{\sum_{m=1}^{2} \|v^m\|^2_{H^{1/2}(\Gamma_s^m)}}.$$

Let

$$\mathcal{K} := \left\{ (v^1,\, v^2) \in \mathcal{V} :\ v^2(x) - v^1(x) \geq 0 \quad \text{for } x \in \Gamma_c \right\} \tag{4.9}$$

and

$$v_i^m := v^m|_{\Gamma_i^m} \quad \text{for } (v^1,\, v^2) \in \mathcal{V}.$$

Definition 4.1 *Function* $(u^1,\, u^2) \in \mathcal{K}$ *is a* **`boundary weak solution`** *of (4.5)–(4.7) if*

$$\sum_{m=1}^{2} \sum_{i=1}^{p^m} \int_{\Gamma_i^m} (S_i^m u_i^m)(x)(v_i^m - u_i^m)(x)\, \mathrm{d}s \geq \sum_{m=1}^{2} \sum_{i=1}^{p^m} \int_{\Gamma_i^m} (N_i^m f)(x)(v_i^m - u_i^m)(x)\, \mathrm{d}s \tag{4.10}$$

for all $(v^1,\, v^2) \in \mathcal{K}$.

The operators S_i^m and N_i^m are the local Steklov–Poincaré and Newton operators, respectively, corresponding to the local domains Ω_i^m with boundaries Γ_i^m.

Now let us make the following ideas and examine the situation thoroughly. First, let $(u^1,\, u^2) \in \mathcal{K}$ be a smooth solution of (4.10). We further assume that the function f and all the boundaries Γ^m and Γ_i^m are smooth, too.

`i)` There exists a unique weak solution of the problem

$$\begin{aligned} -\triangle \widetilde{u}_i^m(x) &= f(x) \qquad \text{for } x \in \Omega_i^m, \\ \widetilde{u}_i^m(x) &= u_i^m(x) \quad \text{for } x \in \Gamma_i^m \end{aligned} \tag{4.11}$$

that is also a classical solution of (4.11). Due to (4.4) we get, by (3.19),

$$t_i^m(x) = \frac{\partial \widetilde{u}_i^m}{\partial \underline{n}_i^m}(x) = (S_i^m u_i^m)(x) - (N_i^m f)(x) \quad \text{for } x \in \Gamma_i^m.$$

Hence for all $(v^1,\, v^2) \in \mathcal{K}$

$$\sum_{m=1}^{2} \sum_{i=1}^{p^m} \int\limits_{\Gamma_i^m} t_i^m(x)(v_i^m - u_i^m)(x)\,\mathrm{d}s \geq 0.$$

`ii)` Let $k \in \{1, \ldots, p^1\}$ and $l \in \{1, \ldots, p^2\}$ be arbitrary such that $\Gamma_k^1 \cap \Gamma_l^2 \neq \emptyset$. Let

$$\chi \in \left\{ v := \widetilde{v}|_{\Gamma_s^1 \cup \Gamma_s^2} :\ \widetilde{v} \in C^\infty(\overline{\Omega^1 \cup \Omega^2}) \ \text{ and } \ \operatorname{supp} v \subset \Gamma_k^1 \cap \Gamma_l^2 \right\}.$$

Then

$$(v^1,\, v^2) := \left(u^1 + \chi|_{\Gamma_s^1}\,,\, u^2 + \chi|_{\Gamma_s^2} \right) \in \mathcal{K}$$

and thus

$$\int\limits_{\Gamma_k^1 \cap \Gamma_l^2} t_k^1(x)\,\chi(x)\,\mathrm{d}s \ + \int\limits_{\Gamma_k^1 \cap \Gamma_l^2} t_l^2(x)\,\chi(x)\,\mathrm{d}s = \int\limits_{\Gamma_k^1 \cap \Gamma_l^2} (t_k^1(x) + t_l^2(x))\,\chi(x)\,\mathrm{d}s \geq 0.$$

Since χ may be chosen arbitrarily, we obtain

$$t_k^1(x) + t_l^2(x) = 0 \qquad \text{for all } x \in \Gamma_k^1 \cap \Gamma_l^2$$

(compare with the fourth condition of (4.6)).

In what follows, we shall need the definition

$$\mathcal{D}(\Gamma_*) := \left\{ v := \widetilde{v}|_{\Gamma_s^2} :\ \widetilde{v} \in C^\infty(\overline{\Omega^2}) \ \text{ and } \ \operatorname{supp} v \subset \Gamma_* \right\}$$

for every $\Gamma_* \subset \Gamma_s^2$.

`iii)` Let $k \in \{1, \ldots, p^1\}$ and $l \in \{1, \ldots, p^2\}$ be arbitrary such that $\Gamma_k^1 \cap \Gamma_l^2 \neq \emptyset$. Let

$$\chi \in \left\{ v \in \mathcal{D}(\Gamma_k^1 \cap \Gamma_l^2) :\ v(x) \geq 0 \quad \text{for } x \in \operatorname{supp} v \right\}.$$

Then

$$(v^1,\, v^2) := (u^1,\, u^2 + \chi) \in \mathcal{K}$$

and thus

$$\int\limits_{\Gamma_k^1 \cap \Gamma_l^2} t_l^2(x)\chi(x)\,\mathrm{d}s \geq 0.$$

Since $\chi \geq 0$ may be chosen arbitrarily, we get

$$t_l^2(x) \geq 0 \qquad \text{for all } x \in \Gamma_k^1 \cap \Gamma_l^2$$

(compare with the second condition of (4.6)). Furthermore, let $\overline{x} \in \Gamma_k^1 \cap \Gamma_l^2$ satisfy $u_l^2(\overline{x}) - u_k^1(\overline{x}) > 0$. Then for a small enough $\varepsilon > 0$ we have

$$u_l^2(x) - u_k^1(x) > 0 \qquad \text{for all } x \in \mathcal{U}_\varepsilon(\overline{x}) := \left\{ x \in \Gamma_k^1 \cap \Gamma_l^2 : \ \|\overline{x} - x\| < \varepsilon \right\}.$$

Let $\chi \in \mathcal{D}(\mathcal{U}_\varepsilon(\overline{x}))$. Obviously,

$$(v^1,\, v^2) := (u^1,\, u^2 + r\,\chi) \in \mathcal{K}$$

for all $r \in (-\delta,\, \delta)$ if $\delta > 0$ is small enough. Thus

$$r \int\limits_{\mathcal{U}_\varepsilon(\overline{x})} t_l^2(x)\,\chi(x)\,\mathrm{d}s \geq 0.$$

Since χ may be chosen arbitrarily, we obtain

$$t_l^2(x) = 0 \qquad \text{for all } x \in \mathcal{U}_\varepsilon(\overline{x}),$$

i.e.

$$t_l^2(x) = 0 \quad \text{for all } x \in \Gamma_k^1 \cap \Gamma_l^2 \quad \text{such that} \quad u_l^2(x) - u_k^1(x) > 0$$

(compare with the third condition of (4.6)). The first condition of (4.6) is clearly satisfied due to the choice of $\mathcal{K}$.

`iv)` Let $l \in \{1, \ldots, p^2\}$ be arbitrary such that $\Gamma_l^2 \cap \Gamma_f^2 \neq \emptyset$. Let $\chi \in \mathcal{D}(\Gamma_l^2 \cap \Gamma_f^2)$. Then

$$(v^1,\, v^2) := (u^1,\, u^2 + \chi) \in \mathcal{K}$$

and thus

$$\int\limits_{\Gamma_l^2 \cap \Gamma_f^2} t_l^2(x)\,\chi(x)\,\mathrm{d}s \geq 0.$$

Since χ may be chosen arbitrarily, we get

$$t_l^2(x) = 0 \qquad \text{for all } x \in \Gamma_l^2 \cap \Gamma_f^2$$

(compare with the third condition of (4.5)). In the case of the left membrane we proceed analogously. The second condition of (4.5) follows immediately from the choice of $\mathcal{K}$.

`v)` Let $k,\, l \in \{1, \ldots, p^2\}$ be arbitrary such that $k \neq l$ and $\Gamma_{kl}^2 \neq \emptyset$. Let $\chi \in \mathcal{D}(\Gamma_{kl}^2)$. Then

$$(v^1,\, v^2) := (u^1,\, u^2 + \chi) \in \mathcal{K}$$

and thus

$$\int_{\Gamma_{kl}^2} t_k^2(x)\,\chi(x)\,\mathrm{d}s \;+\int_{\Gamma_{kl}^2} t_l^2(x)\,\chi(x)\,\mathrm{d}s = \int_{\Gamma_{kl}^2} (t_k^2(x)+t_l^2(x))\,\chi(x)\,\mathrm{d}s \geq 0.$$

Since χ may be chosen arbitrarily, we get

$$t_k^2(x)+t_l^2(x)=0 \qquad \text{for all } x\in\Gamma_{kl}^2$$

(compare with the second condition of (4.7)). In the case of the left membrane we proceed analogously. The first condition of (4.7) follows from the assumed smoothness of $(u^1,\,u^2)$.

We have just checked that if we assume smoothness of the boundary weak solution $(u^1,\,u^2)$ of (4.5)–(4.7), the function f, and all the boundaries Γ^m and Γ_i^m, then the unique solution of the system of the Dirichlet boundary value problems (4.11) is a classical solution of (4.5)–(4.7).

4.1.4 Minimization of energy functional

Theorem 4.1 *Let $\mathcal{V}$ and $\mathcal{K}$ be given by (4.8) and (4.9), respectively. Then $\mathcal{K}$ is a nonempty, closed, and convex set that is not a subspace of $\mathcal{V}$.*

`Proof. i)` In order to prove the closeness of $\mathcal{K}$, we have to show that

$$\left.\begin{aligned} \{(v_n^1,\,v_n^2)\}_{n=1}^{\infty} &\subset \mathcal{K} \\ (v_n^1,\,v_n^2) &\to (v^1,\,v^2) \text{ in } \mathcal{V} \end{aligned}\right\} \Rightarrow (v^1,\,v^2)\in\mathcal{K}.$$

Since for all $n\in\mathbb{N}$ we have $v_n^2-v_n^1\geq 0$ on Γ_c, we obtain

$$v^2(x)-v^1(x)=\lim(v_n^2(x)-v_n^1(x))\geq 0 \qquad \text{for } x\in\Gamma_c,$$

i.e. $(v^1,\,v^2)\in\mathcal{K}$.

`ii)` Let $(u^1,\,u^2)$, $(v^1,\,v^2)\in\mathcal{K}$, $t\in(0,\,1)$, and $x\in\Gamma_c$ be arbitrary. To prove the convexity of $\mathcal{K}$, we have to show that $t(u^1,\,u^2)+(1-t)(v^1,\,v^2)\in\mathcal{K}$. This is straightforward:

$$tu^2(x)+(1-t)v^2(x)-(tu^1(x)+(1-t)v^1(x))=$$

$$t(u^2(x)-u^1(x))+(1-t)(v^2(x)-v^1(x))\geq 0.$$

`iii)` Since for $(u^1,\,u^2)\in\mathcal{K}$ satisfying $u^2(x)-u^1(x)>0$ for $x\in\Gamma_c$ it holds that $-1\cdot(u^1,\,u^2)\notin\mathcal{K}$, we get that $\mathcal{K}$ is not a subspace of $\mathcal{V}$.

□

Now let us use the notations

$$u := (u^1,\, u^2) \quad \text{and} \quad v := (v^1,\, v^2).$$

Furthermore, let

$$\mathcal{A}(u,\, v) := \sum_{m=1}^{2} \sum_{i=1}^{p^m} \langle S_i^m u_i^m,\, v_i^m \rangle_{L^2(\Gamma_i^m)} \qquad \text{for } u,\, v \in \mathcal{V}$$

and

$$\mathcal{F}(v) := \sum_{m=1}^{2} \sum_{i=1}^{p^m} \langle N_i^m f,\, v_i^m \rangle_{L^2(\Gamma_i^m)} \qquad \text{for } v \in \mathcal{V}.$$

Theorem 4.2 *The variational inequality (4.10), i.e. the problem to find $u \in \mathcal{K}$ satisfying*

$$\mathcal{A}(u,\, v-u) \geq \mathcal{F}(v-u) \qquad \textit{for all } v \in \mathcal{K}$$

is equivalent to the problem: find $u \in \mathcal{K}$ such that

$$\mathcal{J}(u) = \min \{\mathcal{J}(v) :\ v \in \mathcal{K}\}, \tag{4.12}$$

where $\mathcal{J}$ is the energy functional (2.5), i.e.

$$\mathcal{J}(v) := \frac{1}{2}\,\mathcal{A}(v,\, v) - \mathcal{F}(v).$$

`Proof.` According to Theorems 2.7 and 4.1, it is sufficient to show that $\mathcal{A}$ is a bounded, symmetric, and semi–elliptic bilinear form on $\mathcal{V}$ and that $\mathcal{F}$ is a linear, bounded functional on $\mathcal{V}$.

`i)` Since S_i^m is linear on $H^{1/2}(\Gamma_i^m)$ (see Theorem 3.9), $\mathcal{A}$ is indeed a bilinear form on $\mathcal{V}$. Since S_i^m is bounded on $H^{1/2}(\Gamma_i^m)$ (see Theorem 3.9), there are constants $M_i^m > 0$ such that for all $u,\, v \in \mathcal{V}$

$$\begin{aligned}
|\mathcal{A}(u,\, v)| &\leq \sum_{m=1}^{2} \sum_{i=1}^{p^m} \left| \langle S_i^m u_i^m,\, v_i^m \rangle_{L^2(\Gamma_i^m)} \right| \\
&\leq \sum_{m=1}^{2} \sum_{i=1}^{p^m} M_i^m \, \|u_i^m\|_{H^{1/2}(\Gamma_i^m)} \, \|v_i^m\|_{H^{1/2}(\Gamma_i^m)} \\
&\leq \sum_{m=1}^{2} \sum_{i=1}^{p^m} M_i^m \, \|u\|_{\mathcal{V}} \, \|v\|_{\mathcal{V}} \\
&\leq (p^1 + p^2) \max_{\substack{m\,=\,1,\,2 \\ i\,=\,1,\,\ldots,\,p^m}} M_i^m \ \ \|u\|_{\mathcal{V}} \, \|v\|_{\mathcal{V}},
\end{aligned}$$

and therefore $\mathcal{A}$ is bounded on $\mathcal{V}$. The symmetry and semi–ellipticity of $\mathcal{A}$ on $\mathcal{V}$ follow immediately from the symmetry and semi–ellipticity of S_i^m on $H^{1/2}(\Gamma_i^m)$, respectively (see Theorem 3.9).

`ii)` The linearity of $\mathcal{F}$ on $\mathcal{V}$ is obvious. Since N_i^m is bounded on $L^2(\Omega_i^m)$ (see Theorem 3.10), there are constants $L_i^m > 0$ such that for all $v \in \mathcal{V}$

$$\begin{aligned}
|\mathcal{F}(v)| &\leq \sum_{m=1}^{2}\sum_{i=1}^{p^m} \left|\langle N_i^m f,\, v_i^m\rangle_{L^2(\Gamma_i^m)}\right| \\
&\leq \sum_{m=1}^{2}\sum_{i=1}^{p^m} L_i^m \,\|f\|_{L^2(\Omega_i^m)}\, \|v_i^m\|_{H^{1/2}(\Gamma_i^m)} \\
&\leq \sum_{m=1}^{2}\sum_{i=1}^{p^m} L_i^m \,\|f\|_{L^2(\Omega^1\cup\Omega^2)}\, \|v\|_{\mathcal{V}} \\
&\leq (p^1+p^2) \max_{\substack{m=1,2\\ i=1,\ldots,p^m}} L_i^m \;\; \|f\|_{L^2(\Omega^1\cup\Omega^2)}\, \|v\|_{\mathcal{V}}\,,
\end{aligned}$$

and thus the functional $\mathcal{F}$ is bounded on $\mathcal{V}$.

□

Now let us recall assumptions (4.1) and (4.4) which, by Lemma 3.2, give

$$\sum_{i=1}^{p^2} \left\langle N_i^2 f,\, 1\right\rangle_{L^2(\Gamma_i^2)} = \sum_{i=1}^{p^2} \langle f,\, 1\rangle_{L^2(\Omega_i^2)} = \int_{\Omega^2} f(x)\,\mathrm{d}x < 0. \tag{4.13}$$

Inequality (4.13) shall be useful in the proof of the following theorem.

Theorem 4.3 *There is a unique solution of the problem (4.12).*

`Proof. i)` Let us first prove the uniqueness. Let $u, w \in \mathcal{K}$ be two distinct solutions of (4.12), i.e. $u \neq w$. Then for all $v \in \mathcal{K}$

$$\begin{aligned}
\mathcal{A}(u,\, v-u) &\geq \mathcal{F}(v-u), \\
\mathcal{A}(w,\, v-w) &\geq \mathcal{F}(v-w).
\end{aligned}$$

If we substitute v by w and then by u, we have

$$\begin{aligned}
\mathcal{A}(u,\, w-u) &\geq \mathcal{F}(w-u), \\
\mathcal{A}(w,\, u-w) &\geq \mathcal{F}(u-w)
\end{aligned}$$

and by adding these two inequalities and changing the sign, we get

$$\mathcal{A}(w-u,\, w-u) \leq 0,$$

i.e.

$$\sum_{m=1}^{2}\sum_{i=1}^{p^m}\langle S_i^m(w_i^m-u_i^m),\, w_i^m-u_i^m\rangle_{L^2(\Gamma_i^m)}\le 0.$$

By Theorem 3.8 and (3.22), we have that

$$S_i^m v=0 \qquad \text{for all } v\in \operatorname{Ker} D_i^m, \tag{4.14}$$

and therefore

$$\langle S_i^m v,\, v\rangle_{L^2(\Gamma_i^m)}=0 \qquad \text{for all } v\in \operatorname{Ker} D_i^m.$$

Now let $v\in H^{1/2}(\Gamma_i^m)\setminus \operatorname{Ker} D_i^m$ be arbitrary. Then we can write $v=k+\widetilde{v}$, where $k\in \operatorname{Ker} D_i^m$ and $0\neq \widetilde{v}\in H^{1/2}(\Gamma_i^m)_{/\operatorname{Ker} D_i^m}$. Due to (4.14) and the symmetry and $H^{1/2}(\Gamma_i^m)_{/\operatorname{Ker} D_i^m}$–ellipticity of S_i^m (see Theorem 3.9), we obtain

$$\langle S_i^m v,\, v\rangle_{L^2(\Gamma_i^m)}=\langle S_i^m\widetilde{v},\,\widetilde{v}\rangle_{L^2(\Gamma_i^m)}\ge \alpha^{D_i^m}\,\|\widetilde{v}\|^2_{H^{1/2}(\Gamma_i^m)}>0.$$

Thus we get that

$$w_i^m-u_i^m\in \operatorname{Ker} D_i^m.$$

Theorem 3.8 yields

$$\operatorname{Ker} D_i^m=\operatorname{Span}\{1\},$$

and therefore

$$w_i^m(x)=u_i^m(x)+c \quad \text{for } x\in\Gamma_i^m,\ \ c\in\mathbb{R}.$$

Hence

$$(w^1,\,w^2)=(u^1+c^1,\,u^2+c^2),\quad (c^1,\,c^2)\in\mathbb{R}^2\setminus\{(0,\,0)\}.$$

However, since $w^1(x)=u^1(x)=0$ for $x\in\Gamma_u^1$, we get

$$(w^1,\,w^2)=(u^1,\,u^2+c^2),\quad c^2\in\mathbb{R}\setminus\{0\}.$$

Then

$$\mathcal{J}(u+(0,\,c^2))=\mathcal{J}(u)\ \Rightarrow\ \mathcal{F}(u+(0,\,c^2))=\mathcal{F}(u)\ \Rightarrow\ \sum_{i=1}^{p^2}\left\langle N_i^2 f,\,1\right\rangle_{L^2(\Gamma_i^2)}=0$$

which contradicts (4.13). Thus there is at most one solution of (4.12).

`ii)` From the boundedness, symmetry, and semi–ellipticity of the bilinear form $\mathcal{A}$ on $\mathcal{V}$ and from the boundedness of the linear functional $\mathcal{F}$ on $\mathcal{V}$ (all

shown in the proof of the latter theorem), we get, by Theorem 2.6, that $\mathcal{J}$ is continuous and convex on $\mathcal{V}$. By Theorems 2.4 and 4.1, we see that to prove the existence of a solution of problem (4.12), it is sufficient to show that $\mathcal{J}$ is coercive on $\mathcal{K}$.

Let us at least prove the coercivity of $\mathcal{J}$ on $\mathcal{K}$ for

$$p^1 = p^2 = 1.$$

We have

$$\mathcal{J}(v) = \sum_{m=1}^{2} \underbrace{\frac{1}{2}\left\langle S^m v^m,\, v^m\right\rangle_{L^2(\Gamma^m)} - \left\langle N^m f,\, v^m\right\rangle_{L^2(\Gamma^m)}}_{=:\mathcal{J}^m(v)} \qquad \text{for } v := (v^1, v^2) \in \mathcal{K}.$$

Since $v^1 \in H_0^{1/2}(\Gamma^1, \Gamma_u^1)$, we get due to Theorems 3.9 and 3.10

$$\mathcal{J}^1(v) \geq \frac{1}{2}\alpha^{D^1}\|v^1\|^2_{H^{1/2}(\Gamma^1)} - c^1\|v^1\|_{H^{1/2}(\Gamma^1)}. \tag{4.15}$$

Let us further put $v^2 = \widetilde{v} + k$, where $\widetilde{v} \in H^{1/2}(\Gamma^2)_{/\operatorname{Ker} D^2}$ and $k \in \operatorname{Ker} D^2 = \operatorname{Span}\{1\}$ (see Theorem 3.8). We can write $k = k^+ - k^-$, where $k^+ := \max\{0, k\}$ and $k^- := \max\{0, -k\}$. Theorems 3.9 and 3.10 and observation (3.22) then yield

$$\begin{aligned}
\mathcal{J}^2(v) &\geq \frac{1}{2}\left\langle S^2\widetilde{v},\, \widetilde{v}\right\rangle_{L^2(\Gamma^2)} - c^2\|\widetilde{v}\|_{H^{1/2}(\Gamma^2)} - k\left\langle N^2 f,\, 1\right\rangle_{L^2(\Gamma^2)} \\
&\geq \frac{1}{2}\alpha^{D^2}\|\widetilde{v}\|^2_{H^{1/2}(\Gamma^2)} - c^2\|\widetilde{v}\|_{H^{1/2}(\Gamma^2)} \\
&\quad -k^+\left\langle N^2 f,\, 1\right\rangle_{L^2(\Gamma^2)} + k^-\left\langle N^2 f,\, 1\right\rangle_{L^2(\Gamma^2)}.
\end{aligned} \tag{4.16}$$

For a while, let us assume $k \leq 0$. Then, by the definition of $\mathcal{K}$ (4.9),

$$v^1(x) \leq \widetilde{v}(x) - k^- \qquad \text{for all } x \in \Gamma_c,$$

which implies

$$k^- \leq \widetilde{v}(x) - v^1(x) \qquad \text{for all } x \in \Gamma_c.$$

Thus

$$\|\widetilde{v} - v^1\|_{L^2(\Gamma_c)} \geq k^-\sqrt{\operatorname{meas}\Gamma_c}$$

and we obtain

$$k^- \leq \frac{\|\widetilde{v}\|_{L^2(\Gamma_c)} + \|v^1\|_{L^2(\Gamma_c)}}{\sqrt{\operatorname{meas}\Gamma_c}} \leq \frac{\|\widetilde{v}\|_{H^{1/2}(\Gamma^2)} + \|v^1\|_{H^{1/2}(\Gamma^1)}}{\sqrt{\operatorname{meas}\Gamma_c}}. \tag{4.17}$$

Due to (4.13), (4.15), (4.16), and (4.17), we can write

$$\begin{aligned}\mathcal{J}(v) \;\geq\; & \frac{1}{2}\alpha^{D^1}\|v^1\|^2_{H^{1/2}(\Gamma^1)} + \left(\frac{\langle N^2 f,\, 1\rangle_{L^2(\Gamma^2)}}{\sqrt{\operatorname{meas}\Gamma_c}} - c^1\right)\|v^1\|_{H^{1/2}(\Gamma^1)} \\ & + \frac{1}{2}\alpha^{D^2}\|\widetilde{v}\|^2_{H^{1/2}(\Gamma^2)} + \left(\frac{\langle N^2 f,\, 1\rangle_{L^2(\Gamma^2)}}{\sqrt{\operatorname{meas}\Gamma_c}} - c^2\right)\|\widetilde{v}\|_{H^{1/2}(\Gamma^2)} \\ & + k^+\left|\langle N^2 f,\, 1\rangle_{L^2(\Gamma^2)}\right|. \qquad (4.18)\end{aligned}$$

Now let $\|v\|_{\mathcal{V}} \to \infty$. In the following ideas, we shall use the estimate (4.18). If $\|v^1\|_{H^{1/2}(\Gamma^1)} \to \infty$, then $\mathcal{J}(v) \to \infty$. The same holds if $\|\widetilde{v}\|_{H^{1/2}(\Gamma^2)} \to \infty$. In case $\|v^1\|_{H^{1/2}(\Gamma^1)}$ and $\|\widetilde{v}\|_{H^{1/2}(\Gamma^2)}$ are bounded, so is k^- due to (4.17), and therefore $k^+ \to \infty$, which causes (see (4.13)) $\mathcal{J}(v)$ blows up.

□

4.2 Linear homogeneous isotropic elastostatics

4.2.1 Model contact problem

Let us consider an elastic body occupying the reference configuration $\Omega := (0,\, a)^3$, $a > 0$, with the boundary denoted by Γ. In addition, we define

$$\begin{aligned}\Gamma_u &:= \{x \in \Gamma :\; x_2 = 0\}, \\ \Gamma_c &:= \{x \in \Gamma :\; x_3 = 0\}, \\ \Gamma_f &:= \Gamma \setminus \{\Gamma_u \cup \Gamma_c\}\end{aligned}$$

and assume the situation when the body is fixed on Γ_u, free on Γ_f and inside Ω it is loaded by volume forces with the density $\underline{f}$. Under the body, there is a rigid obstacle $P_d := \{x \in \mathbb{R}^3 :\; x_3 \leq d\}$, $d < 0$, so that we shall consider contact conditions prescribed on the part Γ_c. For better comprehension, we refer to Figure 4.2.

Let us formulate our model problem in terms of displacements. We shall pay attention to finding a sufficiently smooth displacement field $\underline{u}$ such that

$$\begin{aligned}-\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\sigma_{ij}(\underline{u}, x) &= f_i(x) \quad \text{for } x \in \Omega,\; i = 1, 2, 3, \\ \underline{u}(x) &= \underline{0} \quad \text{for } x \in \Gamma_u, \qquad (4.19) \\ t_i(x) := \sum_{j=1}^{3}\sigma_{ij}(\underline{u}, x)n_j(x) &= 0 \quad \text{for } x \in \Gamma_f,\; i = 1, 2, 3,\end{aligned}$$

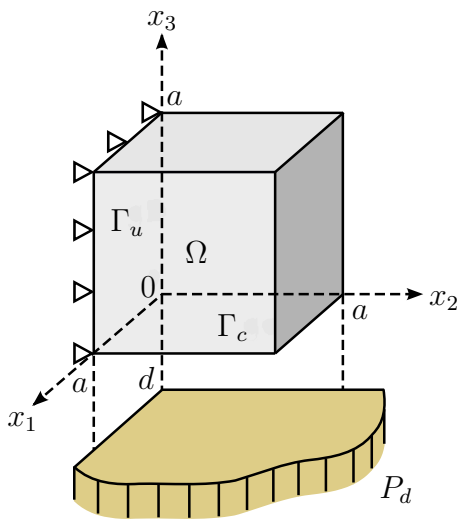

Figure 4.2: Geometry of the 3D model contact problem

together with the contact conditions

$$\left.\begin{array}{rcl} u_3(x) & \geq & d \\ t_3(x) & \geq & 0 \\ (u_3(x)-d)\, t_3(x) & = & 0 \end{array}\right\} \text{ for } x \in \Gamma_c \tag{4.20}$$

with $n_j(x)$ denoting jth component of the exterior unit normal vector $\underline{n}(x)$ of Ω that is defined for almost all $x \in \Gamma$.

We shall assume that the body is made of homogeneous isotropic material, i.e. the stress tensor $\{\sigma_{ij}(\underline{u}, x)\}_{i,j=1}^{3}$ complies with the Hook law (3.25), where the strain tensor $\{e_{ij}(\underline{u}, x)\}_{i,j=1}^{3}$ is given by (3.26) and E and ν are material constants, to be specific, $E > 0$ is the Young modulus and $\nu \in (0, 1/2)$ is the Poisson ratio. Since we assume ν is bounded away from $1/2$, we deal here with the case of compressible elastostatics. Furthermore, we consider the density $\underline{f}$ of internal forces belonging to $[L^2(\Omega)]^3$.

Due to the first contact condition, the body is not allowed to penetrate the obstacle. Furthermore, at points, where contact occurs, we allow positive normal boundary stress, while at points, where the cube does not touch the obstacle, the normal stress has to be zero. Let us note that the contact points are a priori unknown.

4.2.2 Domain decomposition

Now let us decompose Ω into $p \in \mathbb{N}$ non–overlapping Lipschitz subdomains:

$$\overline{\Omega} = \bigcup_{m=1}^{p} \overline{\Omega_m}, \quad \Omega_m \cap \Omega_n = \emptyset \quad \text{for } m \neq n.$$

Furthermore, let us denote

$$\Gamma_m := \partial\Omega_m, \quad \Gamma_{mn} := \Gamma_m \cap \Gamma_n \quad \text{for } m \neq n, \quad \Gamma_s := \bigcup_{m=1}^{p} \Gamma_m$$

and call Γ_s a skeleton of Ω. Moreover, let $\underline{n}_m(x) := (n_{m,1}(x), n_{m,2}(x), n_{m,3}(x))$ denote the exterior unit normal vector of Ω_m at the point $x \in \Gamma_m$.

Now instead of (4.19), (4.20) we consider the system of local boundary value problems

$$\begin{aligned} -\sum_{j=1}^{3} \frac{\partial}{\partial x_j} \sigma_{ij}(\underline{u}_m, x) &= f_i(x) && \text{for } x \in \Omega_m,\ i = 1,2,3, \\ \underline{u}_m(x) &= \underline{0} && \text{for } x \in \Gamma_u \cap \Gamma_m, \\ t_{m,i}(x) := \sum_{j=1}^{3} \sigma_{ij}(\underline{u}_m, x) n_{m,j}(x) &= 0 && \text{for } x \in \Gamma_f \cap \Gamma_m,\ i = 1,2,3, \end{aligned} \tag{4.21}$$

together with the local contact conditions

$$\left.\begin{aligned} u_{m,3}(x) &\geq d \\ t_{m,3}(x) &\geq 0 \\ (u_{m,3}(x) - d)\, t_{m,3}(x) &= 0 \end{aligned}\right\} \text{ for } x \in \Gamma_c \cap \Gamma_m \tag{4.22}$$

and with the so–called transmission conditions

$$\left.\begin{aligned} \underline{u}_m(x) &= \underline{u}_n(x) \\ \underline{t}_m(x) + \underline{t}_n(x) &= \underline{0} \end{aligned}\right\} \text{ for } x \in \Gamma_{mn}, \tag{4.23}$$

which link the local problems together and ensure "smoothness" across artificial interfaces Γ_{mn}.

4.2.3 Boundary weak formulation

First, let us recall that $H^{1/2}(\Gamma_s)$ is the trace space of $H^1(\Omega)$ restricted to the skeleton Γ_s^m equipped with the norm

$$\|v\|_{H^{1/2}(\Gamma_s)} := \sqrt{\sum_{m=1}^{p} \left\| v|_{\Gamma_m} \right\|^2_{H^{1/2}(\Gamma_m)}}.$$

Moreover,

$$H_0^{1/2}(\Gamma_s,\,\Gamma_u) := \left\{ v \in H^{1/2}(\Gamma_s) :\ v(x) = 0 \quad \text{for } x \in \Gamma_u \right\}.$$

Now let

$$\mathcal{K} := \left\{ \underline{v} \in [H_0^{1/2}(\Gamma_s, \Gamma_u)]^3 :\ v_3(x) \geq d \quad \text{for } x \in \Gamma_c \right\} \tag{4.24}$$

and

$$\underline{v}_m := \underline{v}|_{\Gamma_m} \quad \text{for } \underline{v} \in \mathcal{K}.$$

Definition 4.2 *Function $\underline{u} \in \mathcal{K}$ is a* **`boundary weak solution`** *of (4.21)–(4.23) if*

$$\sum_{m=1}^{p} \left\langle S_m^{Lamé} \underline{u}_m,\, \underline{v}_m - \underline{u}_m \right\rangle_{\Gamma_m} \geq \sum_{m=1}^{p} \left\langle N_m^{Lamé} \underline{f},\, \underline{v}_m - \underline{u}_m \right\rangle_{\Gamma_m} \quad \textit{for all } \underline{v} \in \mathcal{K} \tag{4.25}$$

with

$$\langle \underline{u},\, \underline{v} \rangle_{\Gamma_m} := \sum_{i=1}^{3} \langle u_i,\, v_i \rangle_{L^2(\Gamma_m)}.$$

The operators $S_m^{\text{Lamé}}$ and $N_m^{\text{Lamé}}$ are the local Steklov–Poincaré and Newton operators, respectively, corresponding to the local domains Ω_m with boundaries Γ_m.

By using analogous ideas presented after Definition 4.1, we can check that if the boundary weak solution $\underline{u}$ of (4.21)–(4.23), the function $\underline{f}$, and all the local boundaries Γ_m and the boundary Γ are smooth, then the unique solution of the system of the Dirichlet boundary value problems

$$\begin{aligned} -\sum_{j=1}^{3} \tfrac{\partial}{\partial x_j} \sigma_{ij}(\widetilde{\underline{u}}_m, x) &= f_i(x) \quad \text{for } x \in \Omega_m,\ i = 1,2,3, \\ \widetilde{\underline{u}}_m(x) &= \underline{u}_m(x) \quad \text{for } x \in \Gamma_m \end{aligned}$$

is a classical solution of (4.21)–(4.23).

4.2.4 Minimization of energy functional

Theorem 4.4 *Assume $\mathcal{K}$ is given by (4.24). Then $\mathcal{K}$ is a closed, convex, nonempty set that is not a subspace of $[H_0^{1/2}(\Gamma_s,\, \Gamma_u)]^3$.*

`Proof.` The proof is analogous to that of Theorem 4.1. □

Now let us introduce the following notations

$$\mathcal{A}(\underline{u},\, \underline{v}) := \sum_{m=1}^{p} \left\langle S_m^{\text{Lamé}} \underline{u}_m,\, \underline{v}_m \right\rangle_{\Gamma_m} \qquad \text{for } \underline{u},\, \underline{v} \in [H_0^{1/2}(\Gamma_s,\, \Gamma_u)]^3$$

and

$$\mathcal{F}(\underline{v}) := \sum_{m=1}^{p} \left\langle N_m^{\text{Lamé}} \underline{f},\, \underline{v}_m \right\rangle_{\Gamma_m} \qquad \text{for } \underline{v} \in [H_0^{1/2}(\Gamma_s,\, \Gamma_u)]^3.$$

Theorem 4.5 *The variational inequality (4.25), i.e. the problem to find* $\underline{u} \in \mathcal{K}$ *satisfying*

$$\mathcal{A}(\underline{u},\, \underline{v} - \underline{u}) \geq \mathcal{F}(\underline{v} - \underline{u}) \qquad \textit{for all } \underline{v} \in \mathcal{K}$$

has a unique solution.

`Proof.` Due to Theorems 2.5 and 4.4, it suffices to prove that $\mathcal{A}$ is a bilinear form that is bounded and elliptic on $[H_0^{1/2}(\Gamma_s,\, \Gamma_u)]^3$ and $\mathcal{F}$ is a bounded linear functional on $[H_0^{1/2}(\Gamma_s,\, \Gamma_u)]^3$.

`i)` Since $S_m^{\text{Lamé}}$ is linear on $[H^{1/2}(\Gamma_m)]^3$ (see Theorem 3.19), $\mathcal{A}$ is indeed a bilinear form on $[H_0^{1/2}(\Gamma_s,\, \Gamma_u)]^3$. The linearity of $\mathcal{F}$ on $[H_0^{1/2}(\Gamma_s,\, \Gamma_u)]^3$ is obvious. By Theorems 3.19 and 3.20, there are constants $M_m > 0$ and $L_m > 0$ such that for all $\underline{u},\, \underline{v} \in [H_0^{1/2}(\Gamma_s,\, \Gamma_u)]^3$

$$\begin{aligned} |\mathcal{A}(\underline{u},\, \underline{v})| &\leq \sum_{m=1}^{p} \left| \left\langle S_m^{\text{Lamé}} \underline{u}_m,\, \underline{v}_m \right\rangle_{\Gamma_m} \right| \\ &\leq \sum_{m=1}^{p} M_m \, \|\underline{u}_m\|_{[H^{1/2}(\Gamma_m)]^3} \, \|\underline{v}_m\|_{[H^{1/2}(\Gamma_m)]^3} \\ &\leq p \max_{m\,=\,1,\,\ldots,\,p} M_m \, \|\underline{u}\|_{[H^{1/2}(\Gamma_s)]^3} \, \|\underline{v}\|_{[H^{1/2}(\Gamma_s)]^3} \end{aligned}$$

and

$$\begin{aligned} |\mathcal{F}(\underline{v})| &\leq \sum_{m=1}^{p} \left| \left\langle N_m^{\text{Lamé}} \underline{f},\, \underline{v}_m \right\rangle_{\Gamma_m} \right| \\ &\leq \sum_{m=1}^{p} L_m \, \|\underline{f}\|_{[L^2(\Omega_m)]^3} \, \|\underline{v}_m\|_{[H^{1/2}(\Gamma_m)]^3} \\ &\leq p \max_{m\,=\,1,\,\ldots,\,p} L_m \, \|\underline{f}\|_{[L^2(\Omega)]^3} \, \|\underline{v}\|_{[H^{1/2}(\Gamma_s)]^3} \,. \end{aligned}$$

`ii)` Now the task will be to show the ellipticity of $\mathcal{A}$ on $[H_0^{1/2}(\Gamma_s,\, \Gamma_u)]^3$. Let $\underline{u} \in [H_0^{1/2}(\Gamma_s,\, \Gamma_u)]^3$ and $\underline{u}_m := \underline{u}|_{\Gamma_m}$. Let $\underline{v} \in [H_0^1(\Omega,\, \Gamma_u)]^3$ be such that

for $m = 1, 2, \ldots, p$ the function $\underline{v}_m := \underline{v}|_{\Omega_m}$ is the weak solution of

$$\begin{aligned} -\sum_{j=1}^{3} \tfrac{\partial}{\partial x_j}\sigma_{ij}(\underline{v}_m, x) &= 0 \qquad \text{for } x \in \Omega_m,\ i = 1,2,3, \\ \gamma_0 \underline{v}_m(x) &= \underline{u}_m(x) \quad \text{for } x \in \Gamma_m. \end{aligned}$$

Then for $m = 1, 2, \ldots, p$ we get, by (3.45),

$$\left\langle S_m^{\text{Lamé}} \underline{u}_m, \underline{u}_m \right\rangle_{\Gamma_m} = \int\limits_{\Omega_m} W(\underline{v}_m(x), \underline{v}_m(x))\,\mathrm{d}x.$$

Due to

$$W(\underline{u}(x), \underline{u}(x)) = \lambda \left(\sum_{i=1}^{3} e_{ii}(\underline{u}, x) \right)^2 + 2\mu \sum_{i,j=1}^{3} e_{ij}^2(\underline{u}, x)$$

(see (3.28)) with the positive Lamé constants λ and μ and Korn's inequality (3.27), we obtain

$$\begin{aligned} \mathcal{A}(\underline{u}, \underline{u}) &= \sum_{m=1}^{p} \int\limits_{\Omega_m} W(\underline{v}_m(x), \underline{v}_m(x))\,\mathrm{d}x = \int\limits_{\Omega} W(\underline{v}(x), \underline{v}(x))\,\mathrm{d}x \\ &\geq k_1 \int\limits_{\Omega} \sum_{i,j=1}^{3} e_{ij}^2(\underline{v}, x)\,\mathrm{d}x \\ &\geq k_2 \int\limits_{\Omega} \sum_{i,j=1}^{3} \left(\frac{\partial v_i}{\partial x_j}(x) \right)^2 \mathrm{d}x = k_2 \sum_{i=1}^{3} \int\limits_{\Omega} \|\nabla v_i(x)\|^2\,\mathrm{d}x. \end{aligned}$$

Furthermore, we shall use the Friedrichs theorem 2.1 and estimate (2.2) to get

$$\begin{aligned} \sum_{i=1}^{3} \int\limits_{\Omega} \|\nabla v_i(x)\|^2\,\mathrm{d}x &\geq k_3 \sum_{i=1}^{3} \|v_i\|^2_{H^1(\Omega)} = k_3 \|\underline{v}\|^2_{[H^1(\Omega)]^3} = k_3 \sum_{m=1}^{p} \|\underline{v}_m\|^2_{[H^1(\Omega_m)]^3} \\ &\geq k_4 \sum_{m=1}^{p} \|\underline{u}_m\|^2_{[H^{1/2}(\Gamma_m)]^3} = k_4 \|\underline{u}\|^2_{[H^{1/2}(\Gamma_s)]^3}, \end{aligned}$$

which completes the proof.

□

Due to the symmetry of the bilinear form $\mathcal{A}$ (following by Theorem 3.19), the proof of the latter theorem, and Theorem 2.7, the following proposition is straightforward.

Theorem 4.6 *The variational inequality (4.25), i.e. the problem to find* $\underline{u} \in \mathcal{K}$ *satisfying*

$$\mathcal{A}(\underline{u},\, \underline{v} - \underline{u}) \geq \mathcal{F}(\underline{v} - \underline{u}) \qquad \text{for all } \underline{v} \in \mathcal{K}$$

is equivalent to the problem: find $\underline{u} \in \mathcal{K}$ *such that*

$$\mathcal{J}(\underline{u}) = \min \{\mathcal{J}(\underline{v}) : \ \underline{v} \in \mathcal{K}\}, \tag{4.26}$$

where $\mathcal{J}$ *is the energy functional (2.5), i.e.*

$$\mathcal{J}(\underline{v}) := \frac{1}{2}\,\mathcal{A}(\underline{v},\, \underline{v}) - \mathcal{F}(\underline{v}).$$

Chapter 5

Approximation by boundary elements

Our further analysis covers particularly the model contact problems introduced in Chapter 4. To stay lucid, we shall keep the analyses for the Laplace operator and for the linear homogeneous isotropic elastostatics separate, though, we proceed quite similarly in both cases.

5.1 Basis functions

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded Lipschitz domain with the boundary Γ. We consider an admissible boundary element mesh

$$\Gamma_N = \bigcup_{l=1}^{N} \tau_l, \quad N \in \mathbb{N},$$

where τ_l is either a line segment (for $d = 2$) or a triangle (for $d = 3$). The set of all nodes, i.e. either end points of the line segments or vertices of the triangles, and the set of all elements' mid points shall be denoted by

$$\{x_j\}_{j=1}^{M} \quad \text{and} \quad \{x_k^*\}_{k=1}^{N},$$

respectively. In what follows, we shall use the piecewise constant shape functions defined by

$$\psi_l(x) := \begin{cases} 1 & x \in \tau_l, \\ 0 & \text{elsewhere} \end{cases}$$

for $l = 1, \ldots, N$ and the continuous piecewise linear shape functions defined by

$$\varphi_k(x_j) := \delta_{kj}, \quad \varphi_k \text{ is linear on every } \tau_l$$

Figure 5.1: Piecewise constant (left) and continuous piecewise linear (right) shape function corresponding to 2D

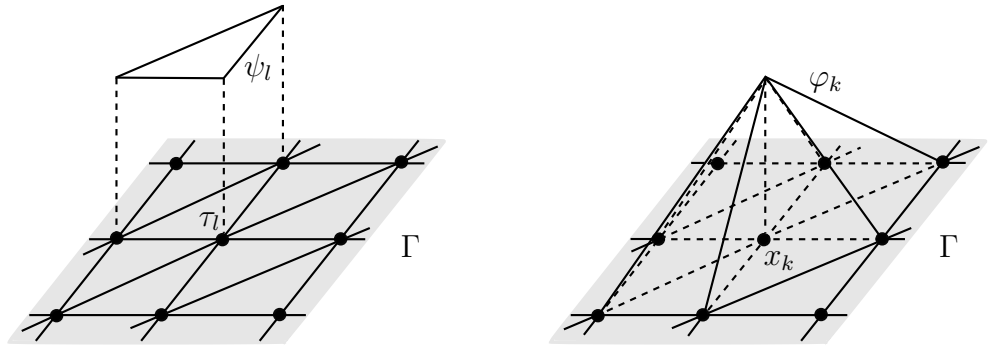

Figure 5.2: Piecewise constant (left) and continuous piecewise linear (right) shape function corresponding to 3D

for $k, j = 1, \ldots, M$, $l = 1, \ldots, N$. Both types of the shape functions corresponding to 2D and 3D are depicted in Figures 5.1 and 5.2, respectively.

5.2 Laplace operator

5.2.1 Approximation of the Steklov–Poincaré and Newton operator

In order to avoid implicit representations of the Steklov–Poincaré and Newton operators, we shall introduce their suitable approximations.

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded Lipschitz domain with the boundary Γ. Moreover, let (3.11) be satisfied for the case $d = 2$. Then for $u \in H^{1/2}(\Gamma)$ we have

$$(Su)(x) = (Du)(x) + (\frac{1}{2}I + K')w(x) \quad \text{for } x \in \Gamma,$$

where $w \in H^{-1/2}(\Gamma)$ is the unique solution of the problem

$$\langle Vw, v\rangle_{L^2(\Gamma)} = \left\langle (\frac{1}{2}I + K)u,\, v \right\rangle_{L^2(\Gamma)} \qquad \text{for all } v \in H^{-1/2}(\Gamma). \tag{5.1}$$

Let

$$Z_h := \text{Span } \{\psi_k\}_{k=1}^N \subset H^{-1/2}(\Gamma)$$

be a finite–dimensional space of shape functions. Then the Galerkin formulation of (5.1) reads: find $w_h \in Z_h$ such that

$$\langle Vw_h,\, v_h\rangle_{L^2(\Gamma)} = \left\langle (\frac{1}{2}I + K)u,\, v_h \right\rangle_{L^2(\Gamma)} \qquad \text{for all } v_h \in Z_h. \tag{5.2}$$

Now we define the approximation $\widetilde{S}$ of S by

$$(\widetilde{S}u)(x) := (Du)(x) + (\frac{1}{2}I + K')w_h(x) \quad \text{for } x \in \Gamma, \tag{5.3}$$

where w_h is the unique solution of (5.2).

For $f \in L^2(\Omega)$ we consider a problem to find $y \in H^{-1/2}(\Gamma)$ such that

$$\langle Vy,\, v\rangle_{L^2(\Gamma)} = \langle N_0 f,\, v\rangle_{L^2(\Gamma)} \qquad \text{for all } v \in H^{-1/2}(\Gamma).$$

The Galerkin formulation of the latter problem reads: find $y_h \in Z_h$ such that

$$\langle Vy_h,\, v_h\rangle_{L^2(\Gamma)} = \langle N_0 f,\, v_h\rangle_{L^2(\Gamma)} \qquad \text{for all } v_h \in Z_h. \tag{5.4}$$

Thus we can define the approximation $\widetilde{N}$ of N as

$$(\widetilde{N}f)(x) := y_h(x) \quad \text{for } x \in \Gamma, \tag{5.5}$$

where y_h solves uniquely (5.4).

Now let us give some properties of such approximations.

Theorem 5.1 *The approximate Steklov–Poincaré operator $\widetilde{S}$ defined by (5.3) is linear, bounded, symmetric, and semi–elliptic on $H^{1/2}(\Gamma)$. Moreover, $\widetilde{S}$ is elliptic on $H_0^{1/2}(\Gamma, \Gamma_u)$ and satisfies the estimate*

$$\|(S - \widetilde{S})u\|_{H^{-1/2}(\Gamma)} \le c \inf_{v_h \in Z_h} \|Su - v_h\|_{H^{-1/2}(\Gamma)}.$$

The approximate Newton operator $\widetilde{N}$ given by (5.5) is linear and bounded on $L^2(\Omega)$ and satisfies the estimate

$$\|(N - \widetilde{N})f\|_{H^{-1/2}(\Gamma)} \le k \inf_{v_h \in Z_h} \|Nf - v_h\|_{H^{-1/2}(\Gamma)}.$$

`Proof.` All the properties except the symmetry of $\widetilde{S}$ are discussed in detail in [44]. The proof of the symmetry of $\widetilde{S}$ proceeds as follows. For any $u \in H^{1/2}(\Gamma)$, we have

$$(\widetilde{S}u)(x) = (Du)(x) + (\frac{1}{2}I + K')w_h^u(x) \quad \text{for } x \in \Gamma,$$

where $w_h^u \in Z_h$ solves uniquely

$$\langle V w_h^u,\, v_h\rangle_{L^2(\Gamma)} = \left\langle (\frac{1}{2}I + K)u,\, v_h \right\rangle_{L^2(\Gamma)} \qquad \text{for all } v_h \in Z_h. \tag{5.6}$$

Now let $u, v \in H^{1/2}(\Gamma)$ be arbitrary and $w_h^u, w_h^v \in Z_h$ be the corresponding unique solutions of (5.6). We get

$$\begin{aligned}
\left\langle (\frac{1}{2}I + K')w_h^u,\, v \right\rangle_{L^2(\Gamma)} &= \left\langle (\frac{1}{2}I + K)v,\, w_h^u \right\rangle_{L^2(\Gamma)} = \langle V w_h^v,\, w_h^u\rangle_{L^2(\Gamma)} \\
&= \langle V w_h^u,\, w_h^v\rangle_{L^2(\Gamma)} = \left\langle (\frac{1}{2}I + K)u,\, w_h^v \right\rangle_{L^2(\Gamma)} \\
&= \left\langle (\frac{1}{2}I + K')w_h^v,\, u \right\rangle_{L^2(\Gamma)}
\end{aligned}$$

Hence and from the symmetry of D on $H^{1/2}(\Gamma)$, we conclude that $\widetilde{S}$ is symmetric on $H^{1/2}(\Gamma)$.

□

Now we shall return to our model contact problem with the Laplace operator and, until otherwise stated, restrict ourselves to the case $d = 2$. Instead of (4.12), we shall consider the following problem: find $u \in \mathcal{K}$ such that

$$\widetilde{\mathcal{J}}(u) = \min\left\{\widetilde{\mathcal{J}}(v) :\ v \in \mathcal{K}\right\}, \tag{5.7}$$

where

$$\widetilde{\mathcal{J}}(v) := \frac{1}{2}\,\widetilde{\mathcal{A}}(v,\, v) - \widetilde{\mathcal{F}}(v)$$

with

$$\widetilde{\mathcal{A}}(u,\, v) := \sum_{m=1}^{2}\sum_{i=1}^{p^m} \left\langle \widetilde{S}_i^m u_i^m,\, v_i^m \right\rangle_{L^2(\Gamma_i^m)} \qquad \text{for } u,\, v \in \mathcal{V}$$

and

$$\widetilde{\mathcal{F}}(v) := \sum_{m=1}^{2}\sum_{i=1}^{p^m} \left\langle \widetilde{N}_i^m f,\, v_i^m \right\rangle_{L^2(\Gamma_i^m)} \qquad \text{for } v \in \mathcal{V}.$$

Let us note that due to Theorem 5.1, $\widetilde{\mathcal{A}}$ is a bounded, symmetric, and semi–elliptic bilinear form on $\mathcal{V}$ and $\widetilde{\mathcal{F}}$ is a bounded linear functional on $\mathcal{V}$.

5.2.2 Application of the Ritz method

Let

$$W_h^m := \text{Span } \{\varphi_k^m\}_{k=1}^{M^m} \subset H^{1/2}(\Gamma_s^m)$$

be a global finite–dimensional trial space on the skeleton Γ_s^m, where the trial functions φ_k^m are the analogues to the continuous piecewise linear test functions introduced in Section 5.1 corresponding to the meshes considered on Γ_s^m. We shall assume that the grids on Γ_s^1 and Γ_s^2 match across Γ_c. Then the Ritz formulation of (5.7) reads: find $u_h \in \mathcal{K}_h$ such that

$$\widetilde{\mathcal{J}}(u_h) = \min \left\{ \widetilde{\mathcal{J}}(v_h) : \ v_h \in \mathcal{K}_h \right\}, \tag{5.8}$$

where

$$\mathcal{K}_h := \Big\{ v_h := (v_h^1, v_h^2) \in W_h^1 \times W_h^2 : \ v_h^2(x_j^2) - v_h^1(x_i^1) \geq 0 \quad \text{for all matching nodes } x_i^1,\ x_j^2 \text{ across } \Gamma_c \text{ and } v_h^1(x_k^1) = 0 \quad \text{for all nodes } x_k^1 \in \Gamma_u^1 \Big\}.$$

Let

$$W_{i,h}^m := \text{Span } \left\{\varphi_k^{m,i}\right\}_{k=1}^{M_i^m}$$

be the restriction of W_h^m onto Γ_i^m. Clearly, for any $\varphi_k^{m,i} \in W_{i,h}^m$ there exists a unique $\varphi_l^m \in W_h^m$ satisfying $\varphi_k^{m,i} = \varphi_l^m|_{\Gamma_i^m}$. For any $v_h^m \in W_h^m$ and its restriction $v_{i,h}^m \in W_{i,h}^m$ we can compute the connectivity matrix $\mathsf{A}_i^m \in \mathbb{R}^{M_i^m \times M^m}$ such that

$$\mathbf{v}_i^m = \mathsf{A}_i^m \mathbf{v}^m, \tag{5.9}$$

where $\mathbf{v}_i^m$ and $\mathbf{v}^m$ are the vectors of coordinates of $v_{i,h}^m$ and v_h^m in the bases $\{\varphi_k^{m,i}\}_{k=1}^{M_i^m}$ and $\{\varphi_k^m\}_{k=1}^{M^m}$, respectively. Problem (5.8) is further equivalent to the problem: find $\mathbf{u} \in K$ such that

$$J_0(\mathbf{u}) = \min \left\{ J_0(\mathbf{v}) : \ \mathbf{v} \in K \right\}, \tag{5.10}$$

where

$$J_0(\mathbf{v}) := \sum_{m=1}^{2} \sum_{i=1}^{p^m} \left[\frac{1}{2} \left(\widetilde{\mathsf{S}}_{i,h}^m \mathsf{A}_i^m \mathbf{v}^m,\, \mathsf{A}_i^m \mathbf{v}^m \right) - \left(\widetilde{\mathbf{R}}_{i,h}^m,\, \mathsf{A}_i^m \mathbf{v}^m \right) \right] \tag{5.11}$$

and

$$K := \Big\{ \mathbf{v} := (\mathbf{v}^1, \mathbf{v}^2) \in \mathbb{R}^{M^1} \times \mathbb{R}^{M^2} : \ \mathbf{v}^2[j] - \mathbf{v}^1[i] \geq 0 \quad \text{for all } i,\ j \text{ corresponding to the matching nodes } x_i^1,\ x_j^2 \text{ across } \Gamma_c \text{ and } \mathbf{v}^1[k] = 0 \quad \text{for all } k \text{ corresponding to the nodes } x_k^1 \in \Gamma_u^1 \Big\}.$$

Here, $\widetilde{\mathsf{S}}_{i,h}^m \in \mathbb{R}^{M_i^m \times M_i^m}$ is the discrete approximate local Steklov–Poincaré operator

$$\widetilde{\mathsf{S}}_{i,h}^m := \mathsf{D}_{i,h}^m + (\frac{1}{2}\mathsf{M}_{i,h}^m + \mathsf{K}_{i,h}^m)^\top (\mathsf{V}_{i,h}^m)^{-1} (\frac{1}{2}\mathsf{M}_{i,h}^m + \mathsf{K}_{i,h}^m)$$

and $\widetilde{\mathbf{R}}_{i,h}^m \in \mathbb{R}^{M_i^m}$ is the discrete approximate local Newton operator

$$\widetilde{\mathbf{R}}_{i,h}^m := (\mathsf{M}_{i,h}^m)^\top (\mathsf{V}_{i,h}^m)^{-1} \mathbf{N}_{0,i,h}^m.$$

The local boundary element matrices $\mathsf{V}_{i,h}^m$, $\mathsf{K}_{i,h}^m$, and $\mathsf{D}_{i,h}^m$ are fully populated. Matrix $\mathsf{V}_{i,h}^m$ is symmetric positive definite and matrix $\mathsf{D}_{i,h}^m$ is symmetric positive semi–definite.

The discrete local single layer potential operator $\mathsf{V}_{i,h}^m \in \mathbb{R}^{N_i^m \times N_i^m}$ is defined by

$$\begin{aligned} \mathsf{V}_{i,h}^m[k,l] &:= \left\langle V_i^m \psi_l^{m,i}, \psi_k^{m,i} \right\rangle_{L^2(\Gamma_i^m)} \\ &= -\frac{1}{2\pi} \int\limits_{\tau_k^{m,i}} \int\limits_{\tau_l^{m,i}} \log \|x-y\| \, \mathrm{d}s_y \, \mathrm{d}s_x, \end{aligned}$$

for $k, l = 1, \ldots, N_i^m$.

For the discrete local double layer potential operator $\mathsf{K}_{i,h}^m \in \mathbb{R}^{N_i^m \times M_i^m}$ we have, by Theorem 3.5, the representation

$$\begin{aligned} \mathsf{K}_{i,h}^m[k,j] &:= \left\langle K_i^m \varphi_j^{m,i}, \psi_k^{m,i} \right\rangle_{L^2(\Gamma_i^m)} \\ &= \frac{1}{2\pi} \int\limits_{\tau_k^{m,i}} \int\limits_{\Gamma_i^m} \frac{(x-y, \, \underline{n}_i^m(y))}{\|x-y\|^2} \varphi_j^{m,i}(y) \, \mathrm{d}s_y \, \mathrm{d}s_x \end{aligned}$$

for $k = 1, \ldots, N_i^m$, $j = 1, \ldots, M_i^m$.

Both matrices $\mathsf{V}_{i,h}^m$ and $\mathsf{K}_{i,h}^m$ may be evaluated by using analytical integration in combination with the numerical integration schemes [31, 42].

For the discrete local hypersingular integral operator $\mathsf{D}_{i,h}^m \in \mathbb{R}^{M_i^m \times M_i^m}$ there is, by Theorem 3.7, the representation

$$\begin{aligned} \mathsf{D}_{i,h}^m[n,j] &:= \left\langle D_i^m \varphi_j^{m,i}, \varphi_n^{m,i} \right\rangle_{L^2(\Gamma_i^m)} \\ &= -\frac{1}{2\pi} \int\limits_{\Gamma_i^m} \operatorname{curl}_{\Gamma_i^m} \varphi_j^{m,i}(x) \int\limits_{\Gamma_i^m} \log \|x-y\| \operatorname{curl}_{\Gamma_i^m} \varphi_n^{m,i}(y) \, \mathrm{d}s_y \, \mathrm{d}s_x \end{aligned}$$

for $n, j = 1, \ldots, M_i^m$ with

$$\operatorname{curl}_{\Gamma_i^m} \varphi_j^{m,i}(x) := n_{i,1}^m(x) \frac{\partial}{\partial x_2} \widetilde{\varphi}_j^{m,i}(x) - n_{i,2}^m(x) \frac{\partial}{\partial x_1} \widetilde{\varphi}_j^{m,i}(x),$$

where $\widetilde{\varphi}_j^{m,i}$ is some locally defined extension of $\varphi_j^{m,i}$ into the neighbourhood of Γ_i^m. If we define the extension $\widetilde{\varphi}_j^{m,i}$ so that it is constant along $\underline{n}_i^m$, we get, due to the fact that $\varphi_j^{m,i}$ is linear on every element $\tau_k^{m,i}$, $\operatorname{curl}_{\Gamma_i^m} \varphi_j^{m,i}$ to be constant on every $\tau_k^{m,i}$. Thus we can avoid the direct evaluation of the matrix $\mathsf{D}_{i,h}^m$ and derive

$$\mathsf{D}_{i,h}^m = (\mathsf{T}_i^m)^\top \mathsf{V}_{i,h}^m \mathsf{T}_i^m$$

with the local transformation matrix $\mathsf{T}_i^m \in \mathbb{R}^{N_i^m \times M_i^m}$ that is given by

$$\mathsf{T}_i^m[k,j] := \operatorname{curl}_{\Gamma_i^m} \varphi_j^{m,i}(x), \quad x \in \tau_k^{m,i},$$

for $k = 1, \ldots, N_i^m$, $j = 1, \ldots, M_i^m$.

The local mass matrix $\mathsf{M}_{i,h}^m \in \mathbb{R}^{N_i^m \times M_i^m}$ is given by

$$\mathsf{M}_{i,h}^m[k,j] := \int_{\tau_k^{m,i}} \varphi_j^{m,i}(x) \, \mathrm{d}s_x$$

for $k = 1, \ldots, N_i^m$, $j = 1, \ldots, M_i^m$.

The vector $\mathbf{N}_{0,i,h}^m \in \mathbb{R}^{N_i^m}$ is defined by

$$\mathbf{N}_{0,i,h}^m[l] := \left\langle N_{0,i}^m f, \, \psi_l^{m,i} \right\rangle_{L^2(\Gamma_i^m)}$$

for $l = 1, \ldots, N_i^m$.

5.2.3 Evaluation of the Newton potential

For evaluation of the vector $\mathbf{N}_{0,i,h}^m$ it is necessary to compute the local Newton potential $N_{0,i}^m f$. We follow an indirect approach that is introduced, e.g., in [43, 44].

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded Lipschitz domain with the boundary Γ and let f be a smooth function defined on Ω. Let u be a particular solution of the Poisson equation

$$-\triangle u(x) = f(x) \quad \text{for } x \in \Omega$$

that is smooth to the boundary Γ. The function u satisfies also the first equation of (3.5), so that

$$(N_0 f)(x) = (\frac{1}{2} I + K) u(x) - (Vt)(x) \quad \text{for } x \in \Gamma$$

with $t := \frac{\partial u}{\partial n}$. We shall now approximate u and t by the functions u_h and t_h defined for $x \in \Gamma$ as

$$u_h(x) := \sum_{j=1}^{M} u(x_j)\varphi_j(x) \quad \text{and} \quad t_h(x) := \sum_{k=1}^{N} t(x_k^*)\psi_k(x).$$

Thus

$$\begin{aligned}\langle N_0 f,\, \psi_l\rangle_{L^2(\Gamma)} \;=\; & \sum_{j=1}^{M}\left(\frac{1}{2}\langle \varphi_j,\, \psi_l\rangle_{L^2(\Gamma)} + \langle K\varphi_j,\, \psi_l\rangle_{L^2(\Gamma)}\right) u(x_j) - \\ & \sum_{k=1}^{N}\langle V\psi_k,\, \psi_l\rangle_{L^2(\Gamma)}\, t(x_k^*).\end{aligned}$$

Returning back to our model contact problem, for a smooth f we get

$$\mathbf{N}_{0,i,h}^m \approx \left(\frac{1}{2}\mathsf{M}_{i,h}^m + \mathsf{K}_{i,h}^m\right)\mathbf{u}_i^m - \mathsf{V}_{i,h}^m \mathbf{t}_i^m,$$

where $\mathbf{u}_i^m$ and $\mathbf{t}_i^m$ are vectors of the interpolation points, i.e.

$$\mathbf{u}_i^m[j] := u_i^m(x_j) \quad \text{and} \quad \mathbf{t}_i^m[k] := t_i^m(x_k^*)$$

for $j = 1, \ldots, M_i^m$, $k = 1, \ldots, N_i^m$. The function u_i^m is a particular solution of

$$-\triangle u_i^m(x) = f(x) \quad \text{for } x \in \Omega_i^m \tag{5.12}$$

smooth to the boundary Γ_i^m and $t_i^m := \frac{\partial u_i^m}{\partial \underline{n}_i^m}$.

Let us note that if the particular solution of (5.12) can be calculated analytically, the situation is quite easy. In the opposite case, one can employ the finite element method to compute an approximation of u_i^m on some fictitious domain $\Omega_{0,i}^m \supset \Omega_i^m$.

5.3 Linear homogeneous isotropic elastostatics

5.3.1 Approximation of the Steklov–Poincaré and Newton operator

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain with the boundary Γ. For $\underline{u} \in [H^{1/2}(\Gamma)]^3$ we have

$$(S^{\text{Lamé}}\underline{u})(x) = (D^{\text{Lamé}}\underline{u})(x) + \left(\frac{1}{2}I + (K^{\text{Lamé}})'\right)\underline{w}(x) \qquad \text{for all } x \in \Gamma,$$

where $\underline{w} \in [H^{-1/2}(\Gamma)]^3$ is the unique solution of the problem

$$\left\langle V^{\text{Lamé}}\underline{w},\, \underline{v}\right\rangle_\Gamma = \left\langle \left(\frac{1}{2}I + K^{\text{Lamé}}\right)\underline{u},\, \underline{v}\right\rangle_\Gamma \qquad \text{for all } \underline{v} \in [H^{-1/2}(\Gamma)]^3. \tag{5.13}$$

Let

$$Z_h := \left[\text{Span } \{\psi_k\}_{k=1}^N\right]^3 \subset [H^{-1/2}(\Gamma)]^3$$

be a finite–dimensional space of shape functions. Then the Galerkin formulation of (5.13) reads: find $\underline{w}_h \in Z_h$ such that

$$\left\langle V^{\text{Lamé}}\underline{w}_h,\, \underline{v}_h\right\rangle_\Gamma = \left\langle \left(\frac{1}{2}I + K^{\text{Lamé}}\right)\underline{u},\, \underline{v}_h\right\rangle_\Gamma \qquad \text{for all } \underline{v}_h \in Z_h. \tag{5.14}$$

Now we define the approximation $\widetilde{S}^{\text{Lamé}}$ of $S^{\text{Lamé}}$ by

$$(\widetilde{S}^{\text{Lamé}}\underline{u})(x) := (D^{\text{Lamé}}\underline{u})(x) + \left(\frac{1}{2}I + (K^{\text{Lamé}})'\right)\underline{w}_h(x) \quad \text{for } x \in \Gamma, \tag{5.15}$$

where w_h is the unique solution of (5.14).

For $\underline{f} \in [L^2(\Omega)]^3$ we consider a problem to find $\underline{y} \in [H^{-1/2}(\Gamma)]^3$ such that

$$\left\langle V^{\text{Lamé}}\underline{y},\, \underline{v}\right\rangle_\Gamma = \left\langle N_0^{\text{Lamé}}\underline{f},\, \underline{v}\right\rangle_\Gamma \qquad \text{for all } \underline{v} \in [H^{-1/2}(\Gamma)]^3.$$

The Galerkin formulation of the latter problem reads: find $\underline{y}_h \in Z_h$ such that

$$\left\langle V^{\text{Lamé}}\underline{y}_h,\, \underline{v}_h\right\rangle_\Gamma = \left\langle N_0^{\text{Lamé}}\underline{f},\, \underline{v}_h\right\rangle_\Gamma \qquad \text{for all } \underline{v}_h \in Z_h. \tag{5.16}$$

Thus we can define the approximation $\widetilde{N}^{\text{Lamé}}$ of $N^{\text{Lamé}}$ as

$$(\widetilde{N}^{\text{Lamé}}\underline{f})(x) := \underline{y}_h(x) \quad \text{for } x \in \Gamma, \tag{5.17}$$

where $\underline{y}_h$ solves uniquely (5.16).

Now let us give the following properties of such approximations.

Theorem 5.2 *The approximate Steklov–Poincaré operator $\widetilde{S}^{Lamé}$ defined by (5.15) is linear, bounded, symmetric, and semi–elliptic on $[H^{1/2}(\Gamma)]^3$. Moreover, $\widetilde{S}^{Lamé}$ is elliptic on $[H_0^{1/2}(\Gamma, \Gamma_u)]^3$ and satisfies the estimate*

$$\|(S^{Lamé} - \widetilde{S}^{Lamé})\underline{u}\|_{[H^{-1/2}(\Gamma)]^3} \le c \inf_{\underline{v}_h \in Z_h} \|S^{Lamé}\underline{u} - \underline{v}_h\|_{[H^{-1/2}(\Gamma)]^3}.$$

The approximate Newton operator $\widetilde{N}^{Lamé}$ given by (5.17) is linear and bounded on $[L^2(\Omega)]^3$ and satisfies the estimate

$$\|(N^{Lamé} - \widetilde{N}^{Lamé})\underline{f}\|_{[H^{-1/2}(\Gamma)]^3} \le k \inf_{\underline{v}_h \in Z_h} \|N^{Lamé}\underline{f} - \underline{v}_h\|_{[H^{-1/2}(\Gamma)]^3}.$$

`Proof.` Derivation of these properties follows that for the case of the Laplace operator, which is discussed in [44]. The proof of the symmetry of $\widetilde{S}^{\text{Lamé}}$ on $[H^{1/2}(\Gamma)]^3$ is analogous that of Theorem 5.1.

□

Now we shall return to our model contact problem of linear homogeneous isotropic elastostatics and, instead of (4.26), consider the following problem: find $\underline{u} \in \mathcal{K}$ such that

$$\widetilde{\mathcal{J}}(\underline{u}) = \min \left\{ \widetilde{\mathcal{J}}(\underline{v}) : \ \underline{v} \in \mathcal{K} \right\}, \tag{5.18}$$

where

$$\widetilde{\mathcal{J}}(\underline{v}) := \frac{1}{2}\, \widetilde{\mathcal{A}}(\underline{v},\, \underline{v}) - \widetilde{\mathcal{F}}(\underline{v})$$

with

$$\widetilde{\mathcal{A}}(\underline{u},\, \underline{v}) := \sum_{m=1}^{p} \left\langle \widetilde{S}_m^{\text{Lamé}} \underline{u}_m,\, \underline{v}_m \right\rangle_{\Gamma_m} \qquad \text{for } \underline{u},\, \underline{v} \in [H_0^{1/2}(\Gamma_s,\, \Gamma_u)]^3$$

and

$$\widetilde{\mathcal{F}}(\underline{v}) := \sum_{m=1}^{p} \left\langle \widetilde{N}_m^{\text{Lamé}} \underline{f},\, \underline{v}_m \right\rangle_{\Gamma_m} \qquad \text{for } \underline{v} \in [H_0^{1/2}(\Gamma_s,\, \Gamma_u)]^3.$$

Let us note that due to Theorem 5.2, $\widetilde{\mathcal{A}}$ is a bounded, symmetric, and semi–elliptic bilinear form on $[H_0^{1/2}(\Gamma_s,\, \Gamma_u)]^3$ and $\widetilde{\mathcal{F}}$ is a bounded linear functional on $[H_0^{1/2}(\Gamma_s,\, \Gamma_u)]^3$.

5.3.2 Application of the Ritz method

Let

$$W_h := \left[\text{Span } \{\varphi_k\}_{k=1}^{M} \right]^3 \subset [H^{1/2}(\Gamma_s)]^3$$

be the global finite–dimensional trial space on the skeleton Γ_s, where the trial functions φ_k are the analogues to the continuous piecewise linear test functions introduced in Section 5.1 corresponding to the mesh considered on Γ_s. Then the Ritz formulation of (5.18) reads: find $\underline{u}_h \in \mathcal{K}_h$ such that

$$\widetilde{\mathcal{J}}(\underline{u}_h) = \min \left\{ \widetilde{\mathcal{J}}(\underline{v}_h) : \ \underline{v}_h \in \mathcal{K}_h \right\}, \tag{5.19}$$

where

$$\mathcal{K}_h := \Big\{ \underline{v}_h \in W_h : \ (\underline{v}_h)_3(x_j) \geq d \quad \text{for all nodes } x_j \in \Gamma_c \text{ and} \\ \underline{v}_h(x_k) = 0 \quad \text{for all nodes } x_k \in \Gamma_u \Big\}.$$

Let

$$W_{m,h} := \left[\text{Span } \{\varphi_k^m\}_{k=1}^{M_m}\right]^3$$

be the restriction of W_h onto Γ_m. Clearly, for any φ_k^m there exists a unique φ_l satisfying $\varphi_k^m = \varphi_l|_{\Gamma_m}$. For any $\underline{v}_h \in W_h$ and its restriction $\underline{v}_{m,h} \in W_{m,h}$ we can compute the connectivity matrix $\mathsf{A}_m \in \mathbb{R}^{M_m \times M}$ such that

$$\mathbf{v}_m = \mathsf{A}_m^{\text{Lamé}} \mathbf{v}, \tag{5.20}$$

$$\mathsf{A}_m^{\text{Lamé}} := \begin{pmatrix} \mathsf{A}_m & \mathsf{O} & \mathsf{O} \\ \mathsf{O} & \mathsf{A}_m & \mathsf{O} \\ \mathsf{O} & \mathsf{O} & \mathsf{A}_m \end{pmatrix}, \quad \mathbf{v}_m := \begin{pmatrix} \mathbf{v}_{m,1} \\ \mathbf{v}_{m,2} \\ \mathbf{v}_{m,3} \end{pmatrix}, \quad \mathbf{v} := \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{pmatrix},$$

where $\mathbf{v}_{m,i}$ and $\mathbf{v}_i$ are the vectors of coordinates of $(\underline{v}_{m,h})_i$ and $(\underline{v}_h)_i$ in the bases $\{\varphi_k^m\}_{k=1}^{M_m}$ and $\{\varphi_k\}_{k=1}^{M}$, respectively. Problem (5.19) is further equivalent to the problem: find $\mathbf{u} \in K$ such that

$$J_0(\mathbf{u}) = \min \{J_0(\mathbf{v}) : \ \mathbf{v} \in K\}, \tag{5.21}$$

where

$$J_0(\mathbf{v}) := \sum_{m=1}^{p} \left[\frac{1}{2}\left(\widetilde{\mathsf{S}}_{m,h}^{\text{Lamé}} \mathsf{A}_m^{\text{Lamé}} \mathbf{v},\, \mathsf{A}_m^{\text{Lamé}} \mathbf{v}\right) - \left(\widetilde{\mathbf{R}}_{m,h}^{\text{Lamé}},\, \mathsf{A}_m^{\text{Lamé}} \mathbf{v}\right)\right] \tag{5.22}$$

and

$$K := \Big\{\mathbf{v} \in \mathbb{R}^{3M} : \ \mathbf{v}[2M + j] \geq d \quad \text{for all indices } j \text{ corresponding to the nodes } x_j \in \Gamma_c \text{ and } \mathbf{v}[iM + k] = 0 \text{ for } i = 0, 1, 2 \text{ and all indices } k \text{ corresponding to the nodes } x_k \in \Gamma_u\Big\}.$$

Here, $\widetilde{\mathsf{S}}_{m,h}^{\text{Lamé}} \in \mathbb{R}^{3M_m \times 3M_m}$ is the discrete approximate local Steklov–Poincaré operator

$$\widetilde{\mathsf{S}}_{m,h}^{\text{Lamé}} := \mathsf{D}_{m,h}^{\text{Lamé}} + (\frac{1}{2}\mathsf{M}_{m,h}^{\text{Lamé}} + \mathsf{K}_{m,h}^{\text{Lamé}})^\top (\mathsf{V}_{m,h}^{\text{Lamé}})^{-1} (\frac{1}{2}\mathsf{M}_{m,h}^{\text{Lamé}} + \mathsf{K}_{m,h}^{\text{Lamé}})$$

and $\widetilde{\mathbf{R}}_{m,h}^{\text{Lamé}} \in \mathbb{R}^{3M_m}$ is the discrete approximate local Newton operator

$$\widetilde{\mathbf{R}}_{m,h}^{\text{Lamé}} := (\mathsf{M}_{m,h}^{\text{Lamé}})^\top (\mathsf{V}_{m,h}^{\text{Lamé}})^{-1} \mathbf{N}_{0,m,h}^{\text{Lamé}}.$$

The local boundary element matrices $\mathsf{V}_{m,h}^{\text{Lamé}}$, $\mathsf{K}_{m,h}^{\text{Lamé}}$, and $\mathsf{D}_{m,h}^{\text{Lamé}}$ are all fully populated. Matrix $\mathsf{V}_{m,h}^{\text{Lamé}}$ is symmetric positive definite and matrix $\mathsf{D}_{m,h}^{\text{Lamé}}$ is symmetric positive semi–definite.

For the discrete local single layer potential operator we have, by Theorem 3.15, the representation

$$\mathsf{V}_{m,h}^{\text{Lamé}} = \frac{1+\nu}{2E(1-\nu)} \cdot$$
$$\left((3-4\nu) \begin{pmatrix} \mathsf{V}_{m,h} & \mathsf{O} & \mathsf{O} \\ \mathsf{O} & \mathsf{V}_{m,h} & \mathsf{O} \\ \mathsf{O} & \mathsf{O} & \mathsf{V}_{m,h} \end{pmatrix} + \begin{pmatrix} \mathsf{V}_{11,m,h} & \mathsf{V}_{12,m,h} & \mathsf{V}_{13,m,h} \\ \mathsf{V}_{12,m,h} & \mathsf{V}_{22,m,h} & \mathsf{V}_{23,m,h} \\ \mathsf{V}_{13,m,h} & \mathsf{V}_{23,m,h} & \mathsf{V}_{33,m,h} \end{pmatrix} \right)$$

with the discrete local single layer potential operator for the Laplace operator $\mathsf{V}_{m,h} \in \mathbb{R}^{N_m \times N_m}$ defined by

$$\mathsf{V}_{m,h}[k,l] := \frac{1}{4\pi} \int\limits_{\tau_k^m} \int\limits_{\tau_l^m} \frac{1}{\|x-y\|} \, \mathrm{d}s_y \, \mathrm{d}s_x$$

and the matrices $\mathsf{V}_{ij,m,h} \in \mathbb{R}^{N_m \times N_m}$ defined by

$$\mathsf{V}_{ij,m,h}[k,l] := \frac{1}{4\pi} \int\limits_{\tau_k^m} \int\limits_{\tau_l^m} \frac{(x_i - y_i)(x_j - y_j)}{\|x-y\|^3} \, \mathrm{d}s_y \, \mathrm{d}s_x$$

for $k, l = 1, \ldots, N_m$, $i, j = 1, 2, 3$, $i \leq j$.

For the discrete local double layer potential operator we have, by Theorem 3.16, the representation

$$\mathsf{K}_{m,h}^{\text{Lamé}} =$$

$$\begin{pmatrix} \mathsf{K}_{m,h} & \mathsf{O} & \mathsf{O} \\ \mathsf{O} & \mathsf{K}_{m,h} & \mathsf{O} \\ \mathsf{O} & \mathsf{O} & \mathsf{K}_{m,h} \end{pmatrix} - \begin{pmatrix} \mathsf{V}_{m,h} & \mathsf{O} & \mathsf{O} \\ \mathsf{O} & \mathsf{V}_{m,h} & \mathsf{O} \\ \mathsf{O} & \mathsf{O} & \mathsf{V}_{m,h} \end{pmatrix} \mathsf{T}_m + \frac{E}{1+\nu} \mathsf{V}_{m,h}^{\text{Lamé}} \, \mathsf{T}_m$$

with the discrete local double layer potential operator for the Laplace operator $\mathsf{K}_{m,h} \in \mathbb{R}^{N_m \times M_m}$ defined by

$$\mathsf{K}_{m,h}[k,n] := \frac{1}{4\pi} \int\limits_{\tau_k^m} \int\limits_{\Gamma_m} \frac{(x-y, \, \underline{n}_m(y))}{\|x-y\|^3} \varphi_n^m(y) \, \mathrm{d}s_y \, \mathrm{d}s_x$$

and the local transformation matrix

$$\mathsf{T}_m := \begin{pmatrix} \mathsf{O} & \mathsf{T}_{12,m} & \mathsf{T}_{13,m} \\ -\mathsf{T}_{12,m} & \mathsf{O} & \mathsf{T}_{23,m} \\ -\mathsf{T}_{13,m} & -\mathsf{T}_{23,m} & \mathsf{O} \end{pmatrix},$$

where the blocks $\mathsf{T}_{ij,m} \in \mathbb{R}^{N_m \times M_m}$ are given by

$$\mathsf{T}_{ij,m}[k,n] := M_{ij}(\partial_x, \underline{n}_m(x))\, \varphi_n^m(x), \quad x \in \tau_k^m,$$

for $k = 1, \ldots, N_m$, $n = 1, \ldots, M_m$, $i, j = 1, 2, 3$, $i < j$.

In computations, we exploit the symmetry of $\mathsf{V}_{m,h}$ and $\mathsf{V}_{ij,m,h}$. Entries of the matrices $\mathsf{V}_{m,h}$, $\mathsf{V}_{ij,m,h}$, and $\mathsf{K}_{m,h}$ may be calculated so that the inner integral is evaluated analytically and the outer one is approximated by using suitable numerical scheme. The detailed description of this procedure may be found in [41].

Analogously to the case of the Laplace operator, one can derive, by Theorem 3.18, the following representation for the discrete local hypersingular integral operator which is based on the local transformation matrix T_m and the matrices $\mathsf{V}_{m,h}$ and $\mathsf{V}_{m,h}^{\text{Lamé}}$:

$$\begin{aligned}
\mathsf{D}_{m,h}^{\text{Lamé}} \;=\; & \mu \begin{pmatrix} \mathsf{X}_m & \mathsf{O} & \mathsf{O} \\ \mathsf{O} & \mathsf{X}_m & \mathsf{O} \\ \mathsf{O} & \mathsf{O} & \mathsf{X}_m \end{pmatrix} + 2\mu \mathsf{T}_m^\top \begin{pmatrix} \mathsf{V}_{m,h} & \mathsf{O} & \mathsf{O} \\ \mathsf{O} & \mathsf{V}_{m,h} & \mathsf{O} \\ \mathsf{O} & \mathsf{O} & \mathsf{V}_{m,h} \end{pmatrix} \mathsf{T}_m - \\
& 4\mu^2\, \mathsf{T}_m^\top\, \mathsf{V}_{m,h}^{\text{Lamé}}\, \mathsf{T}_m + \\
& \mu \begin{pmatrix} \mathsf{Y}_{1,m} & \mathsf{T}_{23,m}^\top \mathsf{V}_{m,h} \mathsf{T}_{13,m} & -\mathsf{T}_{23,m}^\top \mathsf{V}_{m,h} \mathsf{T}_{12,m} \\ \mathsf{T}_{13,m}^\top \mathsf{V}_{m,h} \mathsf{T}_{23,m} & \mathsf{Y}_{2,m} & \mathsf{T}_{13,m}^\top \mathsf{V}_{m,h} \mathsf{T}_{12,m} \\ -\mathsf{T}_{12,m}^\top \mathsf{V}_{m,h} \mathsf{T}_{23,m} & \mathsf{T}_{12,m}^\top \mathsf{V}_{m,h} \mathsf{T}_{13,m} & \mathsf{Y}_{3,m} \end{pmatrix}
\end{aligned}$$

with the blocks

$$\mathsf{X}_m := \mathsf{T}_{23,m}^\top \mathsf{V}_{m,h} \mathsf{T}_{23,m} + \mathsf{T}_{13,m}^\top \mathsf{V}_{m,h} \mathsf{T}_{13,m} + \mathsf{T}_{12,m}^\top \mathsf{V}_{m,h} \mathsf{T}_{12,m}$$

and

$$\begin{aligned}
\mathsf{Y}_{1,m} &:= \mathsf{T}_{12,m}^\top \mathsf{V}_{m,h} \mathsf{T}_{12,m} + \mathsf{T}_{13,m}^\top \mathsf{V}_{m,h} \mathsf{T}_{13,m}, \\
\mathsf{Y}_{2,m} &:= \mathsf{T}_{12,m}^\top \mathsf{V}_{m,h} \mathsf{T}_{12,m} + \mathsf{T}_{23,m}^\top \mathsf{V}_{m,h} \mathsf{T}_{23,m}, \\
\mathsf{Y}_{3,m} &:= \mathsf{T}_{13,m}^\top \mathsf{V}_{m,h} \mathsf{T}_{13,m} + \mathsf{T}_{23,m}^\top \mathsf{V}_{m,h} \mathsf{T}_{23,m}.
\end{aligned}$$

The last matrix which remains to describe is the local mass matrix $\mathsf{M}_{m,h}^{\text{Lamé}}$. It has a form of

$$\mathsf{M}_{m,h}^{\text{Lamé}} = \begin{pmatrix} \mathsf{M}_{m,h} & \mathsf{O} & \mathsf{O} \\ \mathsf{O} & \mathsf{M}_{m,h} & \mathsf{O} \\ \mathsf{O} & \mathsf{O} & \mathsf{M}_{m,h} \end{pmatrix}$$

with the local mass matrix $\mathsf{M}_{m,h} \in \mathbb{R}^{N_m \times M_m}$ defined by

$$\mathsf{M}_{m,h}[k,n] := \int_{\tau_k^m} \varphi_n^m(x)\,\mathrm{d}s_x$$

for $k = 1, \ldots, N_m$ and $n = 1, \ldots, M_m$.

The vector $\mathbf{N}_{0,m,h}^{\text{Lamé}} \in \mathbb{R}^{3N_m}$ is given by

$$\mathbf{N}_{0,m,h}^{\text{Lamé}}[(i-1)N_m + k] := \left\langle (N_{0,m}^{\text{Lamé}} \underline{f})_i,\, \psi_k^m \right\rangle_{L^2(\Gamma_m)}$$

for $k = 1, \ldots, N_m$ and $i = 1, 2, 3$.

5.3.3 Evaluation of the Newton potential

For evaluation of the vector $\mathbf{N}_{0,m,h}^{\text{Lamé}}$ it is necessary to compute the local Newton potential $N_{0,m}^{\text{Lamé}} \underline{f}$. Similarly to the Laplace operator, we shall follow an indirect approach that is introduced, e.g., in [43, 44].

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain with the boundary Γ and let $\underline{f}$ be a smooth function defined on Ω. Let $\underline{u}$ be a particular solution of the equilibrium system

$$-\sum_{j=1}^{3} \frac{\partial}{\partial x_j} \sigma_{ij}(\underline{u}, x) = f_i(x) \quad \text{for } x \in \Omega,\ i = 1, 2, 3,$$

that is smooth to the boundary Γ. The function $\underline{u}$ satisfies also the first equation of (3.30), so that

$$(N_0^{\text{Lamé}} \underline{f})(x) = (\frac{1}{2} I + K^{\text{Lamé}}) \underline{u}(x) - (V^{\text{Lamé}} \underline{t})(x) \quad \text{for } x \in \Gamma$$

with $t_i(x) := \sum_{j=1}^{3} \sigma_{ij}(\underline{u}, x) n_j(x)$.

We shall now approximate $\underline{u}$ and $\underline{t}$ by the functions $\underline{u}_h$ and $\underline{t}_h$ defined for $x \in \Gamma$ as

$$u_{h,i}(x) := \sum_{j=1}^{M} u_i(x_j)\varphi_j(x) \quad \text{and} \quad t_{h,i}(x) := \sum_{k=1}^{N} t_i(x_k^*)\psi_k(x),$$

$i = 1, 2, 3$.

Returning back to our model contact problem, for a smooth $\underline{f}$ we get

$$\mathbf{N}_{0,m,h}^{\text{Lamé}} \approx \left(\frac{1}{2}\mathsf{M}_{m,h}^{\text{Lamé}} + \mathsf{K}_{m,h}^{\text{Lamé}}\right) \begin{pmatrix} \mathbf{u}_{m,1} \\ \mathbf{u}_{m,2} \\ \mathbf{u}_{m,3} \end{pmatrix} - \mathsf{V}_{m,h}^{\text{Lamé}} \begin{pmatrix} \mathbf{t}_{m,1} \\ \mathbf{t}_{m,2} \\ \mathbf{t}_{m,3} \end{pmatrix},$$

where $\mathbf{u}_{m,i}$ and $\mathbf{t}_{m,i}$ are vectors of the interpolation points, i.e.

$$\mathbf{u}_{m,i}[j] := u_{m,i}(x_j) \quad \text{and} \quad \mathbf{t}_{m,i}[k] := t_{m,i}(x_k^*)$$

for $j = 1, \ldots, M_m,\ k = 1, \ldots, N_m$, and $i = 1, 2, 3$. The function $\underline{u}_m$ is a particular solution of

$$-\sum_{j=1}^{3} \frac{\partial}{\partial x_j} \sigma_{ij}(\underline{u}_m, x) = f_i(x) \quad \text{for } x \in \Omega_m,\ i = 1, 2, 3, \tag{5.23}$$

smooth to the boundary Γ_m and $t_{m,i}(x) := \sum_{j=1}^{3} \sigma_{ij}(\underline{u}_m, x) n_{m,j}(x)$.

Let us note that if the particular solution of the system (5.23) can be calculated analytically, the situation is quite easy. Otherwise one can employ the finite element method to compute an approximation of $\underline{u}_m$ on some fictitious domain $\Omega_{0,m} \supset \Omega_m$.

Chapter 6

Duality principle, Preconditioning by the natural coarse grid

6.1 Primal quadratic programming problem

In order to satisfy the boundary condition of the Dirichlet type which occurs in the both considered contact problems, one could modify the corresponding stiffness matrix and right hand side appropriately. Here we shall follow, instead, the recently proposed Total FETI domain decomposition method [20] and define additional equality constraints to satisfy the Dirichlet boundary condition. Such an analogue to Total FETI shall be called Total BETI and it was introduced by Of [38] as All floating BETI. This approach is advantageous for two reasons. Firstly, all the subdomains can be treated in the same way, and secondly, we enrich the kernel of the stiffness matrix which shall be used in construction of our preconditioner.

6.1.1 Laplace operator

Now considering (5.9), we can read the energy functional (5.11) as a function of the variables $\mathbf{v}_i^m$, where $m = 1, 2$ and $i = 1, \ldots, p^m$. Moreover, let us use

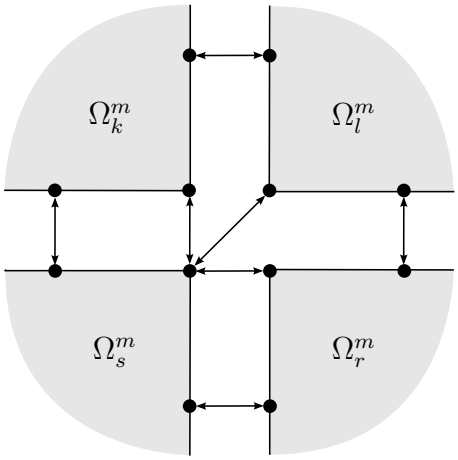

Figure 6.1: Matrix B_C: continuity condition across artificial interfaces

the notation

$$\mathbf{v} := \begin{pmatrix} \mathbf{v}_1^1 \\ \vdots \\ \mathbf{v}_{p^1}^1 \\ \mathbf{v}_1^2 \\ \vdots \\ \mathbf{v}_{p^2}^2 \end{pmatrix}.$$

To satisfy the prescribed continuity condition across all artificial interfaces Γ_{ij}^m, we define the equality constraints on the vector $\mathbf{v}$ by

$$\mathsf{B}_C\mathbf{v} = \mathbf{0},$$

where every row of B_C is associated with a pair of matching nodes on Γ_{ij}^m (see Figure 6.1). Each row consists of a single 1 and single -1 at the appropriate positions and zeros elsewhere.

To describe the non–penetration condition across the interface Γ_c, we introduce the inequality constraints

$$\mathsf{B}_I\mathbf{v} \leq \mathbf{0} =: \mathbf{c}_I,$$

where, similarly to B_C, every row of B_I is associated with a pair of matching nodes on Γ_c (see Figure 6.2). Each row consists of a single 1 and single -1 at the appropriate positions and zeros elsewhere.

In order to satisfy the Dirichlet condition on Γ_u^1, let us define another equality constraints

$$\mathsf{B}_D\mathbf{v} = \mathbf{0},$$

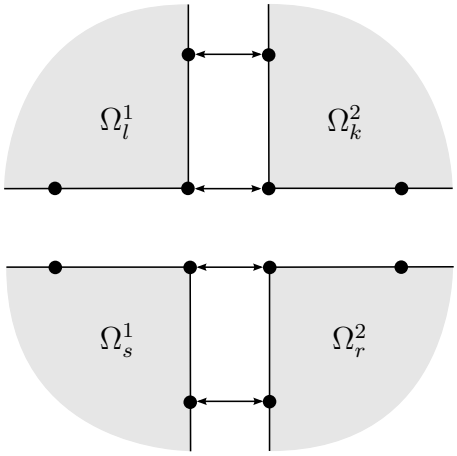

Figure 6.2: Matrix B_I: contact condition across Γ_c

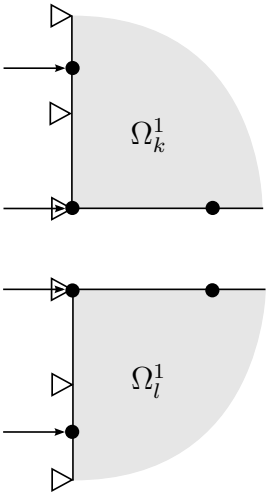

Figure 6.3: Matrix B_D: Dirichlet condition along Γ_u^1

where the rows of the matrix B_D are related to the nodes on Γ_u^1 (see Figure 6.3). Entries of each row are equal to zero except a single 1 at the position corresponding to the node with prescribed zero displacement.

At this moment, we can reformulate (5.10) as the primal quadratic programming problem:

$$\text{minimize} \quad J(\mathbf{v}) \quad \text{subject to} \quad \mathsf{B}_I\mathbf{v} \leq \mathbf{c}_I \quad \text{and} \quad \mathsf{B}_E\mathbf{v} = \mathbf{0}, \tag{6.1}$$

where

$$J(\mathbf{v}) := \frac{1}{2}\mathbf{v}^\top\widetilde{\mathsf{S}}\mathbf{v} - \widetilde{\mathbf{R}}^\top\mathbf{v},$$

$$\widetilde{\mathsf{S}} := \begin{pmatrix} \widetilde{\mathsf{S}}^1_{1,h} & \mathsf{O} & \cdots & \cdots & \cdots & \mathsf{O} \\ \mathsf{O} & \ddots & \mathsf{O} & \cdots & \cdots & \mathsf{O} \\ \mathsf{O} & \mathsf{O} & \widetilde{\mathsf{S}}^1_{p^1,h} & \mathsf{O} & \cdots & \mathsf{O} \\ \mathsf{O} & \cdots & \mathsf{O} & \widetilde{\mathsf{S}}^2_{1,h} & \mathsf{O} & \mathsf{O} \\ \mathsf{O} & \cdots & \cdots & \mathsf{O} & \ddots & \mathsf{O} \\ \mathsf{O} & \cdots & \cdots & \cdots & \mathsf{O} & \widetilde{\mathsf{S}}^2_{p^2,h} \end{pmatrix}, \quad \widetilde{\mathbf{R}} := \begin{pmatrix} \widetilde{\mathbf{R}}^1_{1,h} \\ \vdots \\ \widetilde{\mathbf{R}}^1_{p^1,h} \\ \widetilde{\mathbf{R}}^2_{1,h} \\ \vdots \\ \widetilde{\mathbf{R}}^2_{p^2,h} \end{pmatrix}, \tag{6.2}$$

and

$$\mathsf{B}_E := \begin{pmatrix} \mathsf{B}_C \\ \mathsf{B}_D \end{pmatrix}.$$

Note that we allow here some redundancy when constructing matrices B_I and B_D. This increase in number of the rows of B_I and B_D is, however, insignificant comparing to the total number of constraints. Moreover, let us state beforehand that our algorithm for the solution of the resulting problem allows linearly dependent rows of the matrix of all constraints B (see (6.6)). If we prefer to get rid of these redundancies, we can do it either geometrically or by subsequent elimination of the dependent rows from the matrix B.

6.1.2 Linear homogeneous isotropic elastostatics

Considering (5.20), we can understand the energy functional (5.22) as a function of the variables $\mathbf{v}_m$, where $m = 1, \ldots, p$. Moreover, let us use the notation

$$\mathbf{v} := \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_p \end{pmatrix}.$$

In the similar way as it is described in Subsection 6.1.1, we shall now define constraints on the vector $\mathbf{v}$. Constructions of all matrices are analogous

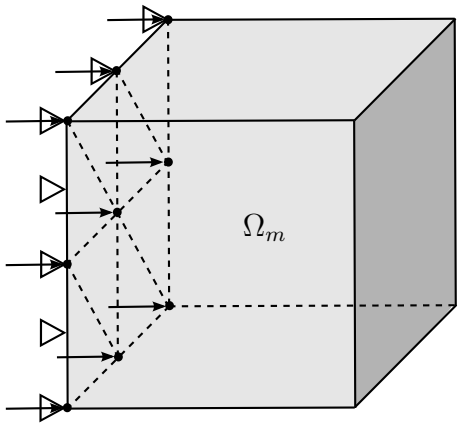

Figure 6.4: Matrix B_D: Dirichlet condition along Γ_u

to those described in Subsection 6.1.1. The only difference is we have to take into account that every $\mathbf{v}_m$ has three blocks corresponding to displacements in the coordinate axes directions.

"Gluing" across the interfaces Γ_{mn} and "fixation" along the part Γ_u are enforced by the equality constraints

$$\mathsf{B}_C\mathbf{v} = \mathbf{0} \quad \text{and} \quad \mathsf{B}_D\mathbf{v} = \mathbf{0},$$

respectively. Every row of B_C consists of a single 1 and single -1 at the appropriate positions and zeros elsewhere. The construction of B_C is slightly more difficult than in Subsection 6.1.1 (for example, for an "interior" corner node we have seven equalities prescribed for each coordinate direction). Every row of B_D consists of a single 1 at the appropriate position and zeros elsewhere (see Figure 6.4). When constructing the matrices B_C and B_D, we constrain all three coordinate directions.

To avoid penetration into obstacle, we introduce the inequality constraints

$$\mathsf{B}_I\mathbf{v} \leq \mathbf{c}_I,$$

where we constrain only the corresponding parts of the third blocks of the corresponding vectors $\mathbf{v}_m$. Every row of B_I consists of a single -1 at the appropriate position and zeros elsewhere (see Figure 6.5), while every entry of $\mathbf{c}_I$ equals to $-d$.

Now we can reformulate (5.21) as the primal quadratic programming problem:

$$\text{minimize} \quad J(\mathbf{v}) \quad \text{subject to} \quad \mathsf{B}_I\mathbf{v} \leq \mathbf{c}_I \quad \text{and} \quad \mathsf{B}_E\mathbf{v} = \mathbf{0}, \tag{6.3}$$

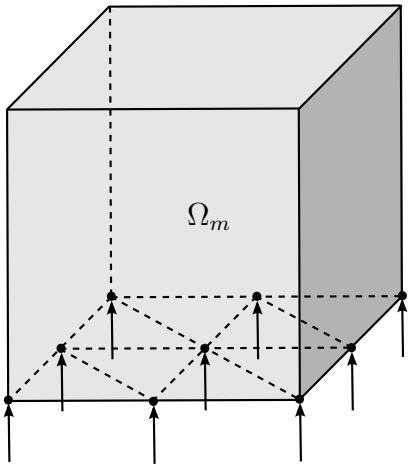

Figure 6.5: Matrix B_I: contact condition across Γ_c ($\Gamma_m \cap \Gamma_c \neq \emptyset$)

where

$$J(\mathbf{v}) := \frac{1}{2}\mathbf{v}^\top \widetilde{\mathsf{S}}\mathbf{v} - \widetilde{\mathbf{R}}^\top \mathbf{v},$$

$$\widetilde{\mathsf{S}} := \begin{pmatrix} \widetilde{\mathsf{S}}^{\text{Lamé}}_{1,h} & \mathsf{O} & \mathsf{O} \\ \mathsf{O} & \ddots & \mathsf{O} \\ \mathsf{O} & \mathsf{O} & \widetilde{\mathsf{S}}^{\text{Lamé}}_{p,h} \end{pmatrix}, \quad \widetilde{\mathbf{R}} := \begin{pmatrix} \widetilde{\mathbf{R}}^{\text{Lamé}}_{1,h} \\ \vdots \\ \widetilde{\mathbf{R}}^{\text{Lamé}}_{p,h} \end{pmatrix}, \tag{6.4}$$

and

$$\mathsf{B}_E := \begin{pmatrix} \mathsf{B}_C \\ \mathsf{B}_D \end{pmatrix}.$$

Finally, let us refer to the last paragraph of Subsection 6.1.1, which holds also for this case.

6.2 Dual quadratic programming problem

Since the primal formulations (6.1) and (6.3) of the considered contact problems are of the same type, we shall, from now on, join the further analyses of our model problems together.

Now the intention is to reduce the dimension, simplify the structure, and improve the conditioning of our problem. We shall use the duality theory, so that we shall eliminate primal variables and replace the general inequality constraints in the primal formulations (6.1) and (6.3) by the bound constraints in the dual formulation.

Let us recall that for a square real matrix A and a vector $\mathbf{b} \in \operatorname{Im}\mathsf{A}$ we define a left generalized inverse A^+ by $\mathsf{A}\mathsf{A}^+\mathsf{A} = \mathsf{A}$. Indeed, $\overline{\mathbf{x}} := \mathsf{A}^+\mathbf{b}$ satisfies $\mathsf{A}\overline{\mathbf{x}} = \mathbf{b}$ since there is a $\mathbf{y}$ such that

$$\mathsf{A}\overline{\mathbf{x}} = \mathsf{A}\mathsf{A}^+\mathbf{b} = \mathsf{A}\mathsf{A}^+\mathsf{A}\mathbf{y} = \mathsf{A}\mathbf{y} = \mathbf{b}.$$

The concept of 'generalized inverse' comes from the fact that A^+ behaves on $\operatorname{Im}\mathsf{A}$ like the inverse matrix A^{-1}.

Let us now start with the observation that the symmetric blocks $\widetilde{\mathsf{S}}^m_{i,h}$ and $\widetilde{\mathsf{S}}^{\text{Lamé}}_{m,h}$ of the stiffness matrix $\widetilde{\mathsf{S}}$ (given by (6.2) or (6.4)) are positive semi–definite due to the lack of the Dirichlet boundary condition, and therefore they are singular. Let $\widetilde{\mathsf{S}}^{m,+}_{i,h}$ and $\widetilde{\mathsf{S}}^{\text{Lamé},+}_{m,h}$ be symmetric left generalized inverses of the matrices $\widetilde{\mathsf{S}}^m_{i,h}$ and $\widetilde{\mathsf{S}}^{\text{Lamé}}_{m,h}$, respectively, so that the matrix $\widetilde{\mathsf{S}}^+$ defined by

$$\widetilde{\mathsf{S}}^+ := \begin{pmatrix} \widetilde{\mathsf{S}}^{1,+}_{1,h} & \mathsf{O} & \cdots & \cdots & \cdots & \mathsf{O} \\ \mathsf{O} & \ddots & \mathsf{O} & \cdots & \cdots & \mathsf{O} \\ \mathsf{O} & \mathsf{O} & \widetilde{\mathsf{S}}^{1,+}_{p^1,h} & \mathsf{O} & \cdots & \mathsf{O} \\ \mathsf{O} & \cdots & \mathsf{O} & \widetilde{\mathsf{S}}^{2,+}_{1,h} & \mathsf{O} & \mathsf{O} \\ \mathsf{O} & \cdots & \cdots & \mathsf{O} & \ddots & \mathsf{O} \\ \mathsf{O} & \cdots & \cdots & \cdots & \mathsf{O} & \widetilde{\mathsf{S}}^{2,+}_{p^2,h} \end{pmatrix}$$

or

$$\widetilde{\mathsf{S}}^+ := \begin{pmatrix} \widetilde{\mathsf{S}}^{\text{Lamé},+}_{1,h} & \mathsf{O} & \mathsf{O} \\ \mathsf{O} & \ddots & \mathsf{O} \\ \mathsf{O} & \mathsf{O} & \widetilde{\mathsf{S}}^{\text{Lamé},+}_{p,h} \end{pmatrix}$$

(depending on the choice of the above model contact problem) satisfies

$$\widetilde{\mathsf{S}} = \widetilde{\mathsf{S}}\widetilde{\mathsf{S}}^+\widetilde{\mathsf{S}}.$$

Thus $\widetilde{\mathsf{S}}^+$ is a symmetric left generalized inverse of $\widetilde{\mathsf{S}}$.

Since the null space of $\widetilde{\mathsf{S}}$ is non–trivial, we can define a matrix R as a full column rank matrix whose columns span the null space of $\widetilde{\mathsf{S}}$, i.e.

$$\operatorname{Im}\mathsf{R} = \operatorname{Ker}\widetilde{\mathsf{S}}.$$

We shall assume that

$$\mathsf{R} := \begin{pmatrix} \mathsf{R}_1^1 & \mathsf{O} & \cdots & \cdots & \cdots & \mathsf{O} \\ \mathsf{O} & \ddots & \mathsf{O} & \cdots & \cdots & \mathsf{O} \\ \mathsf{O} & \mathsf{O} & \mathsf{R}_{p^1}^1 & \mathsf{O} & \cdots & \mathsf{O} \\ \mathsf{O} & \cdots & \mathsf{O} & \mathsf{R}_1^2 & \mathsf{O} & \mathsf{O} \\ \mathsf{O} & \cdots & \cdots & \mathsf{O} & \ddots & \mathsf{O} \\ \mathsf{O} & \cdots & \cdots & \cdots & \mathsf{O} & \mathsf{R}_{p^2}^2 \end{pmatrix} \quad \text{or} \quad \mathsf{R} := \begin{pmatrix} \mathsf{R}_1^{\text{Lamé}} & \mathsf{O} & \mathsf{O} \\ \mathsf{O} & \ddots & \mathsf{O} \\ \mathsf{O} & \mathsf{O} & \mathsf{R}_p^{\text{Lamé}} \end{pmatrix}$$

(depending on the choice of the above model contact problem), where R_i^m and $\mathsf{R}_m^{\text{Lamé}}$ correspond to the kernels of $\widetilde{\mathsf{S}}_{i,h}^m$ and $\widetilde{\mathsf{S}}_{m,h}^{\text{Lamé}}$, respectively. Moreover, we have

$$\mathsf{R}_i^m = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{and} \quad \mathsf{R}_m^{\text{Lamé}} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & -\mathbf{x}_2^m & \mathbf{0} & \mathbf{x}_3^m \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{x}_1^m & -\mathbf{x}_3^m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{x}_2^m & -\mathbf{x}_1^m \end{pmatrix}, \tag{6.5}$$

where $\mathbf{x}_i^m$ is a vector of ith coordinates of all nodes located on Γ_m.

By introducing vectors of the Lagrange multipliers $\boldsymbol{\lambda}_I$ and $\boldsymbol{\lambda}_E$ associated with the inequalities and equalities, respectively, and denoting

$$\boldsymbol{\lambda} := \begin{pmatrix} \boldsymbol{\lambda}_I \\ \boldsymbol{\lambda}_E \end{pmatrix}, \quad \mathsf{B} := \begin{pmatrix} \mathsf{B}_I \\ \mathsf{B}_E \end{pmatrix}, \quad \text{and} \quad \mathbf{c} := \begin{pmatrix} \mathbf{c}_I \\ \mathbf{0} \end{pmatrix}, \tag{6.6}$$

we can define the Lagrangian associated with problems (6.1) and (6.3) by

$$L(\mathbf{v}, \boldsymbol{\lambda}) := \frac{1}{2}\mathbf{v}^\top \widetilde{\mathsf{S}}\mathbf{v} - \widetilde{\mathbf{R}}^\top \mathbf{v} + \boldsymbol{\lambda}^\top(\mathsf{B}\mathbf{v} - \mathbf{c}). \tag{6.7}$$

It can be observed [2] that (6.1) and (6.3) are equivalent to the following inf–sup problem: find $(\mathbf{u}, \overline{\boldsymbol{\lambda}})$ such that $\overline{\boldsymbol{\lambda}}_I \geq \mathbf{0}$ and

$$L(\mathbf{u}, \overline{\boldsymbol{\lambda}}) = \sup_{\boldsymbol{\lambda}_I \geq \mathbf{0}} \inf_{\mathbf{v}} L(\mathbf{v}, \boldsymbol{\lambda}). \tag{6.8}$$

Keeping $\boldsymbol{\lambda}$ fixed, the necessary condition for a minimizer $\mathbf{v}$ of L with respect to the first variable is

$$\widetilde{\mathsf{S}}\mathbf{v} = \widetilde{\mathbf{R}} - \mathsf{B}^\top \boldsymbol{\lambda}.$$

The latter system is solvable if and only if

$$\widetilde{\mathbf{R}} - \mathsf{B}^\top \boldsymbol{\lambda} \in \operatorname{Im} \widetilde{\mathsf{S}}, \tag{6.9}$$

which can be rewritten as

$$\mathsf{R}^\top(\widetilde{\mathbf{R}} - \mathsf{B}^\top \boldsymbol{\lambda}) = \mathbf{0}.$$

If $\boldsymbol{\lambda}$ satisfies (6.9), we can express the minimizer $\mathbf{v}$ as

$$\mathbf{v}(\boldsymbol{\lambda}) = \widetilde{\mathsf{S}}^{+}(\widetilde{\mathbf{R}} - \mathsf{B}^{\top}\boldsymbol{\lambda}). \tag{6.10}$$

Insertion of (6.10) into the Lagrangian (6.7), omission of a constant term, and change of signs lead to the so–called dual problem:

$$\text{minimize} \quad \Theta(\boldsymbol{\lambda}) \quad \text{subject to} \quad \boldsymbol{\lambda}_I \geq \mathbf{0} \quad \text{and} \quad \widetilde{\mathsf{G}}\boldsymbol{\lambda} = \widetilde{\mathbf{e}}, \tag{6.11}$$

where

$$\Theta(\boldsymbol{\lambda}) := \frac{1}{2}\boldsymbol{\lambda}^{\top}\mathsf{F}\boldsymbol{\lambda} - \boldsymbol{\lambda}^{\top}\widetilde{\mathbf{d}}$$

and

$$\mathsf{F} := \mathsf{B}\widetilde{\mathsf{S}}^{+}\mathsf{B}^{\top}, \quad \widetilde{\mathbf{d}} := \mathsf{B}\widetilde{\mathsf{S}}^{+}\widetilde{\mathbf{R}} - \mathbf{c}, \quad \widetilde{\mathsf{G}} := \mathsf{R}^{\top}\mathsf{B}^{\top}, \quad \widetilde{\mathbf{e}} := \mathsf{R}^{\top}\widetilde{\mathbf{R}}. \tag{6.12}$$

Thus we transformed the original equality and inequality constrained problems (6.1) and (6.3) into bound and equality constrained one. Furthermore, once the solution $\overline{\boldsymbol{\lambda}}$ of (6.11) is known, the solution $\mathbf{u}$ of (6.1) or (6.3) may be evaluated by

$$\mathbf{u} = \widetilde{\mathsf{S}}^{+}(\widetilde{\mathbf{R}} - \mathsf{B}^{\top}\overline{\boldsymbol{\lambda}}) + \mathsf{R}\boldsymbol{\alpha}$$

and the formula [15]

$$\boldsymbol{\alpha} = (\mathsf{R}^{\top}\widetilde{\mathsf{B}}^{\top}\widetilde{\mathsf{B}}\mathsf{R})^{-1}\mathsf{R}^{\top}\widetilde{\mathsf{B}}^{\top}\left(\widetilde{\mathbf{c}} - \widetilde{\mathsf{B}}\,\widetilde{\mathsf{S}}^{+}(\widetilde{\mathbf{R}} - \mathsf{B}^{\top}\overline{\boldsymbol{\lambda}})\right),$$

where

$$\widetilde{\mathsf{B}} := \begin{pmatrix} \widetilde{\mathsf{B}}_I \\ \mathsf{B}_E \end{pmatrix} \quad \text{and} \quad \widetilde{\mathbf{c}} := \begin{pmatrix} \widetilde{\mathbf{c}}_I \\ \mathbf{0} \end{pmatrix}$$

with the matrix $(\widetilde{\mathsf{B}}_I,\, \widetilde{\mathbf{c}}_I)$ formed by the rows of $(\mathsf{B}_I,\, \mathbf{c}_I)$ corresponding to the positive entries of $\overline{\boldsymbol{\lambda}}_I$.

6.3 Natural coarse grid

As stated before, by passing from (6.1) or (6.3) to (6.11), we obtained a constrained quadratic programming problem of a reduced dimension, better conditioning, and simplified structure, which is now much more suitable for computations than original problems (6.1) and (6.3), however, even further improvement may be achieved by using the so–called natural coarse grid. As a result, we shall get a constrained quadratic programming problem equivalent to (6.11) with the corresponding augmented Lagrangian whose Hessian has

a spectrum ensuring optimal convergence of the conjugate gradient method. Let us now describe this technique that was originally proposed by Farhat, Mandel, and Roux [27].

Let us introduce a non–singular matrix T defining orthonormalization of the rows of $\widetilde{\mathsf{G}}$, so that the matrix

$$\mathsf{G} := \mathsf{T}\widetilde{\mathsf{G}}$$

satisfies $\mathsf{G}\mathsf{G}^\top = \mathsf{I}$. By using the notation

$$\mathbf{e} := \mathsf{T}\widetilde{\mathbf{e}},$$

we can rewrite our dual problem (6.11) as:

$$\text{minimize} \quad \Theta(\boldsymbol{\lambda}) \quad \text{subject to} \quad \boldsymbol{\lambda}_I \geq \mathbf{0} \quad \text{and} \quad \mathsf{G}\boldsymbol{\lambda} = \mathbf{e}. \tag{6.13}$$

In order to homogenize equality constraints, i.e. to get equality constraints with zeros on the right hand side, we shall look for the solution of (6.13) in the form

$$\boldsymbol{\lambda} = \boldsymbol{\mu} + \widetilde{\boldsymbol{\lambda}},$$

where $\widetilde{\boldsymbol{\lambda}}$ is a vector satisfying $\mathsf{G}\widetilde{\boldsymbol{\lambda}} = \mathbf{e}$. The following lemma shall be useful when we shall show optimality of our algorithm.

Lemma 6.1 *There is even a* $\widetilde{\boldsymbol{\lambda}}$ *satisfying* $\mathsf{G}\widetilde{\boldsymbol{\lambda}} = \mathbf{e}$ *such that* $\widetilde{\boldsymbol{\lambda}}_I \geq \mathbf{0}$.

`Proof.` See [17, 18].

□

Since

$$\Theta(\boldsymbol{\lambda}) = \frac{1}{2}\boldsymbol{\lambda}^\top \mathsf{F}\boldsymbol{\lambda} - \boldsymbol{\lambda}^\top \widetilde{\mathbf{d}} = \frac{1}{2}\boldsymbol{\mu}^\top \mathsf{F}\boldsymbol{\mu} - \boldsymbol{\mu}^\top(\widetilde{\mathbf{d}} - \mathsf{F}\widetilde{\boldsymbol{\lambda}}) + \frac{1}{2}\widetilde{\boldsymbol{\lambda}}^\top \mathsf{F}\widetilde{\boldsymbol{\lambda}} - \widetilde{\boldsymbol{\lambda}}^\top \widetilde{\mathbf{d}},$$

we can consider (in minimization) the dual function Θ without the last two constant terms. Now we can return to the old notation and reformulate equivalently problem (6.13) as:

$$\text{minimize} \quad \Lambda_0(\boldsymbol{\lambda}) \quad \text{subject to} \quad \boldsymbol{\lambda}_I \geq -\widetilde{\boldsymbol{\lambda}}_I \quad \text{and} \quad \mathsf{G}\boldsymbol{\lambda} = \mathbf{0}, \tag{6.14}$$

where

$$\Lambda_0(\boldsymbol{\lambda}) := \frac{1}{2}\boldsymbol{\lambda}^\top \mathsf{F}\boldsymbol{\lambda} - \boldsymbol{\lambda}^\top \mathbf{d} \tag{6.15}$$

and $\mathbf{d} := \widetilde{\mathbf{d}} - \mathsf{F}\widetilde{\boldsymbol{\lambda}}$.

Another step is the definition of the following matrices

$$\mathsf{Q} := \mathsf{G}^\top \mathsf{G} \quad \text{and} \quad \mathsf{P} := \mathsf{I} - \mathsf{Q}.$$

It is easy to check that Q and P are orthogonal projectors on $\operatorname{Im}\mathsf{G}^\top$ and $\operatorname{Ker}\mathsf{G}$, respectively, so that

$$\operatorname{Im}\mathsf{Q} = \operatorname{Im}\mathsf{G}^\top \quad \text{and} \quad \operatorname{Im}\mathsf{P} = \operatorname{Ker}\mathsf{G}.$$

Note that

$$\mathsf{P}\boldsymbol{\lambda} = \boldsymbol{\lambda} \qquad \text{for all } \boldsymbol{\lambda} \in \operatorname{Ker}\mathsf{G}.$$

Problem (6.14) is then equivalent to the problem:

$$\text{minimize} \quad \Lambda(\boldsymbol{\lambda}) \quad \text{subject to} \quad \boldsymbol{\lambda}_I \geq -\widetilde{\boldsymbol{\lambda}}_I \quad \text{and} \quad \mathsf{G}\boldsymbol{\lambda} = \mathbf{0}, \tag{6.16}$$

where

$$\Lambda(\boldsymbol{\lambda}) := \frac{1}{2}\boldsymbol{\lambda}^\top\mathsf{PFP}\boldsymbol{\lambda} - \boldsymbol{\lambda}^\top\mathsf{P}\mathbf{d}.$$

Finally, we introduce the augmented Lagrangian associated with problem (6.16)

$$L(\boldsymbol{\lambda}, \boldsymbol{\mu}, \rho) := \frac{1}{2}\boldsymbol{\lambda}^\top(\mathsf{PFP} + \rho\mathsf{Q})\boldsymbol{\lambda} - \boldsymbol{\lambda}^\top\mathsf{P}\mathbf{d} + \boldsymbol{\mu}^\top\mathsf{G}\boldsymbol{\lambda} \tag{6.17}$$

with some penalty factor $\rho > 0$. Let us note that the Hessian $\mathcal{H} := \mathsf{PFP} + \rho\mathsf{Q}$ of (6.17) is decomposed by the projectors Q and P whose image spaces are invariant subspaces of $\mathcal{H}$. Now let $[a,\, b] \subset \mathbb{R}_+$ be an interval containing non–zero elements of the spectrum $\sigma\{\mathsf{PFP}\}$ of PFP. Then $\sigma\{\mathcal{H}\} \subset [a,\, b] \cup \{\rho\}$, so that $\mathcal{H}$ is non–singular, and, by the analysis of Axelsson [1], there is an upper bound on the number of the conjugate gradient iterations n_{CG} that is needed for reduction of the gradient of the augmented Lagrangian (6.17) by a tolerance $\varepsilon > 0$:

$$n_{CG} \leq \frac{1}{2}\operatorname{int}\left(\sqrt{\frac{b}{a}}\log\left(\frac{2}{\varepsilon}\right) + 3\right). \tag{6.18}$$

Note that this bound is independent of the penalization term ρ. In the latter estimate, int has the meaning of the upper integer part.

6.4 Bounds on spectrum

In this section, we shall examine the spectrum of the Hessian F of Λ_0, see (6.15). Our main tool will be the observation of Langer and Steinbach [32] that the local boundary element stiffness matrices $\widetilde{\mathsf{S}}^m_{i,h}$ and $\widetilde{\mathsf{S}}^m_h$ are spectrally equivalent to some Schur complements of the related local finite element stiffness matrices $\mathsf{K}^m_{i,h}$ and K^m_h, respectively.

For the sake of simplicity, let us start with the following notations. Firstly, let us assume that h is fixed, so that we can skip it. In the case of our contact problem with the Laplace operator, we shall identify each subdomain Ω_i^m with the single index $j := (m-1)p^1 + i$ and denote

$$\mathsf{S}_{\mathrm{BEM},j} := \widetilde{\mathsf{S}}_{i,h}^m \quad \text{and} \quad \mathsf{R}_j := \mathsf{R}_i^m$$

for $j = 1, \ldots, p$, $p := p^1 + p^2$. In the case of our contact problem of linear elastostatics, we shall simply define

$$\mathsf{S}_{\mathrm{BEM},j} := \widetilde{\mathsf{S}}_{j,h}^{\mathrm{Lamé}} \quad \text{and} \quad \mathsf{R}_j := \mathsf{R}_j^{\mathrm{Lamé}}$$

for $j = 1, \ldots, p$. Now we can introduce the matrix

$$\mathsf{S}_{\mathrm{BEM}} := \begin{pmatrix} \mathsf{S}_{\mathrm{BEM},1} & \mathsf{O} & \mathsf{O} \\ \mathsf{O} & \ddots & \mathsf{O} \\ \mathsf{O} & \mathsf{O} & \mathsf{S}_{\mathrm{BEM},p} \end{pmatrix}.$$

Observing that the number of columns of B is the same as the number of columns of $\mathsf{S}_{\mathrm{BEM}}$, we can impose the block structure of $\mathsf{S}_{\mathrm{BEM}}$ on the columns of B, so that

$$\mathsf{B} = (\mathsf{B}_1, \mathsf{B}_2, \ldots, \mathsf{B}_p)$$

and the dual stiffness matrix from (6.12) may be written as

$$\mathsf{F} =: \mathsf{F}_{\mathrm{BEM}} = \mathsf{B}\mathsf{S}_{\mathrm{BEM}}^{+}\mathsf{B}^{\top} = \sum_{j=1}^{p} \mathsf{B}_j \mathsf{S}_{\mathrm{BEM},j}^{+} \mathsf{B}_j^{\top}.$$

Let us now specify the FETI counterpart of $\mathsf{F}_{\mathrm{BEM}}$. In each subdomain, let us consider the regular grid which generates the same boundary mesh as that used to form $\mathsf{F}_{\mathrm{BEM}}$, and let us denote by $\mathsf{K}_{\mathrm{FEM},j}$ the local finite element stiffness matrix arising from the application of linear triangular (2D) or tetrahedral (3D) elements. Numbering the unknowns in the interior of the subdomain Ω_j first, we can write

$$\mathsf{K}_{\mathrm{FEM},j} = \begin{pmatrix} \mathsf{K}_{\mathrm{ii},j} & \mathsf{K}_{\mathrm{ib},j} \\ \mathsf{K}_{\mathrm{bi},j} & \mathsf{K}_{\mathrm{bb},j} \end{pmatrix},$$

where the subscripts b and i refer to the subdomain boundary and the interior unknowns, respectively. The finite element Schur complement matrix arising from elimination of the interior variables can be represented in the form

$$\mathsf{S}_{\mathrm{FEM},j} := \mathsf{K}_{\mathrm{bb},j} - \mathsf{K}_{\mathrm{bi},j}\mathsf{K}_{\mathrm{ii},j}^{-1}\mathsf{K}_{\mathrm{ib},j}.$$

Note that the diagonal block $\mathsf{K}_{\mathrm{ii},j}$ is invertible as it can be interpreted as the stiffness matrix of a membrane or body fixed on the whole boundary, but $\mathsf{S}_{\mathrm{FEM},j}$ is singular with the same kernel as $\mathsf{S}_{\mathrm{BEM},j}$. The FETI counterparts of $\mathsf{S}_{\mathrm{BEM}}$ and $\mathsf{F}_{\mathrm{BEM}}$ are defined [27] by

$$\mathsf{S}_{\mathrm{FEM}} := \begin{pmatrix} \mathsf{S}_{\mathrm{FEM},1} & \mathsf{O} & \mathsf{O} \\ \mathsf{O} & \ddots & \mathsf{O} \\ \mathsf{O} & \mathsf{O} & \mathsf{S}_{\mathrm{FEM},p} \end{pmatrix}$$

and

$$\mathsf{F}_{\mathrm{FEM}} := \mathsf{B}\mathsf{S}^{+}_{\mathrm{FEM}}\mathsf{B}^{\top} = \sum_{j=1}^{p} \mathsf{B}_j \mathsf{S}^{+}_{\mathrm{FEM},j} \mathsf{B}_j^{\top},$$

where $\mathsf{S}^{+}_{\mathrm{FEM},j}$ is a symmetric left generalized inverse of $\mathsf{S}_{\mathrm{FEM},j}$. The bounds on the spectrum of $\mathsf{F}_{\mathrm{FEM}}$ were established by Farhat, Mandel, and Roux [27]. To formulate them, let us denote by $\lambda_{\min}(\mathsf{A})$ and $\lambda_{\max}(\mathsf{A})$ the smallest and the largest eigenvalue of a given real symmetric matrix A, respectively.

Theorem 6.1 *There are constants $C_1 > 0$ and $C_2 > 0$ independent of the discretization parameter h and the decomposition parameter H such that*

$$\lambda_{\min}(\mathsf{P}\mathsf{F}_{\mathrm{FEM}}\mathsf{P} \,|\, \operatorname{Im}\mathsf{P}) \geq C_1 \quad \textit{and} \quad \|\mathsf{P}\mathsf{F}_{\mathrm{FEM}}\mathsf{P}\| \leq C_2 \frac{H}{h}. \tag{6.19}$$

`Proof.` See Theorem 3.2 of Farhat, Mandel, and Roux [27].

□

The statement of Theorem 3.2 of Farhat, Mandel, and Roux [27] gives only an upper bound on the spectral condition number $\kappa(\mathsf{P}\mathsf{F}_{\mathrm{FEM}}\mathsf{P} \,|\, \operatorname{Im}\mathsf{P})$. However, the reasoning that precedes and substantiates their estimate proves both bounds of (6.19). The following lemma allows us to carry over the above mentioned bounds on the spectrum of $\mathsf{F}_{\mathrm{FEM}}$ to those on $\mathsf{F}_{\mathrm{BEM}}$.

Lemma 6.2 *There are constants $c > 0$ and $C > 0$ independent of the discretization parameter h and the decomposition parameter H such that*

$$c(\mathsf{S}^{+}_{\mathrm{FEM},j}\mathsf{B}_j^{\top}\boldsymbol{\lambda},\, \mathsf{B}_j^{\top}\boldsymbol{\lambda}) \leq (\mathsf{S}^{+}_{\mathrm{BEM},j}\mathsf{B}_j^{\top}\boldsymbol{\lambda},\, \mathsf{B}_j^{\top}\boldsymbol{\lambda}) \leq C(\mathsf{S}^{+}_{\mathrm{FEM},j}\mathsf{B}_j^{\top}\boldsymbol{\lambda},\, \mathsf{B}_j^{\top}\boldsymbol{\lambda}) \tag{6.20}$$

for any $\boldsymbol{\lambda}$ such that $\mathsf{R}_j^{\top}\mathsf{B}_j^{\top}\boldsymbol{\lambda} = \mathbf{0}$.

`Proof.` See Lemma 3.3 of Langer and Steinbach [32].

□

Langer and Steinbach consider in their proof only the Moore–Penrose generalized inverse $\mathsf{S}^{\#}_{\mathrm{BEM},j}$. However, observing that $\mathsf{R}_j^\top \mathsf{B}_j^\top \boldsymbol{\lambda} = \mathbf{0}$ is equivalent to $\mathsf{B}_j^\top \boldsymbol{\lambda} \in \operatorname{Im} \mathsf{S}_{\mathrm{BEM},j}$, we get that there is a $\mathbf{y}$ so that $\mathsf{B}_j^\top \boldsymbol{\lambda} = \mathsf{S}_{\mathrm{BEM},j}\mathbf{y}$ and

$$\begin{aligned}(\mathsf{S}^{+}_{\mathrm{BEM},j}\mathsf{B}_j^\top \boldsymbol{\lambda},\, \mathsf{B}_j^\top \boldsymbol{\lambda}) &= \mathbf{y}^\top \mathsf{S}_{\mathrm{BEM},j}\mathsf{S}^{+}_{\mathrm{BEM},j}\mathsf{S}_{\mathrm{BEM},j}\mathbf{y} \\ &= \mathbf{y}^\top \mathsf{S}_{\mathrm{BEM},j}\mathbf{y} \\ &= \mathbf{y}^\top \mathsf{S}_{\mathrm{BEM},j}\mathsf{S}^{\#}_{\mathrm{BEM},j}\mathsf{S}_{\mathrm{BEM},j}\mathbf{y} \\ &= (\mathsf{S}^{\#}_{\mathrm{BEM},j}\mathsf{B}_j^\top \boldsymbol{\lambda},\, \mathsf{B}_j^\top \boldsymbol{\lambda}),\end{aligned}$$

so that Lemma 6.2 holds for the left generalized inverse, too. Now we are able to formulate the BETI counterpart of Theorem 6.1.

Theorem 6.2 *There are constants $C_3 > 0$ and $C_4 > 0$ independent of the discretization parameter h and the decomposition parameter H such that*

$$\lambda_{\min}(\mathsf{P}\mathsf{F}_{\mathrm{BEM}}\mathsf{P} \,|\, \operatorname{Im} \mathsf{P}) \geq C_3 \quad \textit{and} \quad \|\mathsf{P}\mathsf{F}_{\mathrm{BEM}}\mathsf{P}\| \leq C_4 \frac{H}{h}. \tag{6.21}$$

`Proof.` Let us assume that $\boldsymbol{\lambda} \in \operatorname{Im} \mathsf{P}$ which means

$$\mathsf{R}_j^\top \mathsf{B}_j^\top \boldsymbol{\lambda} = \mathbf{0}$$

for $j = 1, \ldots, p$, and $\|\boldsymbol{\lambda}\| = 1$. By Theorem 6.1 and Lemma 6.2, we get that there are positive constants c, C, C_1, and C_2 such that

$$\begin{aligned}C_1 \;\leq\; \boldsymbol{\lambda}^\top \mathsf{F}_{\mathrm{FEM}} \boldsymbol{\lambda} \;&=\; \sum_{j=1}^{p} (\mathsf{S}^{+}_{\mathrm{FEM},j}\mathsf{B}_j^\top \boldsymbol{\lambda},\, \mathsf{B}_j^\top \boldsymbol{\lambda}) \\ \leq\; \frac{1}{c}\sum_{j=1}^{p} (\mathsf{S}^{+}_{\mathrm{BEM},j}\mathsf{B}_j^\top \boldsymbol{\lambda},\, \mathsf{B}_j^\top \boldsymbol{\lambda}) \;&\leq\; \frac{C}{c}\sum_{j=1}^{p} (\mathsf{S}^{+}_{\mathrm{FEM},j}\mathsf{B}_j^\top \boldsymbol{\lambda},\, \mathsf{B}_j^\top \boldsymbol{\lambda}) \\ =\; \frac{C}{c}\boldsymbol{\lambda}^\top \mathsf{F}_{\mathrm{FEM}} \boldsymbol{\lambda} \;&\leq\; \frac{C}{c} C_2 \frac{H}{h}.\end{aligned}$$

Since

$$\boldsymbol{\lambda}^\top \mathsf{F}_{\mathrm{BEM}} \boldsymbol{\lambda} = \sum_{j=1}^{p} (\mathsf{S}^{+}_{\mathrm{BEM},j}\mathsf{B}_j^\top \boldsymbol{\lambda},\, \mathsf{B}_j^\top \boldsymbol{\lambda}),$$

we have thus proved that

$$c\, C_1 \leq \boldsymbol{\lambda}^\top \mathsf{F}_{\mathrm{BEM}} \boldsymbol{\lambda} \leq C C_2 \frac{H}{h}.$$

To finish the proof, it is enough to set $C_3 := c\, C_1$ and $C_4 := C C_2$.

□

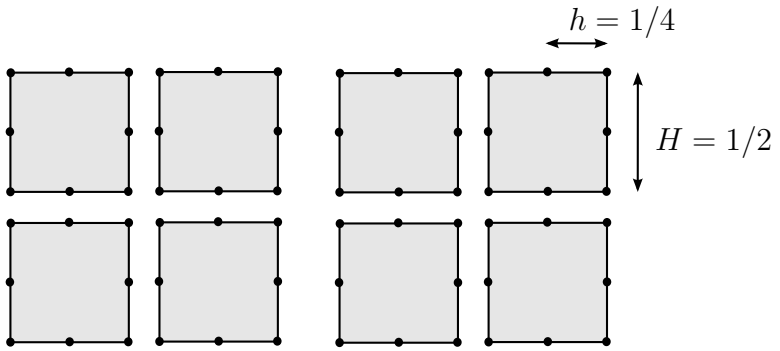

Figure 6.6: Uniform boundary mesh and parameters H and h for the 2D model problem

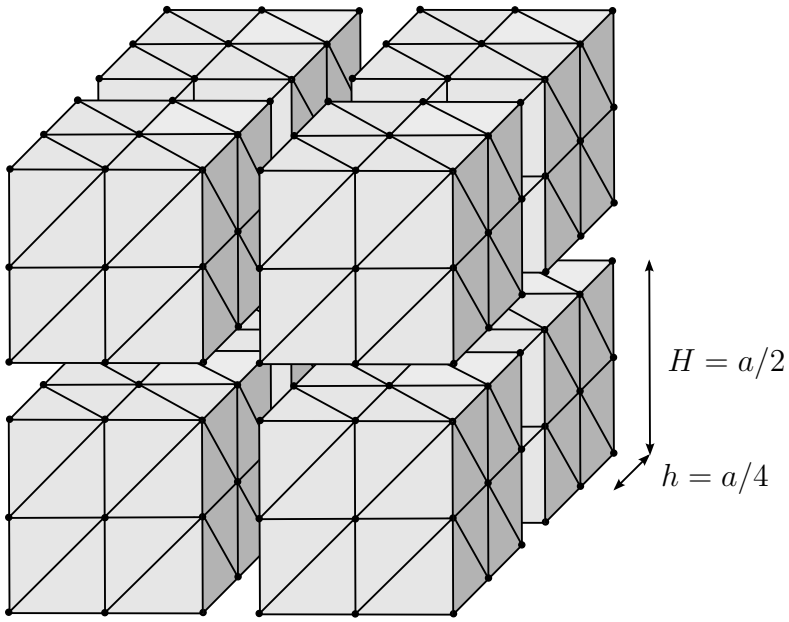

Figure 6.7: Uniform boundary mesh and parameters H and h for the 3D model problem

Langer and Steinbach [32] give stronger polylogarithmic bounds for the preconditioned F, but we cannot use this result since such preconditioning transforms the bound constraints into more general ones.

Thus, by Theorem 6.2, we obtain

$$\kappa(\mathsf{P}\mathsf{F}_{\mathrm{BEM}}\mathsf{P} \,|\, \operatorname{Im}\mathsf{P}) \leq \frac{C_4}{C_3}\frac{H}{h}.$$

It is seen that if we refine the mesh and increase the number of subdomains so that the ratio H/h is kept constant, we have still the same upper bound on the spectral condition number of $\mathsf{P}\mathsf{F}_{\mathrm{BEM}}\mathsf{P} \,|\, \operatorname{Im}\mathsf{P}$. The meaning of the discretization and decomposition parameters h and H, respectively, is illustrated in Figures 6.6 and 6.7.

6.5 Stable evaluation of the left generalized inverse

In Section 6.2, we briefly recalled the concept of the left generalized inverse. If a matrix A is non–singular, its left generalized inverse A^+ coincides with the inverse matrix A^{-1}. If A is singular and symmetric, there is a permutation matrix M such that

$$\mathsf{A} = \mathsf{M} \begin{pmatrix} \mathsf{K} & \mathsf{N} \\ \mathsf{N}^\top & \mathsf{N}^\top \mathsf{K}^{-1} \mathsf{N} \end{pmatrix} \mathsf{M}^\top, \tag{6.22}$$

where K is a non–singular matrix whose dimension is equal to the rank of A, and

$$\mathsf{A}^+ = \mathsf{M} \begin{pmatrix} \mathsf{K}^{-1} & \mathsf{O} \\ \mathsf{O} & \mathsf{O} \end{pmatrix} \mathsf{M}^\top.$$

Let us recall that if A is symmetric positive semi–definite, so is A^+.

Due to our use of Total BETI approach, which enforces fulfilment of the Dirichlet boundary condition by additional equality constraints [20, 38], all the subdomains float. This means we lack the conditions that would prevent rigid body motions. The corresponding blocks

$$\widetilde{\mathsf{S}}^m_{i,h} \in \mathbb{R}^{M^m_i \times M^m_i} \quad \text{and} \quad \widetilde{\mathsf{S}}^{\text{Lamé}}_{m,h} \in \mathbb{R}^{3M_m \times 3M_m}$$

are symmetric positive semi–definite, however, with a priori known kernels (see (6.5)). Now we shall focus on the evaluation (of the actions) of matrices $\widetilde{\mathsf{S}}^{m,+}_{i,h}$ and $\widetilde{\mathsf{S}}^{\text{Lamé},+}_{m,h}$.

In the case of our 2D contact problem with the Laplace operator, the situation is relatively easy. We have to prevent only one rigid motion, namely the vertical translation. This can be done by fixing a single arbitrary node. Thus taking, for instance, the last node in our local numbering, we can write

$$\widetilde{\mathsf{S}}_{i,h}^{m} = \begin{pmatrix} \mathsf{K}_i^m & \mathbf{n}_i^m \\ (\mathbf{n}_i^m)^\top & (\mathbf{n}_i^m)^\top (\mathsf{K}_i^m)^{-1} \mathbf{n}_i^m \end{pmatrix},$$

where K_i^m is non–singular and of the order $M_i^m - 1$, which equals to the rank of $\widetilde{\mathsf{S}}_{i,h}^{m}$, and

$$\widetilde{\mathsf{S}}_{i,h}^{m,+} = \begin{pmatrix} (\mathsf{K}_i^m)^{-1} & \mathbf{0} \\ \mathbf{0}^\top & 0 \end{pmatrix}.$$

The situation in our 3D contact problem of linear elastostatics is more complicated since we have to identify indices whose fixation prevents all six rigid body motions: three translations and three rotations. Let us now briefly sketch the technique by Dostál, Markopoulos, and Menšík described in more detail in [24]. Let us consider any three nodes x_i^m, x_j^m, x_k^m ($x_\ell^m := (x_{\ell,1}^m, x_{\ell,2}^m, x_{\ell,3}^m)$, $\ell = i, j, k$) of the discretization of subdomain Ω^m that are sufficiently far from each other and are not located near any straight line. By using the rows of $\mathsf{R}_m^{\text{Lamé}}$ corresponding to these nodes, we define

$$\mathsf{R}_{m,ijk}^{\text{Lamé}} := \begin{pmatrix} 1 & 0 & 0 & -x_{i,2}^m & 0 & x_{i,3}^m \\ 1 & 0 & 0 & -x_{j,2}^m & 0 & x_{j,3}^m \\ 1 & 0 & 0 & -x_{k,2}^m & 0 & x_{k,3}^m \\ 0 & 1 & 0 & x_{i,1}^m & -x_{i,3}^m & 0 \\ 0 & 1 & 0 & x_{j,1}^m & -x_{j,3}^m & 0 \\ 0 & 1 & 0 & x_{k,1}^m & -x_{k,3}^m & 0 \\ 0 & 0 & 1 & 0 & x_{i,2}^m & -x_{i,1}^m \\ 0 & 0 & 1 & 0 & x_{j,2}^m & -x_{j,1}^m \\ 0 & 0 & 1 & 0 & x_{k,2}^m & -x_{k,1}^m \end{pmatrix}. \tag{6.23}$$

For a while, let A be a symmetric positive semi–definite matrix and L a lower triangular matrix such that

$$\mathsf{A} = \mathsf{L}\mathsf{L}^\top.$$

It can be observed that assuming $\mathbf{e} \in \operatorname{Ker} \mathsf{A}$ and $\ell(\mathbf{e})$ is the largest index of a non–zero entry of $\mathbf{e}$, so that

$$\mathbf{e}[\ell(\mathbf{e})] \neq 0 \quad \text{and} \quad \mathbf{e}[j] = 0 \ \text{ for } j > \ell(\mathbf{e}),$$

then the so–called "modified Cholesky factor" L of A fulfils

$$\mathsf{L}[\ell(\mathbf{e}), \ell(\mathbf{e})] = 0. \tag{6.24}$$

Note that (6.24) implies zero $\ell(\mathbf{e})$th column of L. Moreover, if R_A is a full column rank matrix such that $\operatorname{Ker}\mathsf{A} = \operatorname{Im}\mathsf{R}_\mathsf{A}$, we can use the so–called "modified forward reduction" to find a new R_A satisfying

$$\ell(\mathsf{R}_{\mathsf{A},*1}) < \cdots < \ell(\mathsf{R}_{\mathsf{A},*d}),$$

where $\mathsf{R}_{\mathsf{A},*i}$ denotes an ith column of the new R_A and d is the defect of A. This procedure may be described as follows: transpose R_A, reverse the order of columns, and carry out the standard forward reduction. From the indices of the first non–zero entries on every row of the resulting matrix we can now easily obtain the indices of zero columns of the corresponding factor L. This procedure has been described and tested in [37]. Finally, the key observation of [24] is that the above technique may be applied also to the matrix $\mathsf{R}^{\text{Lamé}}_{m,ijk}$ (6.23) in order to determine the zero columns of the "modified Cholesky factor" of $\widetilde{\mathsf{S}}^{\text{Lamé}}_m$ and thus the row and column indices that shall be eliminated from $\widetilde{\mathsf{S}}^{\text{Lamé}}_m$ to get a non–singular submatrix of $\widetilde{\mathsf{S}}^{\text{Lamé}}_m$. Reasonable conditioning of such submatrix was shown experimentally in [24] for the case of a 3D linear elastostatics problem solved by the finite element method. Similar results for the boundary elements shall be discussed later in Subsection 8.2.1.

Chapter 7

Optimal solvers

In this chapter, we shall describe algorithms for the efficient solution of the bound and equality constrained problem (6.16) that arises from the application of the duality theory and preconditioning by the projectors to the natural coarse grid. These algorithms combine the semi–monotonic augmented Lagrangian method [13] which generates approximations of the Lagrange multipliers for the equality constraints in the outer loop with the working set algorithm for the bound constrained auxiliary problems in the inner loop [25]. Finally, we shall show that the combination of given algorithms is optimal for the solution of problem (6.16).

7.1 Semi–monotonic augmented Lagrangian algorithm for bound and equality constrained minimization

In our implementations, we used the augmented Lagrangian based algorithm called SMALBE. This algorithm is a variant of that proposed by Conn, Gould, and Toint [9] for identifying stationary points of more general problems. The SMALBE algorithm differs from it in the adaptive precision control introduced by Dostál, Friedlander, and Santos [14] and in the control of the penalty parameter introduced by Dostál [13]. The modification presented in [14] was used by Dostál and Horák to develop a scalable FETI based algorithm, as shown experimentally in [16].

The adaptive precision control of the solution $\boldsymbol{\lambda}^k$ of bound constrained auxiliary problems in the inner loop is motivated by an argument that the precision of the solution $\boldsymbol{\lambda}^k$ should be related to its feasibility, i.e. to the value $\|\mathsf{G}\boldsymbol{\lambda}^k\|$, since it does not seem reasonable to solve the auxiliary problems to

high precision at the early stage of computations.

Update of the penalty factor ρ_k arises from the observation that a better penalty approximation corresponds to an increased value of the augmented Lagrangian. Thus whenever the augmented Lagrangian does not increase sufficiently, the penalty factor is multiplied by a parameter $\beta > 1$. Alternatively, we can keep the penalty parameter fixed and increase the precision of the solution of the next auxiliary problem by updating a parameter M. As we will see, the value $M\|\mathsf{G}\boldsymbol{\lambda}^k\|$ specifies the precision of the inner loop solution $\boldsymbol{\lambda}^k$, so that whenever the augmented Lagrangian does not increase enough, M is multiplied by a parameter $\beta < 1$.

Before we give the scheme of the SMALBE algorithm, let us introduce the following notations. The gradient of the augmented Lagrangian (6.17) with respect to the first variable is given by

$$\mathbf{g}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \rho) := \nabla_{\boldsymbol{\lambda}} L(\boldsymbol{\lambda}, \boldsymbol{\mu}, \rho) = \mathsf{PFP}\boldsymbol{\lambda} - \mathsf{Pd} + \mathsf{G}^\top \left(\boldsymbol{\mu} + \rho \mathsf{G}\boldsymbol{\lambda}\right).$$

Let $\mathcal{I}$ denote the set of indices of the bound constrained entries of $\boldsymbol{\lambda}$. The `projected gradient` $\mathbf{g}^P = \mathbf{g}^P(\boldsymbol{\lambda}, \boldsymbol{\mu}, \rho)$ of L at $\boldsymbol{\lambda}$ satisfying $\boldsymbol{\lambda}_{\mathcal{I}} \geq -\widetilde{\boldsymbol{\lambda}}_{\mathcal{I}}$ is then given component–wise by

$$\mathbf{g}_i^P := \begin{cases} \mathbf{g}_i & \text{for} \quad \boldsymbol{\lambda}_i > -\widetilde{\boldsymbol{\lambda}}_i \quad \text{or} \quad i \notin \mathcal{I}, \\ \mathbf{g}_i^- & \text{for} \quad \boldsymbol{\lambda}_i = -\widetilde{\boldsymbol{\lambda}}_i \quad \text{and} \quad i \in \mathcal{I}, \end{cases}$$

where $\mathbf{g}_i^- := \min\{\mathbf{g}_i, 0\}$.

Algorithm 7.1 (`SMALBE` – `Semi-monotonic augmented Lagrangian algorithm for bound and equality constrained problems`)

Step 0: *Initialization of parameters*
Given $\eta > 0,\ \beta > 1,\ M > 0,\ \rho_0 > 0,\ \boldsymbol{\mu}^0$,
set $k := 0$.

Step 1: *Inner iteration with adaptive precision control*
Find $\boldsymbol{\lambda}^k$ such that $\boldsymbol{\lambda}_{\mathcal{I}}^k \geq -\widetilde{\boldsymbol{\lambda}}_{\mathcal{I}}$ and
$\|\mathbf{g}^P(\boldsymbol{\lambda}^k, \boldsymbol{\mu}^k, \rho_k)\| \leq \min\{M\|\mathsf{G}\boldsymbol{\lambda}^k\|, \eta\}$.

Step 2: *Stopping criterion*
If $\|\mathbf{g}^P(\boldsymbol{\lambda}^k, \boldsymbol{\mu}^k, \rho_k)\|$ and $\|\mathsf{G}\boldsymbol{\lambda}^k\|$ are sufficiently small, then
$\boldsymbol{\lambda}^k$ is the solution.

Step 3: *Update of the Lagrange multipliers*
$\boldsymbol{\mu}^{k+1} := \boldsymbol{\mu}^k + \rho_k \mathsf{G}\boldsymbol{\lambda}^k$

Step 4: *Update the penalty parameter*
If $k > 0$ and $L(\boldsymbol{\lambda}^k, \boldsymbol{\mu}^k, \rho_k) < L(\boldsymbol{\lambda}^{k-1}, \boldsymbol{\mu}^{k-1}, \rho_{k-1}) + \frac{\rho_k}{2}\|\mathsf{G}\boldsymbol{\lambda}^k\|^2$, then
$\rho_{k+1} := \beta\rho_k$,
else
$\rho_{k+1} := \rho_k$.

Step 5: Set $k := k + 1$ and return to *Step 1.*

Let us note that all the necessary parameters are listed in *Step 0. Step 1* may be implemented by any algorithm for minimization of the augmented Lagrangian L with respect to $\boldsymbol{\lambda}$ subject to $\boldsymbol{\lambda}_{\mathcal{I}} \geq -\widetilde{\boldsymbol{\lambda}}_{\mathcal{I}}$ which guarantees convergence of the projected gradient to zero. More about the properties and implementation of the SMALBE algorithm may be found in [11, 13].

For the sake of completeness, we also give a scheme of a variant of SMALBE which leaves the initial penalization factor untouched and, instead, updates the parameter M.

Algorithm 7.2 (`Variant of SMALBE with fixed penalty parameter` ρ `and update of` M)

Step 0: *Initialization of parameters*
Given $\eta > 0,\ \beta < 1,\ M_0 > 0,\ \rho > 0,\ \boldsymbol{\mu}^0$,
set $k := 0$.

Step 1: *Inner iteration with adaptive precision control*
Find $\boldsymbol{\lambda}^k$ such that $\boldsymbol{\lambda}^k_{\mathcal{I}} \geq -\widetilde{\boldsymbol{\lambda}}_{\mathcal{I}}$ and
$\|\mathbf{g}^P(\boldsymbol{\lambda}^k, \boldsymbol{\mu}^k, \rho)\| \leq \min\{M_k\|\mathsf{G}\boldsymbol{\lambda}^k\|, \eta\}$.

Step 2: *Stopping criterion*
If $\|\mathbf{g}^P(\boldsymbol{\lambda}^k, \boldsymbol{\mu}^k, \rho)\|$ and $\|\mathsf{G}\boldsymbol{\lambda}^k\|$ are sufficiently small, then
$\boldsymbol{\lambda}^k$ is the solution.

Step 3: *Update of the Lagrange multipliers*
$\boldsymbol{\mu}^{k+1} := \boldsymbol{\mu}^k + \rho\mathsf{G}\boldsymbol{\lambda}^k$

Step 4: *Update of the parameter M*
If $k > 0$ and $L(\boldsymbol{\lambda}^k, \boldsymbol{\mu}^k, \rho) < L(\boldsymbol{\lambda}^{k-1}, \boldsymbol{\mu}^{k-1}, \rho) + \frac{\rho}{2}\|\mathsf{G}\boldsymbol{\lambda}^k\|^2$, then
$M_{k+1} := \beta M_k$,
else
$M_{k+1} := M_k$.

Step 5: Set $k := k + 1$ and return to *Step 1.*

A unique feature of the SMALBE algorithm is its capability to find an approximate solution of problem (6.16) in a number of steps which is bounded in terms of bounds on the spectrum of the Hessian $\mathcal{H} := \mathsf{PFP}+\rho\mathsf{Q}$ [11, 13]. To get a bound on the number of matrix–vector multiplications, it is necessary to have algorithm which can solve the problem

$$\text{minimize} \quad L(\boldsymbol{\lambda}, \boldsymbol{\mu}, \rho) \quad \text{subject to} \quad \boldsymbol{\lambda}_{\mathcal{I}} \geq -\widetilde{\boldsymbol{\lambda}}_{\mathcal{I}} \tag{7.1}$$

with the rate of convergence in terms of the bounds on the spectrum of the Hessian $\mathcal{H}$ of L with respect to $\boldsymbol{\lambda}$. In the following section, we shall describe such an algorithm.

7.2 Algorithm for bound constrained minimization based on gradient projection and proportioning

Let us recall that the unique solution $\overline{\boldsymbol{\lambda}} = \overline{\boldsymbol{\lambda}}(\boldsymbol{\mu}, \rho)$ of (7.1) satisfies the so–called `Karush–Kuhn–Tucker (KKT) conditions`

$$\overline{\boldsymbol{\lambda}}_i = -\widetilde{\boldsymbol{\lambda}}_i \quad \text{and} \quad i \in \mathcal{I} \quad \text{imply} \quad \mathbf{g}_i(\overline{\boldsymbol{\lambda}}) \geq 0$$

and

$$\overline{\boldsymbol{\lambda}}_i > -\widetilde{\boldsymbol{\lambda}}_i \quad \text{or} \quad i \notin \mathcal{I} \quad \text{implies} \quad \mathbf{g}_i(\overline{\boldsymbol{\lambda}}) = 0.$$

Let $\mathcal{A}(\boldsymbol{\lambda})$ and $\mathcal{F}(\boldsymbol{\lambda})$ denote the `active set` and `free set` of indices of $\boldsymbol{\lambda}$, respectively, i.e.

$$\mathcal{A}(\boldsymbol{\lambda}) := \{i \in \mathcal{I} : \ \boldsymbol{\lambda}_i = -\widetilde{\boldsymbol{\lambda}}_i\} \quad \text{and} \quad \mathcal{F}(\boldsymbol{\lambda}) := \{i : \ \boldsymbol{\lambda}_i > -\widetilde{\boldsymbol{\lambda}}_i \ \text{ or } \ i \notin \mathcal{I}\}.$$

To enable an alternative reference to the KKT conditions [2], let us define the `free gradient` $\boldsymbol{\varphi}(\boldsymbol{\lambda})$ and the `chopped gradient` $\boldsymbol{\beta}(\boldsymbol{\lambda})$ by

$$\boldsymbol{\varphi}_i(\boldsymbol{\lambda}) := \begin{cases} \mathbf{g}_i(\boldsymbol{\lambda}) & \text{for} \quad i \in \mathcal{F}(\boldsymbol{\lambda}), \\ 0 & \text{for} \quad i \in \mathcal{A}(\boldsymbol{\lambda}), \end{cases} \quad \text{and} \quad \boldsymbol{\beta}_i(\boldsymbol{\lambda}) := \begin{cases} 0 & \text{for} \quad i \in \mathcal{F}(\boldsymbol{\lambda}), \\ \mathbf{g}_i^-(\boldsymbol{\lambda}) & \text{for} \quad i \in \mathcal{A}(\boldsymbol{\lambda}), \end{cases}$$

so that the KKT conditions are satisfied if and only if the projected gradient

$$\mathbf{g}^P(\boldsymbol{\lambda}) = \boldsymbol{\varphi}(\boldsymbol{\lambda}) + \boldsymbol{\beta}(\boldsymbol{\lambda})$$

is equal to zero. We call $\boldsymbol{\lambda}$ `feasible` if $\boldsymbol{\lambda}_i \geq -\widetilde{\boldsymbol{\lambda}}_i$ for $i \in \mathcal{I}$. The projection P to the set of feasible vectors is defined for any $\boldsymbol{\lambda}$ by

$$P(\boldsymbol{\lambda})_i := \begin{cases} \max\{\boldsymbol{\lambda}_i, -\widetilde{\boldsymbol{\lambda}}_i\} & \text{for } i \in \mathcal{I}, \\ \boldsymbol{\lambda}_i & \text{for } i \notin \mathcal{I}. \end{cases}$$

Let us recall that $\mathcal{H}$ denotes the Hessian of L with respect to $\boldsymbol{\lambda}$. The `expansion step` is defined by

$$\boldsymbol{\lambda}^{k+1} := P\left(\boldsymbol{\lambda}^k - \overline{\alpha}\boldsymbol{\varphi}(\boldsymbol{\lambda}^k)\right)$$

with the steplength $\overline{\alpha} \in (0,\, 2\,\|\mathcal{H}\|^{-1}]$. This step may expand the current active set. To describe it without P, let $\widetilde{\boldsymbol{\varphi}}(\boldsymbol{\lambda})$ be the `reduced free gradient` for any feasible $\boldsymbol{\lambda}$, with entries

$$\widetilde{\boldsymbol{\varphi}}_i = \widetilde{\boldsymbol{\varphi}}_i(\boldsymbol{\lambda}) := \min\left\{\frac{\boldsymbol{\lambda}_i + \widetilde{\boldsymbol{\lambda}}_i}{\overline{\alpha}}, \boldsymbol{\varphi}_i\right\} \quad \text{for } i \in \mathcal{I}, \quad \widetilde{\boldsymbol{\varphi}}_i := \boldsymbol{\varphi}_i \quad \text{for } i \notin \mathcal{I}$$

such that

$$P\left(\boldsymbol{\lambda} - \overline{\alpha}\boldsymbol{\varphi}(\boldsymbol{\lambda})\right) = \boldsymbol{\lambda} - \overline{\alpha}\widetilde{\boldsymbol{\varphi}}(\boldsymbol{\lambda}).$$

If the inequality

$$\|\boldsymbol{\beta}(\boldsymbol{\lambda}^k)\|^2 \le \Gamma^2 \widetilde{\boldsymbol{\varphi}}(\boldsymbol{\lambda}^k)^\top \boldsymbol{\varphi}(\boldsymbol{\lambda}^k) \tag{7.2}$$

holds, then we call the iterate $\boldsymbol{\lambda}^k$ `strictly proportional`. The test (7.2) is used to decide which component of the projected gradient $\mathbf{g}^P(\boldsymbol{\lambda}^k)$ will be reduced in the next step.

The `proportioning step` is defined by

$$\boldsymbol{\lambda}^{k+1} := \boldsymbol{\lambda}^k - \alpha_{CG}\boldsymbol{\beta}(\boldsymbol{\lambda}^k).$$

The steplength α_{CG} is chosen to minimize $L(\boldsymbol{\lambda}^k - \alpha\boldsymbol{\beta}(\boldsymbol{\lambda}^k), \boldsymbol{\mu}^k, \rho_k)$ with respect to α, i.e.

$$\alpha_{CG} := \frac{\boldsymbol{\beta}(\boldsymbol{\lambda}^k)^\top \mathbf{g}(\boldsymbol{\lambda}^k)}{\boldsymbol{\beta}(\boldsymbol{\lambda}^k)^\top \mathcal{H}\boldsymbol{\beta}(\boldsymbol{\lambda}^k)}.$$

The purpose of the proportioning step is to remove indices from the active set.

The `conjugate gradient step` is defined by

$$\boldsymbol{\lambda}^{k+1} := \boldsymbol{\lambda}^k - \alpha_{CG}\mathbf{p}^k,$$

where $\mathbf{p}^k$ is the conjugate gradient direction [1] which is constructed recurrently. The recurrence starts (or restarts) with $\boldsymbol{p}^s := \boldsymbol{\varphi}(\boldsymbol{\lambda}^s)$ whenever $\boldsymbol{\lambda}^s$ is generated by the expansion step or the proportioning step. If $\mathbf{p}^k$ is known, then $\mathbf{p}^{k+1}$ is given by the formulas [1]

$$\mathbf{p}^{k+1} := \boldsymbol{\varphi}(\boldsymbol{\lambda}^{k+1}) - \gamma\mathbf{p}^k, \quad \gamma := \frac{\boldsymbol{\varphi}(\boldsymbol{\lambda}^{k+1})^\top \mathcal{H}\mathbf{p}^k}{(\mathbf{p}^k)^\top \mathcal{H}\mathbf{p}^k}.$$

The conjugate gradient steps are used to carry out the minimization in the face

$$\mathcal{W}_{\mathcal{J}} := \{\boldsymbol{\lambda} :\ \boldsymbol{\lambda}_i = -\widetilde{\boldsymbol{\lambda}}_i \quad \text{for } i \in \mathcal{J}\}, \quad \mathcal{J} := \mathcal{A}(\boldsymbol{\lambda}^s),$$

efficiently. The algorithm that we use may now be described as follows.

Algorithm 7.3 (`MPRGP – Modified proportioning with reduced gradient projections`)

Choose $\boldsymbol{\lambda}^0$ such that $\boldsymbol{\lambda}^0_i \geq -\widetilde{\boldsymbol{\lambda}}_i$ for $i \in \mathcal{I}$, $\overline{\alpha} \in (0,\, 2\|\mathcal{H}\|^{-1}]$, and $\Gamma > 0$.
Set $k := 0$.
For $k \geq 0$ and $\boldsymbol{\lambda}^k$ known, choose $\boldsymbol{\lambda}^{k+1}$ by the following rules:

i) If $\mathbf{g}^P(\boldsymbol{\lambda}^k) = \mathbf{0}$, then set $\boldsymbol{\lambda}^{k+1} := \boldsymbol{\lambda}^k$.

ii) If $\boldsymbol{\lambda}^k$ is strictly proportional and $\mathbf{g}^P(\boldsymbol{\lambda}^k) \neq \mathbf{0}$, then try to generate $\boldsymbol{\lambda}^{k+1}$ by the conjugate gradient step. If $\boldsymbol{\lambda}^{k+1}_i \geq -\widetilde{\boldsymbol{\lambda}}_i$ for $i \in \mathcal{I}$, then accept it, else generate $\boldsymbol{\lambda}^{k+1}$ by the expansion step.

iii) If $\boldsymbol{\lambda}^k$ is not strictly proportional, define $\boldsymbol{\lambda}^{k+1}$ by proportioning.

The MPRGP algorithm has an R–linear rate of convergence in terms of the spectral condition number of the Hessian $\mathcal{H}$ of L [10, 11]. The proof of this rate of convergence for $\overline{\alpha} \in (0,\, \|\mathcal{H}\|^{-1}]$ may be also found in [25]. For more details about the properties and implementation of the MPRGP algorithm, we refer to [11, 25].

7.3 Optimality

In order to show that Algorithm 7.1 with the inner loop implemented by Algorithm 7.3 is optimal for the solution of problem (6.16), let us introduce a new notation that coincides with that used in [12]. We shall use

$$\mathcal{T} := \{(H,h) \in \mathbb{R}^2 :\ 2h \leq H,\ \text{and}\ H/h \in \mathbb{N}\}$$

as the set of indices. Given a constant $C \geq 2$, let us define a subset $\mathcal{T}_C$ of $\mathcal{T}$ by

$$\mathcal{T}_C := \{(H,h) \in \mathbb{R}^2 :\ 2h \leq H,\ H/h \in \mathbb{N},\ \text{and}\ H/h \leq C\}.$$

For any $t \in \mathcal{T}$, we define

$$\begin{aligned} &\mathsf{A}_t := \mathcal{H} = \mathsf{PFP} + \rho\mathsf{Q}, && \mathbf{b}_t := \mathsf{P}\mathbf{d}, \\ &\mathsf{C}_t := \mathsf{G}, && \boldsymbol{\ell}_{t,\mathcal{I}} := -\widetilde{\boldsymbol{\lambda}}_{\mathcal{I}}, \qquad \boldsymbol{\ell}_{t,\mathcal{E}} := -\infty \end{aligned}$$

by the vectors and matrices generated with the discretization and decomposition parameters H and h, respectively, so that problem (6.16) is equivalent to the problem

$$\text{minimize}\quad \Lambda_t(\boldsymbol{\lambda}_t)\quad \text{subject to}\quad \mathsf{C}_t\boldsymbol{\lambda}_t = \mathbf{0}\quad \text{and}\quad \boldsymbol{\lambda}_t \geq \boldsymbol{\ell}_t \tag{7.3}$$

with

$$\Lambda_t(\boldsymbol{\lambda}_t) := \frac{1}{2}\boldsymbol{\lambda}_t^\top \mathsf{A}_t \boldsymbol{\lambda}_t - \mathbf{b}_t^\top \boldsymbol{\lambda}_t.$$

By using these definitions, Lemma 6.1, and $\mathsf{G}\mathsf{G}^\top = \mathsf{I}$, we get

$$\|\mathsf{C}_t\| \le 1 \quad \text{and} \quad \|\boldsymbol{\ell}_t^+\| = 0, \tag{7.4}$$

where for any vector $\mathbf{v}$ with the entries $\mathbf{v}_i$ we define a vector $\mathbf{v}^+$ by $\mathbf{v}_i^+ := \max\{\mathbf{v}_i, 0\}$. Moreover, it follows by Theorem 6.2 that for any $C \ge 2$ there are constants $a_{\max}^C \ge a_{\min}^C > 0$ such that

$$a_{\min}^C \le \lambda_{\min}(\mathsf{A}_t) \le \lambda_{\max}(\mathsf{A}_t) \le a_{\max}^C \tag{7.5}$$

for any $t \in \mathcal{T}_C$. Furthermore, there are positive constants C_1 and C_2 such that $a_{\min}^C \ge C_1$ and $a_{\max}^C \le C_2 C$. In particular, it follows that the assumptions of Theorem 5 (i.e. the inequalities in (7.4) and (7.5)) of [12] are satisfied for any set of indices $\mathcal{T}_C$, $C \ge 2$, so that we have the following result:

Theorem 7.1 *Let $C \ge 2$ and $\varepsilon > 0$ denote given constants, let $\{\boldsymbol{\lambda}_t^k\}, \{\boldsymbol{\mu}_t^k\}$, and $\{\rho_{t,k}\}$ be generated by Algorithm 7.1 (SMALBE) for (7.3) with*

$$\|\boldsymbol{b}_t\| \ge \eta_t > 0, \ \beta > 1, \ M > 0, \ \rho_{t,0} := \rho_0 > 0, \ \text{and } \boldsymbol{\mu}_t^0 := \mathbf{0}.$$

Let $s \ge 0$ denote the smallest integer such that

$$\beta^s \rho_0 \ge \frac{M^2}{a_{\min}^C}$$

and assume that Step 1 of Algorithm 7.1 is implemented by means of Algorithm 7.3 (MPRGP) with parameters $\Gamma > 0$ and $\overline{\alpha} \in (0, (a_{\max}^C + \beta^s \rho_0)^{-1}]$, so that it generates the iterates $\boldsymbol{\lambda}_t^{k,0}, \boldsymbol{\lambda}_t^{k,1}, \ldots, \boldsymbol{\lambda}_t^{k,l} =: \boldsymbol{\lambda}_t^k$ for the solution of (7.3) starting from $\boldsymbol{\lambda}_t^{k,0} := \boldsymbol{\lambda}_t^{k-1}$ with $\boldsymbol{\lambda}_t^{-1} := \mathbf{0}$, where $l = l(t,k)$ is the first index satisfying

$$\|\boldsymbol{g}^P(\boldsymbol{\lambda}_t^{k,l}, \boldsymbol{\mu}_t^k, \rho_{t,k})\| \le M \|\mathsf{C}_t \boldsymbol{\lambda}_t^{k,l}\|$$

or

$$\|\boldsymbol{g}^P(\boldsymbol{\lambda}_t^{k,l}, \boldsymbol{\mu}_t^k, \rho_{t,k})\| \le \varepsilon \|\boldsymbol{b}_t\| \min\{1, M^{-1}\}.$$

Then for any $t \in \mathcal{T}_C$ and problem (7.3), Algorithm 7.1 generates an approximate solution $\boldsymbol{\lambda}_t^{k_t}$ which satisfies

$$M^{-1} \|\boldsymbol{g}^P(\boldsymbol{\lambda}_t^{k_t}, \boldsymbol{\mu}_t^{k_t}, \rho_{t,k_t})\| \le \|\mathsf{C}_t \boldsymbol{\lambda}_t^{k_t}\| \le \varepsilon \|\boldsymbol{b}_t\|$$

at $O(1)$ matrix–vector multiplications by the Hessian of the augmented Lagrangian L_t for (7.3) and $\rho_{t,k_t} \le \beta^s \rho_0$.

The same statement as that of Theorem 7.1 may be proved also for the variant of SMALBE with constant penalty and updated M. To sketch the idea behind, let us note that if the penalty parameter reaches a value that is large enough, then the sufficient increase of the augmented Lagrangian is ensured in every next step, namely, if

$$\rho_{t,k} \geq \frac{M^2}{\lambda_{\min}(\mathsf{A}_t)}, \tag{7.6}$$

then

$$L(\boldsymbol{\lambda}^{t,k+1}, \boldsymbol{\mu}^{t,k+1}, \rho_{k+1}) \geq L(\boldsymbol{\lambda}^{t,k}, \boldsymbol{\mu}^{t,k}, \rho_{t,k}) + \frac{\rho_{t,k+1}}{2}\|\mathsf{C}_t\boldsymbol{\lambda}^{t,k+1}\|^2.$$

Thus if the algorithm keeps the penalty factor fixed, we have the left hand side of (7.6) constant and in finite number of steps we obtain the fulfilment of (7.6) by decreasing the value of parameter M. Further analysis of the optimality then follows that of the original SMALBE.

Detailed proof of the statement of Theorem 7.1 for the case $\overline{\alpha} \in (0, 2(a^C_{\max} + \beta^s\rho_0)^{-1}]$ may be found in [11] (see the chapter 'Bound and Equality Constrained Minimization' of [11]).

Chapter 8

Numerical experiments

In this chapter, we shall present the obtained numerical results. Let us note that our implementations were carried out in Matlab 7.1.

8.1 Laplace operator

We shall now deal with the 2D model multibody contact problem (4.2), (4.3). First of all, let us define the function f which was used in our numerical experiments:

$$f(x) := \begin{cases} -3 & \text{for } x \in (0,\, 1) \times [0.75,\, 1), \\ -1 & \text{for } x \in (1,\, 2) \times (0,\, 0.25], \\ 0 & \text{elsewhere in } \Omega^1 \cup \Omega^2; \end{cases}$$

see also Figure 8.1. Note that f satisfies condition (4.1).

The domains Ω^1 and Ω^2 were both decomposed into identical square subdomains with the side length H. We gradually chose decompositions into $2 \cdot 2^2$, $2 \cdot 4^2$, $2 \cdot 8^2$, and $2 \cdot 16^2$ squares which correspond to the side lengths $H := 1/2,\ 1/4,\ 1/8$, and $1/16$. All subdomain boundaries were further discretized by the same uniform meshes with the element size h. The boundary flux and boundary vertical displacement were approximated by the piecewise constant and continuous piecewise linear trial functions $\psi_l^{m,i}$ and $\varphi_k^{m,i}$, respectively (see Figure 5.1). The deformed membranes for the choice of parameters $h := 1/512$ and $H := 1/8$ are shown in Figure 8.2.

The resulting bound and equality constrained quadratic programming problem (6.16) was solved by the pair of optimal algorithms SMALBE and MPRGP (see Sections 7.1 and 7.2, respectively). In every outer iteration, if the increase of the augmented Lagrangian was not sufficient, we increased the value of the penalization parameter by multiplying it by a parameter

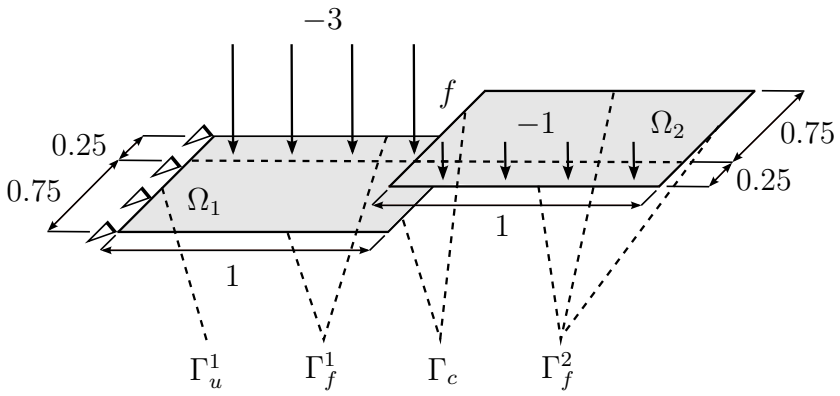

Figure 8.1: 2D model contact problem

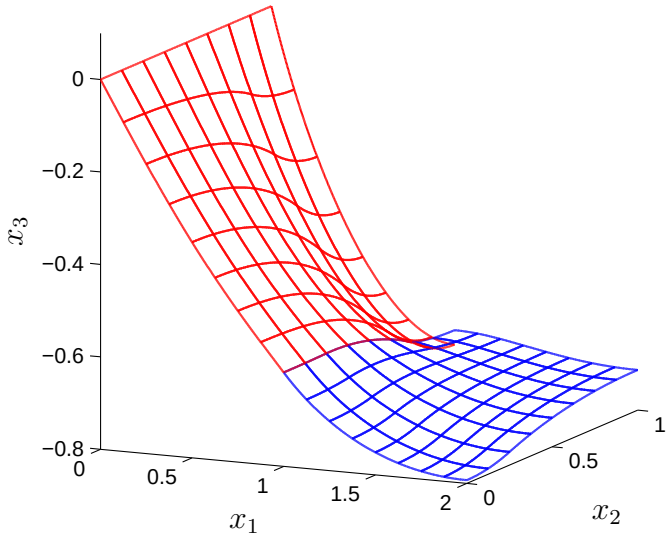

Figure 8.2: Membranes after deformation

	H			
H/h	1/2	1/4	1/8	1/16
64	2,048/778/2 58	8,192/3,622/2 67	3,2768/15,502/2 68	131,072/64,030/3 97
32	1,024/394/2 45	4,096/1,830/2 45	16,384/7,822/3 67	65,536/32,286/4 83
16	512/202/2 33	2,048/934/2 34	8,192/3,982/3 45	32,768/16,414/4 61
8	256/106/2 26	1,024/486/2 25	4,096/2,062/3 33	16,384/8,478/3 38
4	128/58/2 22	512/262/2 27	2,048/1,102/3 32	8,192/4,510/3 34

Table 8.1: Performance for varying decomposition and discretization

$\beta > 1$, so that

$$\rho_{k+1} := \beta \rho_k.$$

Let us also show choices of the SMALBE parameters used in our implementation:

$$\rho_0 := 10\|\mathsf{PFP}\|, \quad M := 1, \quad \beta := 10, \quad \eta := \|\mathsf{Pd}\|, \quad \boldsymbol{\mu}^0 := \mathbf{0}.$$

The initial approximation $\boldsymbol{\lambda}^0$ was set to $\max\{-\widetilde{\boldsymbol{\lambda}}, 0.5\mathsf{B}\widetilde{\mathbf{R}}_h\}$. The SMALBE stopping criterion was chosen as

$$\max\{\|\mathbf{g}^P(\boldsymbol{\lambda}^k, \boldsymbol{\mu}^k, \rho_k)\|, \|\mathsf{G}\boldsymbol{\lambda}^k\|\} \leq 10^{-4}\|\mathsf{Pd}\|.$$

For the MPRGP loops, we used the parameters

$$\overline{\alpha} := \frac{2}{\rho_k}, \quad \Gamma := 1.$$

Now let us show how the presented method works. In Table 8.1, we demonstrate the numerical scalability of the discussed pair of algorithms. The numbers appearing in this table have the following meaning: the upper row of each cell shows the corresponding primal dimension / dual dimension / number of the outer iterations and the lower row gives the number of the conjugate gradient (CG) iterations. Indeed, it can be seen that our method does not deteriorate when we keep the number of subdomain boundary nodes fixed and increase the number of subdomains. In other words,

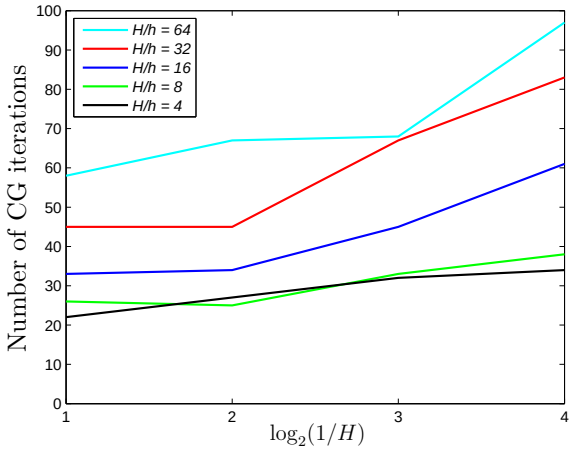

Figure 8.3: Graph of numerical scalability for the 2D model contact problem

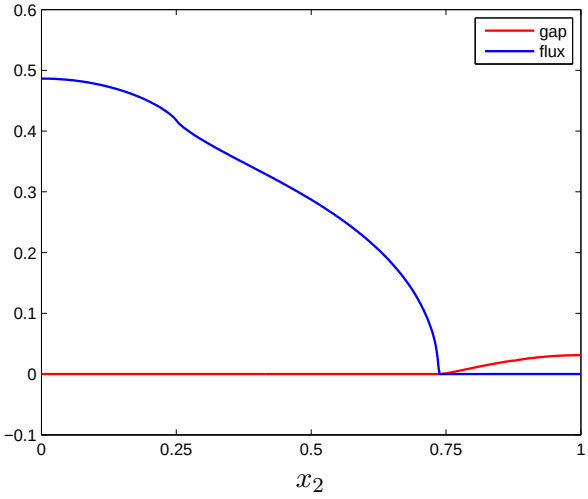

Figure 8.4: Approximation of the flux t^2 and the resulting gap between the membranes along $\Gamma_c := \{(1, x_2) : \ x_2 \in [0, 1]\}$

we observe that the numbers of the CG iterations in the rows vary only moderately. Furthermore, the graph illustrating the numerical scalability is depicted in Figure 8.3.

For the sake of completeness, in Figure 8.4, we also show the computed approximation of the boundary flux t^2 on the contact boundary Γ_c obtained for the choices $h := 1/512$ and $H := 1/8$. The blue curve stands for the contact flux while the red one means the resulting gap between the deformed membranes along Γ_c. It can be observed that the depicted flux corresponds to the problem definition.

8.2 Linear homogeneous isotropic elastostatics

8.2.1 Solution of 3D model contact problem

We shall present the numerical experiments for the model problem (4.19), (4.20). To recall the situation, we refer to Figure 4.2.

The elastic body is represented by the cube $\Omega := (0, 10)^3$ with the sizes given in millimeters. The material constants are defined by the following values: Young's modulus $E := 1.14 \cdot 10^5$ [MPa] and Poisson's ratio $\nu := 0.24$. This choice of the material parameters corresponds to steel. The body is fixed in all directions along the Dirichlet part of the boundary $\Gamma_u := [0, 10] \times \{0\} \times [0, 10]$. The body may touch the rigid plane obstacle along the contact part of the boundary $\Gamma_c := [0, 10] \times [0, 10] \times \{0\}$. The initial distance $|d|$ between the cube and the rigid obstacle is set to 3 [mm]. The remaining part of the boundary of the cube is free, i.e. it is neither loaded by any boundary forces nor fixed in any direction. The density of the internal forces is defined for any $x \in \Omega$ by $\underline{f}(x) := (0, 0, -2.1 \cdot 10^3)$ [N/mm^3], and therefore the volume forces may be interpreted, for instance, as a gravity.

The body was decomposed into identical cubic subdomains with the edge length H. We gradually chose decompositions into 2^3, 3^3, ..., 6^3 cubes which correspond to $H := 10/2,\ 10/3,\ \ldots,\ 10/6$. All subdomain boundaries were further discretized by the same triangular uniform meshes characterized by the discretization parameter h. Every component of the boundary stress and displacement was approximated by the piecewise constant and continuous piecewise linear trial functions ψ_l^m and φ_k^m, respectively (see Figure 5.2). The deformed body for the choice of parameters $h := 1/2$ and $H := 10/5$ is depicted in Figure 8.5. Splitting into subdomains is indicated by the chess–board on the surface.

The resulting bound and equality constrained quadratic programming

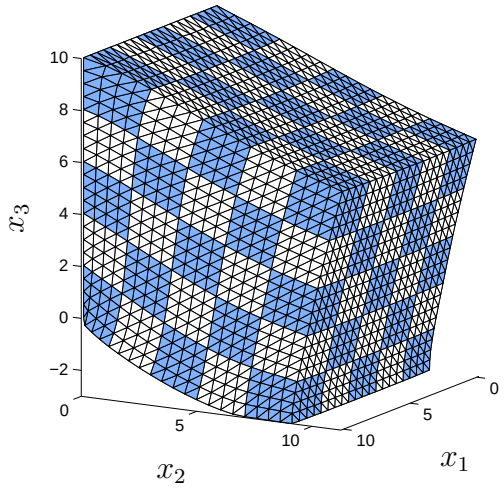

Figure 8.5: Body after deformation

problem (6.16) was solved by the SMALBE and MPRGP algorithms (see Sections 7.1 and 7.2, respectively). In every SMALBE iteration, if the increase of the augmented Lagrangian was not sufficient, we improved the precision of the next auxiliary problem solution by reducing the parameter M_k so that

$$M_{k+1} := \beta M_k, \quad \beta < 1. \tag{8.1}$$

Here we show the used choices of the SMALBE parameters:

$$\rho := \|\mathsf{PFP}\|, \quad M_0 := 1, \quad \beta := \frac{1}{10}, \quad \eta := \|\mathsf{Pd}\|, \quad \boldsymbol{\mu}^0 := \mathbf{0}.$$

The initial approximation $\boldsymbol{\lambda}^0$ was set to $\max\{-\widetilde{\boldsymbol{\lambda}}, 0.5\mathsf{B}\widetilde{\mathbf{R}}_h\}$. The stopping criterion was chosen as

$$\max\{\|\mathbf{g}^P(\boldsymbol{\lambda}^k, \boldsymbol{\mu}^k, \rho)\|, \|\mathsf{G}\boldsymbol{\lambda}^k\|\} \leq 10^{-4}\|\mathsf{Pd}\|.$$

The MPRGP algorithm used the parameters

$$\overline{\alpha} := \frac{2}{\rho}, \quad \Gamma := 1.$$

The numerical scalability of the discussed pair of algorithms is shown in Table 8.2. The upper row of each cell of the table shows the corresponding primal dimension / dual dimension / number of the outer iterations.

	H				
H/h	10/2	10/3	10/4	10/5	10/6
9	11,712/5,023/17 130	39,528/18,744/13 139	93,696/43,441/12 137	183,000/92,992/13 115	316,224/163,275/17 133
8	9,264/4,053/15 124	31,266/15,090/13 134	74,112/37,341/12 137	144,750/74,712/15 140	250,128/131,109/15 180
7	7,104/3,187/15 88	23,976/11,832/14 145	56,832/29,233/11 120	111,000/58,432/15 156	191,808/102,471/14 182
6	5,232/2,425/15 93	17,658/8,970/16 101	41,856/22,117/13 145	81,750/44,152/13 140	141,264/77,361/14 163
5	3,648/1,767/16 95	12,312/6,504/136 94	29,184/15,993/14 131	57,000/31,872/15 138	98,496/55,779/15 147
4	2,352/1,213/15 102	7,938/4,434/14 120	18,816/10,861/14 101	36,750/21,592/15 147	63,504/37,725/14 145
3	1,344/763/15 93	4,536/2,760/18 83	10,752/6,720/17 92	21,000/13,312/17 131	36,288/23,199/22 157
2	624/417/16 111	2,106/1,482/19 94	4,992/3,573/17 140	9,750/7,032/16 75	16,848/12,201/24 130

Table 8.2: Performance for varying decomposition and discretization

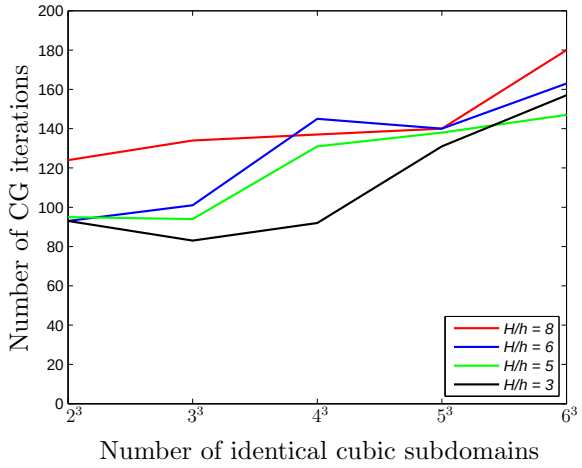

Figure 8.6: Graph of numerical scalability for the 3D model contact problem

The lower row gives the number of the CG iterations. Examining the numbers of iterations in the rows, which correspond to the fixed ratio H/h, we conclude that the number of iterations is constant up to the oscillations that are caused by the non–linearity of our problem. Furthermore, the graph illustrating the numerical scalability is depicted in Figure 8.6.

In Figure 8.7, we show the computed approximations of the normal boundary stress and normal boundary displacement on Γ_c obtained for the choices $h := 1/2$ and $H := 10/5$. The white line stands for the "contact border". It can be seen that at points on Γ_c with x_2–coordinate larger or equal to 8, the contact occurs. It is also seen that the contact stress t_3 is zero at points, where the body is not in contact with the obstacle, and positive at points, where the body touches the obstacle.

Now we shall give the results on the stable evaluation of the left generalized inverse discussed in Section 6.5. Since we decomposed the original cube into identical subdomains and used the same uniform meshes for all subdomains, all the local stiffness matrices $\widetilde{\mathsf{S}}^{\text{Lamé}}_{m,h}$ coincide. Thus we shall deal, for instance, only with the left generalized inverse $\widetilde{\mathsf{S}}^{\text{Lamé},+}_{1,h}$ of the matrix $\widetilde{\mathsf{S}}^{\text{Lamé}}_{1,h}$. At first, let us introduce the notation

$$err(\widetilde{\mathsf{S}}^{\text{Lamé},+}_{1,h}) := \frac{\|\widetilde{\mathsf{S}}^{\text{Lamé}}_{1,h}\,\widetilde{\mathsf{S}}^{\text{Lamé},+}_{1,h}\,\widetilde{\mathsf{S}}^{\text{Lamé}}_{1,h} - \widetilde{\mathsf{S}}^{\text{Lamé}}_{1,h}\|}{\|\widetilde{\mathsf{S}}^{\text{Lamé}}_{1,h}\|}$$

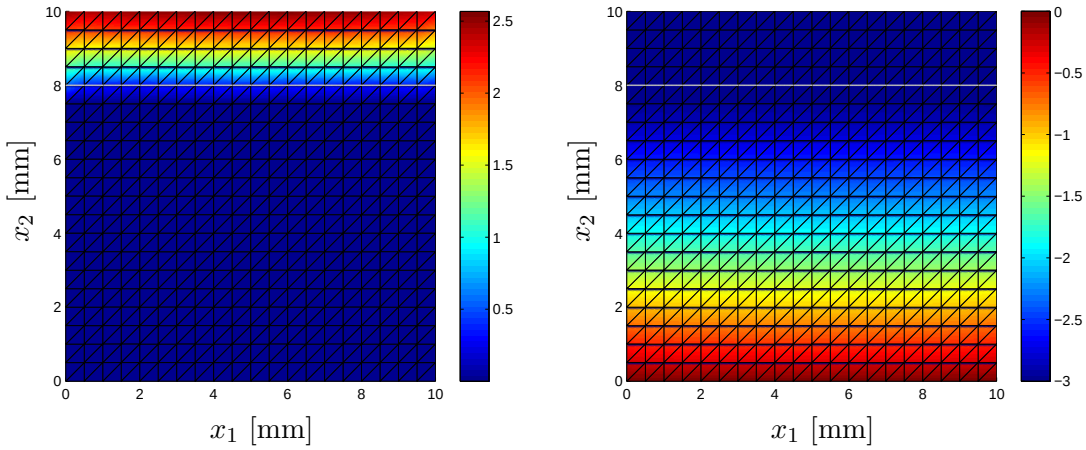

Figure 8.7: Computed approximations of t_3 [10^4 N/mm^2] (left) and u_3 [mm] (right) on Γ_c

for the relative error of the obtained left generalized inverse and the notations

$$\kappa(\mathsf{K}) := \frac{\lambda_{\max}(\mathsf{K})}{\lambda_{\min}(\mathsf{K})} \quad \text{and} \quad \overline{\kappa}(\widetilde{\mathsf{S}}_{1,h}^{\text{Lamé}}) := \frac{\lambda_{\max}(\widetilde{\mathsf{S}}_{1,h}^{\text{Lamé}})}{\lambda_7(\widetilde{\mathsf{S}}_{1,h}^{\text{Lamé}})}$$

for the spectral condition number of the corresponding non–singular part K of $\widetilde{\mathsf{S}}_{1,h}^{\text{Lamé}}$ and the ratio of the maximum and the least non–zero eigenvalue of $\widetilde{\mathsf{S}}_{1,h}^{\text{Lamé}}$, respectively. To recall the meaning of K, we refer to (6.22). Let us also recall that M_1 and N_1 stand for the numbers of nodes and elements on the boundary Γ_1, respectively. Performance of the technique presented in Section 6.5 is shown in Table 8.3. It may be seen that both the relative error and spectral condition number of the non–singular part K are reasonable. The choice of nodes x_i^1, x_j^1, and x_k^1 was made as it is depicted in Figure 8.8 (the node x_i^1 was either the mid point of the corresponding edge or the nearest node with the smaller x_1–coordinate). "Removed" indices from the stiffness matrix $\widetilde{\mathsf{S}}_{1,h}^{\text{Lamé}}$ were the same for all rows of Table 8.3 and they are depicted as arrows in Figure 8.8.

8.2.2 Solution of 3D Hertz problem

We shall now be concerned with a variant of the well–known Hertz problem. Let us consider two elastic bodies Ω^1 and Ω^2 made of isotropic and homogeneous materials. Assume that the body denoted by Ω^1 is a part of a ball

M_1	N_1	$err(\widetilde{\mathsf{S}}_{1,h}^{\text{Lamé},+})$	$\kappa(\mathsf{K})$	$\overline{\kappa}(\widetilde{\mathsf{S}}_{1,h}^{\text{Lamé}})$
26	48	$1.7 \cdot 10^{-7}$	$3 \cdot 10^{2}$	15
56	108	$1.3 \cdot 10^{-7}$	$7 \cdot 10^{2}$	21
98	192	$1.2 \cdot 10^{-7}$	$1.2 \cdot 10^{3}$	24
152	300	$1 \cdot 10^{-7}$	$1.7 \cdot 10^{3}$	27
218	432	$9.4 \cdot 10^{-8}$	$2.4 \cdot 10^{3}$	31
296	588	$8.3 \cdot 10^{-8}$	$3.4 \cdot 10^{3}$	36
386	768	$7.4 \cdot 10^{-8}$	$4.4 \cdot 10^{3}$	41
488	972	$6.7 \cdot 10^{-8}$	$5.6 \cdot 10^{3}$	45

Table 8.3: Stable evaluation of the left generalized inverse

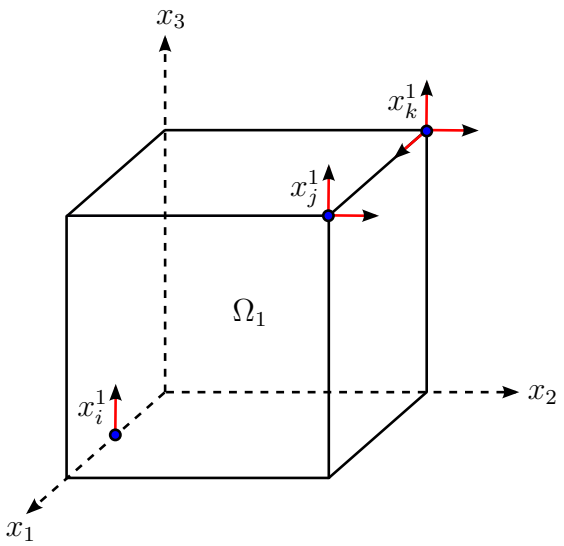

Figure 8.8: "Removed" indices from the local stiffness matrix

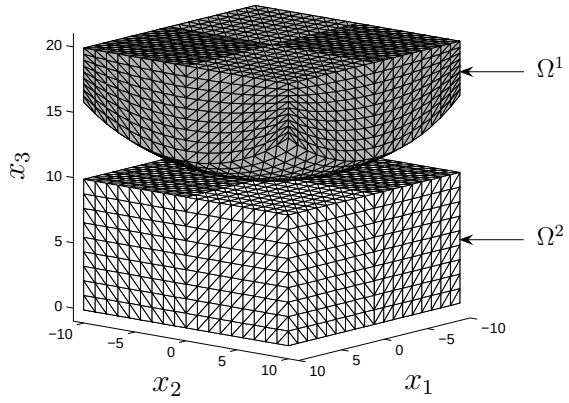

Figure 8.9: Initial geometry

with radius 20 [mm] and the body Ω^2 is a cuboid with the length and depth 20 [mm] and height 10 [mm] . Let us state the following definitions:

$$\Omega^1 := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \quad (x_1, x_2) \in (-10, 10)^2 \text{ and } x_3 \in \left(-\sqrt{20^2 - x_1^2 - x_2^2} + 30, 20 \right) \right\}$$

and

$$\Omega^2 := (-10, 10) \times (-10, 10) \times (0, 10).$$

For the sake of lucidity, the considered geometry is depicted in Figure 8.9 (we used some "quite accurate" polyhedral approximation of the upper body). Let us suppose that the body represented by Ω^1 is made of iron with the Young's modulus $E^1 := 2.1 \cdot 10^5$ [MPa] and Poisson's ratio $\nu^1 := 0.29$ and the body represented by Ω^2 is made of aluminium with the Young's modulus $E^2 := 7 \cdot 10^4$ [MPa] and Poisson's ratio $\nu^2 := 0.35$.

Now let us consider the following situation. The boundary $\Gamma^1 := \partial\Omega^1$ is decomposed into two parts

$$\Gamma_c^1 := \left\{ (x_1, x_2, x_3) \in \Gamma^1 : \; x_3 = -\sqrt{20^2 - x_1^2 - x_2^2} + 30 \right\}$$

and

$$\Gamma_f^1 := \Gamma^1 \setminus \Gamma_c^1$$

and the boundary $\Gamma^2 := \partial\Omega^2$ is decomposed into three parts

$$\Gamma_c^2 := [-10, 10] \times [-10, 10] \times \{10\},$$

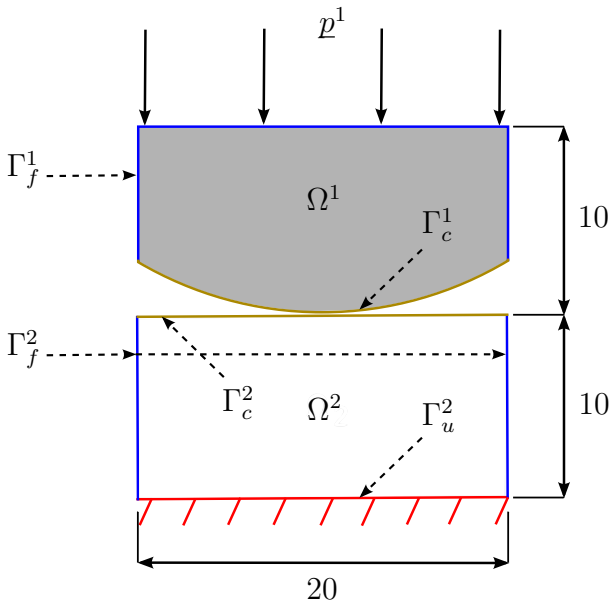

Figure 8.10: 2D analogue to the considered Hertz problem

$$\Gamma_u^2 := [-10, 10] \times [-10, 10] \times \{0\},$$

and

$$\Gamma_f^2 := \Gamma^2 \setminus \{\Gamma_c^2 \cup \Gamma_u^2\}.$$

We shall assume that the cuboid Ω^2 is fixed in all directions along Γ_u^2. On the part Γ_f^1, there acts a boundary stress $\underline{p}^1$ that is defined as

$$\underline{p}^1(x) := \begin{cases} (0, 0, -2 \cdot 10^3) & \text{for } x := (x_1, x_2, x_3) \in \Gamma_f^1 : \ x_3 = 20, \\ \underline{0} & \text{elsewhere on } \Gamma_f^1. \end{cases}$$

The stress $\underline{p}^1$ is given in [N/mm^2]. We see that the upper body is pressed down along its top side and stays free, i.e. unfixed and unloaded, on the remaining part of Γ_f^1. On the other hand, we assume that the cuboid is free on the whole part Γ_f^2, and therefore the boundary stress $\underline{p}^2$ shall be defined by the zero vector everywhere on Γ_f^2. There are no volume forces acting in the interior of Ω^1 and Ω^2. Finally, we can see that the "bottom part" Γ_c^1 of the upper body will come in contact with the top side Γ_c^2 of the cuboid. For better comprehension, we refer to Figure 8.10, which depicts a 2D analogue to our problem.

Now we intend to formulate the problem in terms of displacements. We

shall look for a sufficiently smooth $(\underline{u}^1, \underline{u}^2)$ satisfying

$$\begin{aligned} -\sum_{j=1}^{3} \tfrac{\partial}{\partial x_j}\sigma_{ij}(\underline{u}^m, x) &= 0 && \text{for } x \in \Omega^m,\ i = 1,2,3,\ m = 1,2, \\ \underline{u}^2(x) &= \underline{0} && \text{for } x \in \Gamma_u^2, \\ t_i^m(x) := \sum_{j=1}^{3} \sigma_{ij}(\underline{u}^m, x) n_j^m(x) &= p_i^m(x) && \text{for } x \in \Gamma_f^m,\ i = 1,2,3,\ m = 1,2, \end{aligned} \tag{8.2}$$

together with the contact conditions

$$\begin{aligned} u_3^2(x^2) - u_3^1(x^1) &\le x_3^1 - 10 \\ t_3^1(x^1) &\ge 0 \\ \big(u_3^2(x^2) - u_3^1(x^1) - (x_3^1 - 10)\big)\, t_3^1(x^1) &= 0 \\ t_3^1(x^1) + t_3^2(x^2) &= 0 \end{aligned} \tag{8.3}$$

that hold for $\Gamma_c^1 \ni x^1 := (a, b, x_3^1)$, $\Gamma_c^2 \ni x^2 := (a, b, 10)$, $(a, b) \in [-10, 10]^2$. By $n_j^m(x)$ we mean jth component of the exterior unit normal vector $\underline{n}^m(x)$ of Ω^m that is defined for almost all $x \in \Gamma^m$. Since we assume that the both materials are homogeneous and isotropic, the stress tensor $\{\sigma_{ij}(\underline{u}^m, x)\}_{i,j=1}^3$ complies with the Hook law (3.25), where the strain tensor $\{e_{ij}(\underline{u}^m, x)\}_{i,j=1}^3$ is given by (3.26).

Let us now briefly describe the contact conditions (8.3). The first condition says that the bodies are not allowed to penetrate each other. Furthermore, at points, where contact occurs, the upper body may press the lower one down, while at points Γ_c^1, where the bodies do not touch, the normal boundary stress corresponding to Ω^1 has to be zero. The fourth condition stands for the "action and reaction" condition.

Now due to the symmetry in our geometry and data, the following simplification may be achieved. Instead of the original bodies Ω^1 and Ω^2, we shall consider only their quarters $\widetilde{\Omega}^1$ and $\widetilde{\Omega}^2$ as they are sketched in Figure 8.11. To be precise, let us note that

$$\widetilde{\Omega}^1 := \Big\{ (x_1, x_2, x_3) \in \mathbb{R}^3 :\ (x_1, x_2) \in (0, 10)^2 \text{ and } x_3 \in \Big(-\sqrt{20^2 - x_1^2 - x_2^2} + 30, 20 \Big) \Big\} \tag{8.4}$$

and

$$\widetilde{\Omega}^2 := (0, 10)^3.$$

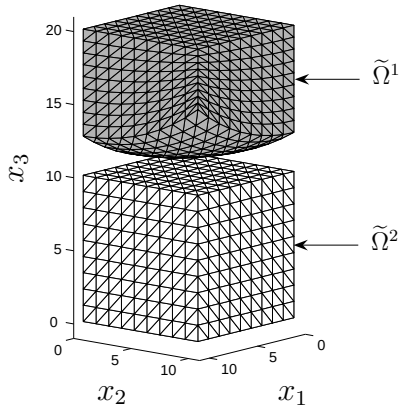

Figure 8.11: "Reduced" initial geometry

To deal still with the same problem, we now have to modify our data slightly and prescribe some additional Dirichlet conditions. Let $\widetilde{\Gamma}^1 := \partial\widetilde{\Omega}^1$ and

$$\widetilde{\Gamma}^1_{u,x_1} := \{(x_1, x_2, x_3) \in \widetilde{\Gamma}^1 : \ x_1 = 0\},$$

$$\widetilde{\Gamma}^1_{u,x_2} := \{(x_1, x_2, x_3) \in \widetilde{\Gamma}^1 : \ x_2 = 0\},$$

$$\widetilde{\Gamma}^1_c := \left\{(x_1, x_2, x_3) \in \widetilde{\Gamma}^1 : \ x_3 = -\sqrt{20^2 - x_1^2 - x_2^2} + 30\right\},$$

and

$$\widetilde{\Gamma}^1_f := \widetilde{\Gamma}^1 \setminus \left\{\widetilde{\Gamma}^1_c \cup \widetilde{\Gamma}^1_{u,x_1} \cup \widetilde{\Gamma}^1_{u,x_2}\right\}.$$

Furthermore, let $\widetilde{\Gamma}^2 := \partial\widetilde{\Omega}^2$ and

$$\widetilde{\Gamma}^2_{u,x_1} := \{0\} \times [0, 10] \times [0, 10], \quad \widetilde{\Gamma}^2_{u,x_2} := [0, 10] \times \{0\} \times [0, 10],$$

$$\widetilde{\Gamma}^2_u := [0, 10] \times [0, 10] \times \{0\}, \quad \widetilde{\Gamma}^2_c := [0, 10] \times [0, 10] \times \{10\},$$

and

$$\widetilde{\Gamma}^2_f := \widetilde{\Gamma}^2 \setminus \left\{\widetilde{\Gamma}^2_c \cup \widetilde{\Gamma}^2_{u,x_1} \cup \widetilde{\Gamma}^2_{u,x_2} \cup \widetilde{\Gamma}^2_u\right\}.$$

Now we shall formulate a new problem. The only difference to the original problem will be an additional fixation along the parts $\widetilde{\Gamma}^m_{u,x_i}$ which will prevent

the deformation in the corresponding direction. The task is to find sufficiently smooth $(\underline{\widetilde{u}}^1, \underline{\widetilde{u}}^2)$ such that

$$\begin{aligned}
-\sum_{j=1}^{3} \tfrac{\partial}{\partial x_j}\sigma_{ij}(\underline{\widetilde{u}}^m, x) &= 0 && \text{for } x \in \widetilde{\Omega}^m,\ i=1,2,3,\ m=1,2,\\
\underline{\widetilde{u}}^2(x) &= \underline{0} && \text{for } x \in \widetilde{\Gamma}^2_u,\\
\widetilde{u}^m_i(x) &= 0 && \text{for } x \in \widetilde{\Gamma}^m_{u,x_i},\ i=1,2,\ m=1,2,\\
\widetilde{t}^m_i(x) := \sum_{j=1}^{3} \sigma_{ij}(\underline{\widetilde{u}}^m, x)\widetilde{n}^m_j(x) &= p^m_i(x) && \text{for } x \in \widetilde{\Gamma}^m_f,\ i=1,2,3,\ m=1,2,
\end{aligned} \tag{8.5}$$

together with the contact conditions

$$\begin{aligned}
\widetilde{u}^2_3(x^2) - \widetilde{u}^1_3(x^1) &\le x^1_3 - 10\\
\widetilde{t}^1_3(x^1) &\ge 0\\
\big(\widetilde{u}^2_3(x^2) - \widetilde{u}^1_3(x^1) - (x^1_3 - 10)\big)\,\widetilde{t}^1_3(x^1) &= 0\\
\widetilde{t}^1_3(x^1) + \widetilde{t}^2_3(x^2) &= 0
\end{aligned} \tag{8.6}$$

that hold for $\widetilde{\Gamma}^1_c \ni x^1 := (a, b, x^1_3)$, $\widetilde{\Gamma}^2_c \ni x^2 := (a, b, 10)$, $(a, b) \in [0, 10]^2$. By $\widetilde{n}^m_j(x)$ we mean jth component of the exterior unit normal vector $\underline{\widetilde{n}}^m(x)$ of $\widetilde{\Omega}^m$ that is defined for almost all $x \in \widetilde{\Gamma}^m$.

We solve the "reduced" problem (8.5), (8.6) analogously to our model problem of linear elastostatics (4.19), (4.20), i.e. we start with a non–overlapping domain decomposition, then continue with the boundary element discretization by using the Ritz method and application of the duality theory and orthogonal projectors to the natural coarse grid. We end up with the resulting bound and equality constrained quadratic programming problem (6.16).

At this moment, we shall give the obtained numerical results. We used the decompositions into $2 \cdot 2^3$ and $2 \cdot 3^3$ subdomains characterized by the decomposition parameters $H := 10/2$ and $H := 10/3$, respectively. Let us note that the subdomains corresponding to $\widetilde{\Omega}^2$ were identical cubic regions, while the decompositions of the body $\widetilde{\Omega}^1$ are depicted by using the chess–board in Figure 8.12 (we still use some "quite accurate" polyhedral approximation of $\widetilde{\Omega}^1$). All subdomain boundaries were further discretized by triangular meshes with the discretization parameter h. All subdomain boundaries corresponding to $\widetilde{\Omega}^2$ were meshed uniformly with the same grid. For the approximation of every component of the boundary stress and displacement, we used the piecewise constant and continuous piecewise linear trial functions ψ^m_l and φ^m_k, respectively (see Figure 5.2). The computed deformation for the choice

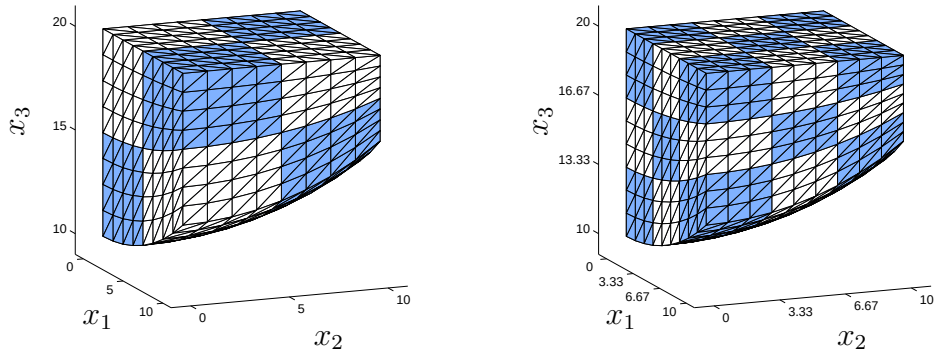

Figure 8.12: Splitting $\widetilde{\Omega}^1$ into 2^3 (left) and 3^3 (right) subdomains

of parameters $h := 10/9$ and $H := 10/3$ is depicted in Figure 8.13. Splitting into subdomains is indicated by the chess–boards on both bodies' surfaces.

The resulting problem (6.16) was solved by algorithms SMALBE and MPRGP described in Sections 7.1 and 7.2, respectively. We used the variant of SMALBE which decreases the parameter M whenever the augmented Lagrangian does not increase enough, i.e. uses the update (8.1). Except the initial setting of M, we used the very same parameters and stopping criterion for the SMALBE and MPRGP algorithms as in the previous subsection. In this case, we put

$$M_0 := \frac{1}{2}.$$

In Table 8.4, we can see the performance of our algorithms. As in the previous examples, the upper row of each cell of the table shows the corresponding primal dimension / dual dimension / number of the outer iterations. The lower row gives the number of the CG iterations. It can be seen that the algorithms work worse than in Subsection 8.2.1, since we deal here with the moving contact interface and the identification of contact points is more difficult.

Now we shall give some additional figures to study better the obtained solution. Let us start with Figure 8.14, where we depict the initial gap and the gap between the bodies after deformation. In Figure 8.15, we show the computed approximations of the normal boundary stress $\widetilde{t}_3^2$ and normal boundary displacement $\widetilde{u}_3^2$ on $\widetilde{\Gamma}_c^2$. The white curves in Figures 8.14 and 8.15 mean the "contact border". The depicted situations correspond to the choices $h := 10/9$ and $H := 10/3$.

Let us also briefly mention the evaluation of the left generalized inverses.

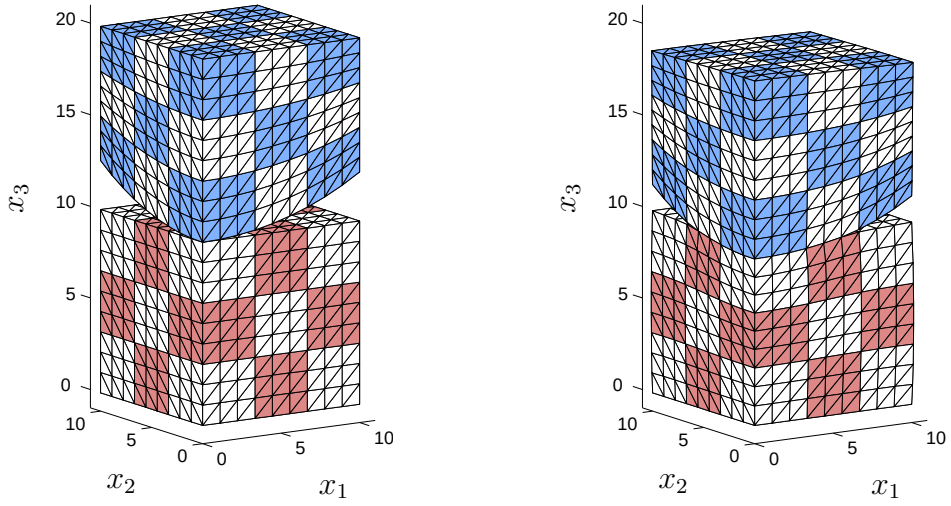

Figure 8.13: Bodies before (left) and after (right) deformation

	H	
H/h	10/2	10/3
5	7,248/3,534/21 162	24,624/13,008/23 221
4	4,704/2,426/24 272	15,876/8,868/24 338
3	2,688/1,526/25 255	9,072/5,520/30 327
2	1,248/834/28 236	4,212/2,964/29 304

Table 8.4: Performance for varying decomposition and discretization

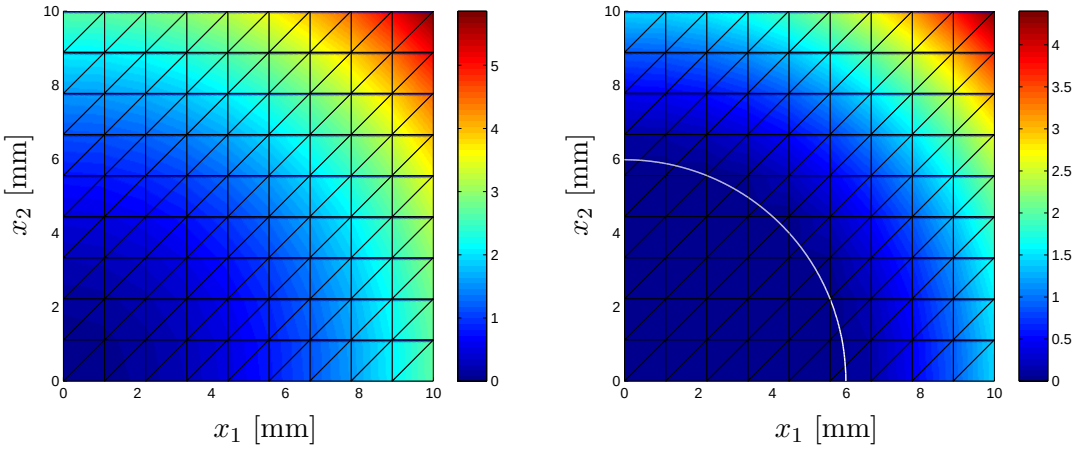

Figure 8.14: Gap between the bodies before (left) and after (right) deformation

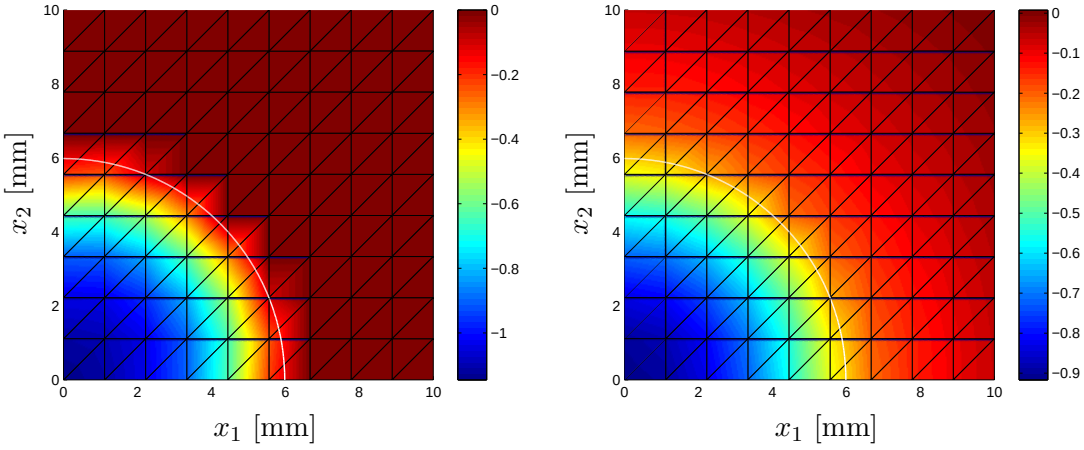

Figure 8.15: Computed approximations of $\widetilde{t}_3^2$ [10^4 N/mm^2] (left) and $\widetilde{u}_3^2$ [mm] (right) on $\widetilde{\Gamma}_c^2$

Since we decomposed the lower body into identical subdomains which were all meshed by the same grid, we get that all the local stiffness matrices corresponding to $\widetilde{\Omega}^2$ coincide. On the other hand, since the subdomains of $\widetilde{\Omega}^1$ differ, so do the local stiffness matrices corresponding to the upper body. For every considered case of discretization and domain decomposition, we computed the characteristics which are given in Table 8.3, i.e. the relative error of the left generalized inverse, spectral condition number of the related non–singular part, and ratio of the maximum and the least non–zero eigenvalue of the associated stiffness matrix. Let us note that the obtained numbers were of the very same order as they are given in Table 8.3.

Conclusion

In this thesis, we presented solutions of 2D and 3D model contact problems described by the Laplace operator and the system of linear homogeneous isotropic elastostatics, respectively, by using the BETI method preconditioned by the natural coarse grid. Our approach is based on the observation (following from the analyses of Langer and Steinbach [32] and Farhat et al. [27]) that when the mesh is refined and the number of subdomains is increased so that the ratio H/h of the decomposition parameter H and the discretization parameter h is kept constant, the bounds on the spectrum of the preconditioned dual stiffness matrix (arising from the application of boundary elements) do not change. The latter result enables us to extend the work of Dostál and Horák [19] to the case of boundary elements and develop a scalable BETI based algorithm for the variational inequalities, whose solution to a prescribed precision can be found in a number of matrix–vector multiplications that is bounded independently of the discretization provided the ratio H/h is bounded. The presented results of our numerical experiments are in a good agreement with the theory.

There are many issues which are closely connected to this topic and to which we could devote ourselves in future. For instance, further improvement could be achieved by application of the standard BETI preconditioners [32] to the solution of auxiliary linear problem in the inner loop of our algorithm. Also overcoming the drawback represented by the densely populated matrices by using some Fast BE technique would improve significantly efficiency of the discussed method. Our approach could be also adapted to the solution of problems with Coulomb friction [26, 21].

Bibliography

[1] Axelsson, O.: Iterative Solution Methods. Cambridge University Press, Cambridge (1994).

[2] Bertsekas, D. P.: Nonlinear Optimization. Athena Scientific, Belmont (1999).

[3] Börm, S., Grasedyck, L., Hackbusch, W.: Hierarchical Matrices. Max Planck Institute for Mathematics in the Sciences, Leipzig (2003).

[4] Bouchala, J., Dostál, Z., Sadowská, M.: Solution of boundary variational inequalities by combining fast quadratic programming algorithms with symmetric BEM. Advances in Boundary Integral Methods – Proceedings of the Fifth UK Conference on Boundary Integral Methods, University of Liverpool, 221–228 (2005).

[5] Bouchala, J., Dostál, Z., Sadowská, M.: Solving 2D Contact Problem by Boundary Element Tearing and Interconnecting Method. Advances in Boundary Integral Methods – Proceedings of the Sixth UK Conference on Boundary Integral Methods, Durham University, 63–70 (2007).

[6] Bouchala, J., Dostál, Z., Sadowská, M.: Scalable BETI for Variational Inequalities. Lecture Notes in Computational Science and Engineering – selection of 71 refereed papers presented at the 17th International Conference on Domain Decomposition Methods, Springer – Verlag Berlin Heidelberg, 167–174 (2008).

[7] Bouchala, J., Dostál, Z., Sadowská, M.: Theoretically supported scalable BETI method for variational inequalities. Computing **82**, 53–75 (2008).

[8] Costabel, M.: Boundary integral operators on Lipschitz domains: Elementary results. SIAM J Math Anal **19**, 613–626 (1988).

[9] Conn, A. R., Gould, N. I. M., Toint, Ph. L.: A globally convergent augmented Lagrangian algorithm for optimization with general constraints and simple bounds. SIAM J Numer Anal **28**, 545–572 (1991).

[10] Domorádová, M., Dostál, Z., Sadowská, M.: Superrelaxation in minimizing quadratic functions subject to bound constraints. In preparation.

[11] Dostál, Z.: Optimal Quadratic Programming Algorithms with Applications to Variational Inequalities. Springer Monograph (to appear).

[12] Dostál, Z.: An optimal algorithm for bound and equality constrained quadratic programming problems with bounded spectrum. Computing **78**, 311–328 (2006).

[13] Dostál, Z.: Inexact semimonotonic Augmented Lagrangians with optimal feasibility convergence for quadratic programming with simple bounds and equality constraints. SIAM J Numer Anal **45**, 500–513 (2007).

[14] Dostál, Z., Friedlander, A., Santos, S. A.: Augmented Lagrangians with adaptive precision control for quadratic programming with simple bounds and equality constraints. SIAM J Optim **13**, 1120–1140 (2003).

[15] Dostál, Z., Gomes, F. A. M., Santos, S. A.: Duality based domain decomposition with natural coarse space for variational inequalities. J Comput Appl Math **126**, 397–415 (2000).

[16] Dostál, Z., Horák, D.: Scalability and FETI based algorithm for large discretized variational inequalities. Math Comput Sim **61**, 347–357 (2003).

[17] Dostál, Z., Horák, D.: Scalable FETI with optimal dual penalty for a variational inequality. Numer Lin Alg Appl **11**, 455–472 (2004).

[18] Dostál, Z., Horák, D.: Scalable FETI with optimal dual penalty for semicoercive variational inequalities. Contemp Math **329**, 79–88 (2003).

[19] Dostál, Z., Horák, D.: Theoretically supported scalable FETI for numerical solution of variational inequalities. SIAM J Numer Anal **45**, 500–513 (2007).

[20] Dostál, Z., Horák, D., Kučera, R.: Total FETI – an easier implementable variant of the FETI method for numerical solution of elliptic PDE. Commun Numer Meth Eng **22**, 1155–1162 (2006).

[21] Dostál, Z., Horák, D., Kučera, R., Vondrák, V., Haslinger, J., Dobiáš, J., Pták, S: FETI based algorithms for contact problems: scalability, large displacements and 3D Coulomb friction. Comput Meth Appl Mech Eng **194**, 395–409 (2005).

[22] Dostál, Z., Horák, D., Stefanica, D.: A scalable FETI–DP algorithm for coercive variational inequalities. IMACS J Appl Numer Math **54**, 378–390 (2005).

[23] Dostál, Z., Malík, J., Friedlander, A., Santos, S. A.: Analysis of semicoercive contact problems using symmetric BEM and augmented Lagrangians. Eng Anal Bound El **18**, 195–201 (1996).

[24] Dostál, Z., Markopoulos, A., Menšík, M.: Evaluating the action of a generalized inverse of the stiffness matrix of a floating structure with rigid modes. Submitted.

[25] Dostál, Z., Schöberl, J.: Minimizing Quadratic Functions Subject to Bound Constraints with the Rate of Convergence and Finite Termination. Comput Opt Appl **30**, 23–43 (2005).

[26] Eck, C., Steinbach, O., Wendland, W. L.: A symmetric boundary element method for contact problems with friction. Math Comput Sim **50** 43–61 (1999).

[27] Farhat, C., Mandel, J., Roux, F.-X.: Optimal convergence properties of the FETI domain decomposition method. Comput Meth Appl Mech Eng **115**, 365–385 (1994).

[28] Farhat, C., Roux, F.-X.: A method of finite element tearing and interconnecting and its parallel solution algorithm. Int J Numer Meth Eng **32**, 1205–1227 (1991).

[29] Farhat, C., Roux, F.-X.: An unconventional domain decomposition method for an efficient parallel solution of large–scale finite element systems. SIAM J Sci Comput **13**, 379–396 (1992).

[30] Glowinski, R., Lions, J. L., Tremolieres, R.: Numerical Analysis of Variational Inequalities, North–Holland, Amsterdam (1981).

[31] Gray, L. J.: Evaluation of singular and hypersingular Galerkin integrals: direct limits and symbolic computation. Singular integrals in the boundary element method: advances in boundary elements, Comput Mech Publ, Chapter 2 (1998).

[32] Langer, U., Steinbach, O.: Boundary element tearing and interconnecting methods. Computing **71** 205–228 (2003).

[33] Haslinger, J.: Metoda konečných prvků pro řešení eliptických rovnic a nerovnic. MFF UK, Praha (1980). (In Czech)

[34] Haslinger, J., Hlaváček, I., Lovíšek, J., Nečas, J.: Riešenie variačných nerovností v mechanike. Alfa (1982). (In Slovak)

[35] Horák, D.: FETI based domain decomposition methods for variational inequalities. PhD Thesis, VŠB – Technical University of Ostrava (2005).

[36] McLean, W.: Strongly Elliptic Systems and Boundary Integral Equations. Cambridge University Press (2000).

[37] Menšík, M.: Cholesky decomposition of positive semidefinite matrices with known kernel. Bc Thesis, VŠB – Technical University of Ostrava, Ostrava (2007). (In Czech)

[38] Of, G.: BETI - Gebietszerlegungsmethoden mit schnellen Randelementverfahren und Anwendungen. PhD Thesis, University of Stuttgart (2006). (In German)

[39] Of, G., Steinbach, O., Wendland, W. L.: Applications of a fast multipole Galerkin in boundary element method in linear elastostatics, Comput Visual Sci **8**, 201–209 (2005).

[40] Of, G., Steinbach, O., Wendland, W. L.: The fast multipole method for the symmetric boundary integral formulation, IMA J Numer Anal **26**, 272–296 (2006).

[41] Rjasanow, S., Steinbach, O.: The Fast Solution of Boundary Integral Equations. Mathematical and Analytical Techniques with Applications to Engineering, Springer – New York (2007).

[42] Sadowská, M.: Solution of variational inequalities by the boundary integral equations. MSc Thesis, VŠB – Technical University of Ostrava (2005). (In Czech)

[43] Steinbach, O.: Fast evaluation of Newton potentials in boundary element methods. East–West J Numer Math **7**, 211–222 (1999).

[44] Steinbach, O.: Stability Estimates for Hybrid Coupled Domain Decomposition Methods. Lecture Notes in Mathematics, Springer – Verlag Berlin Heidelberg (2003).

[45] Steinbach, O.: Numerical Approximation Methods for Elliptic Boundary Value Problems. Finite and Boundary Elements. Springer – New York (2008).

[46] Toselli, A., Widlund, O. B.: Domain Decomposition Methods – Algorithms and Theory. Springer – Verlag Berlin Heidelberg (2005).