LINEÁRNÍ ALGEBRA 1

1.) Komplexní čísla

množina komplexních čísel : $\mathbb{C} = \{ x + iy \mid x, y \in \mathbb{R} \}$

i … imaginární jednotka : $i^2 = -1$

operace na množině $\mathbb{C}$:

a) sčítání : $(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$

b) násobení : $(a_1 + ib_1) \cdot (a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)$

Př.: Převeďte na algebraický tvar (tj. na tvar $z = a + ib$, kde $a, b \in \mathbb{R}$).

1. $(1+3i)^2 = 1^2 + 6i + (3i)^2 = \underline{\underline{-8+6i}}$

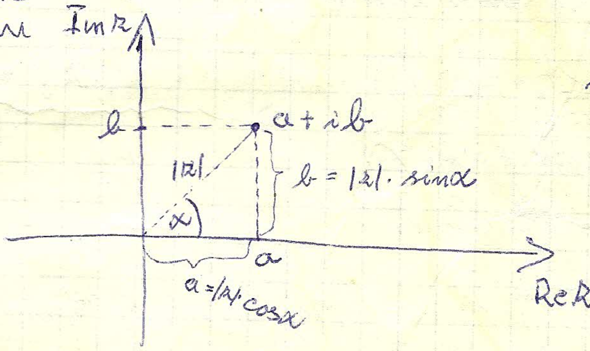
2. $\dfrac{1}{3-2i} = \dfrac{1}{3-2i} \cdot \dfrac{3+2i}{3+2i} = \dfrac{3+2i}{9+4} = \dfrac{3+2i}{13} = \underline{\underline{\dfrac{3}{13} + \dfrac{2}{13} i}}$

   $(a-b)(a+b) = a^2 - b^2$

3. $\dfrac{6+8i}{1-i} = \dfrac{(6+8i)(1+i)}{(1-i)(1+i)} = \dfrac{(6+8i)(1+i)}{2} = (3+4i)(1+i) = \underline{\underline{-1+7i}}$

4. $(1+i)^{156} = ?$ možno udělat podle binomické věty, ale jednodušší je to pomocí :

Goniometrický tvar komplexního čísla



z Pyth. věty: $z = \sqrt{a^2 + b^2}$

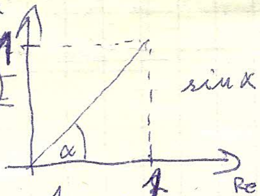
$a + ib = |z| \cos\alpha + i |z| \sin\alpha$

$$\boxed{a + ib = |z|(\cos\alpha + i \sin\alpha)}$$

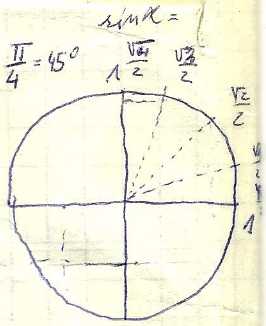
Využití : V tomto tvaru je možno komplexní čísla snadno násobit, dělit, umocňovat a odmocňovat.

Př. Převeďte na goniom. tvar

1. ~~$1+i\sqrt{2} = \sqrt{2+2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right) = \sqrt{6}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$~~



$\sin\alpha = \frac{\sqrt{2}}{2}\,\frac{1}{\sqrt{2}} \Rightarrow \alpha = \frac{\pi}{4} = 45°$



~~2.~~

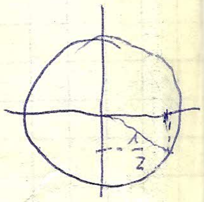
1. $z = 1+i \Rightarrow |z| = \sqrt{1^2+1^2} = \sqrt{2}$, $\cos\alpha = \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}}$, $\sin\alpha = \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} \Rightarrow \alpha = \frac{\pi}{4} = 45°$

$\Rightarrow z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

2. $2+2\sqrt{3}\,i \Rightarrow |z| = \sqrt{4+12} = \sqrt{16} = 4$, $\cos\alpha = \frac{2}{4} = \frac{1}{2}$, $\sin\alpha = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2} \Rightarrow \alpha = \frac{\pi}{3}$

$\Rightarrow z = 4\cdot\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

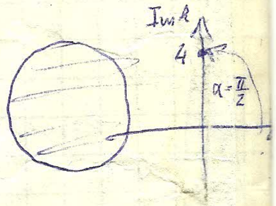
3. $3-\sqrt{3}\,i \Rightarrow |z| = \sqrt{9+3} = \sqrt{12} = 2\sqrt{3}$, $\sin\alpha = \frac{-\sqrt{3}}{2\sqrt{3}} = \frac{-1}{2}$, $\cos\alpha = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$ $\Rightarrow \alpha = 2\pi - \frac{\pi}{6}$, $\alpha = \frac{11}{6}\pi$



$\Rightarrow z = 2\sqrt{3}\left(\cos\left(\frac{11}{6}\pi\right) + i\sin\left(\frac{11}{6}\pi\right)\right)$

4. $0+4i \Rightarrow |z| = \sqrt{0^2+4^2} = 4$, $\Rightarrow \alpha = \frac{\pi}{2} \Rightarrow$

$z = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$



Pozor!:

5. $0-4i \Rightarrow |z| = \sqrt{0^2+4^2} = 4$, $\alpha = -\frac{\pi}{2} \Rightarrow \alpha = \frac{3}{2}\pi$

$z = 4\left(\cos\frac{3}{2}\pi + i\sin\frac{3}{2}\pi\right)$

Násobení a dělení v goniometrickém tvaru

$z_1 = |z_1|(\cos\alpha_1 + i\sin\alpha_1)$

$z_2 = |z_2|(\cos\alpha_2 + i\sin\alpha_2)$

$\Rightarrow$

$$\boxed{\begin{aligned} z_1\cdot z_2 &= |z_1||z_2|\left(\cos(\alpha_1+\alpha_2) + i\sin(\alpha_1+\alpha_2)\right) \\ \frac{z_1}{z_2} &= \frac{|z_1|}{|z_2|}\left(\cos(\alpha_1-\alpha_2) + i\sin(\alpha_1-\alpha_2)\right) \end{aligned}}$$

Př:

1. $z_1 = 3(\cos 20° + i\sin 20°)$
$z_2 = 6(\cos 45° + i\sin 45°)$

$\Rightarrow z_1\cdot z_2 = 18(\cos 65° + i\sin 65°)$

$\frac{z_1}{z_2} = \frac{1}{2}\left(\cos(-25°) + i\sin(-25°)\right) =$

$= \frac{1}{2}(\cos 335° + i\sin 335°)$

Umocňování a odmocňování v goniometrickém tvaru

Pro libovolné přirozené číslo n platí: Jestliže $z = |z| \cdot (\cos\alpha + i \sin\alpha)$ pak

$$\boxed{\begin{array}{l} z^n = |z|^n \left(\cos(n\alpha) + i \cdot \sin(n\alpha) \right) \\[2ex] \sqrt[n]{z} = \sqrt[n]{|z|} \left(\cos \frac{\alpha + 2k\pi}{n} + i \sin \frac{\alpha + 2k\pi}{n} \right) \quad \alpha \text{ v radiánech}, \ k = 0, 1, \dots, n-1 \end{array}}$$

α ve stupních $\Rightarrow$ $\frac{\alpha + k \cdot 360^\circ}{n}$

Př.

1. $z = 64 (\cos 60^\circ + i \sin 60^\circ) \quad \Rightarrow$

a) $z^2 = 64^2 (\cos 120^\circ + i \sin 120^\circ) \qquad \left(= 64^2 \left(\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right) = \frac{64^2}{2} \cdot \sqrt{3} - i \frac{64^2}{2} \right)$

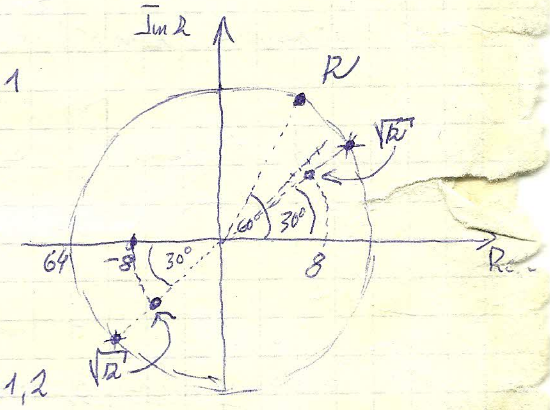
$z^3 = 64^3 (\cos 180^\circ + i \sin 180^\circ) \qquad \left(= 64^3 (-1 + i \cdot 0) = -64^3 \right)$

$\vdots$

b) $\sqrt[2]{z} = \sqrt{64} \left(\cos \frac{60 + \cancel{2k\pi}\ k \cdot 360^\circ}{2} + i \sin \frac{60 + \cancel{2k\pi}\ k \cdot 360^\circ}{2} \right), \ k = 0, 1$

$\Rightarrow \sqrt[2]{z} = 8 (\cos 30^\circ + i \sin 30^\circ)$

$\Rightarrow \sqrt[2]{z} = 8 (\cos 210^\circ + i \sin 210^\circ)$

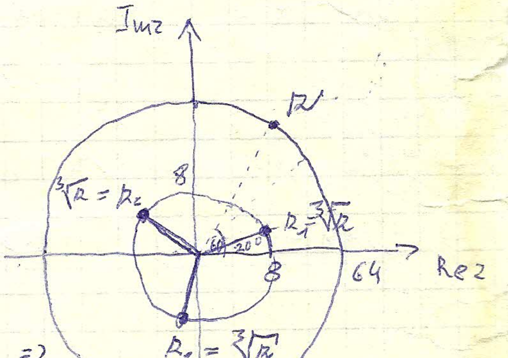


$\sqrt[3]{z} = \sqrt[3]{64} \left(\cos \frac{60^\circ + k \cdot 360^\circ}{3} + i \sin \frac{60^\circ + k \cdot 360^\circ}{3} \right), \ k = 0, 1, 2$

$\sqrt[3]{z} = 4 \cdot (\cos 20^\circ + i \sin 20^\circ)$

$\sqrt[3]{z} = 4 \cdot (\cos 140^\circ + i \sin 140^\circ)$

$\sqrt[3]{z} = 4 \cdot (\cos 260^\circ + i \sin 260^\circ)$



2. $z = (1+i)^3 \quad \Rightarrow \quad \sqrt[4]{z} = ?$

$z = \left(\sqrt{1^2 + 1^2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right)^3 = (\sqrt{2})^3 \left(\cos \frac{3}{4}\pi + i \sin \frac{3}{4}\pi \right) \Rightarrow$

$\sqrt[4]{z} = \sqrt[4]{\sqrt{2}^3} \left(\cos \frac{3}{16}\pi + i \sin \frac{3}{16}\pi \right) = 2^{\frac{3}{8}} \left(\cos \frac{3}{16}\pi + i \sin \frac{3}{16}\pi \right)$

$= 2^{\frac{3}{8}} \left(\cos \left(\frac{3}{16}\pi + \frac{2\pi}{4} \right) + i \sin \left(\frac{3}{16}\pi + \frac{2\pi}{4} \right) \right) = 2^{\frac{3}{8}} \left(\cos \frac{11}{16}\pi + i \sin \frac{11}{16}\pi \right)$

$= 2^{\frac{3}{8}} \left(\cos \frac{19}{16}\pi + i \sin \frac{19}{16}\pi \right)$

$= 2^{\frac{3}{8}} \left(\cos \frac{27\pi}{16} + i \sin \frac{27\pi}{16} \right)$

Př: Určete :

a) $\sqrt[2]{z}$, kde $z = 15\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

b) $\sqrt[3]{z}$, kde $z = 12\left(\cos 18^\circ + i\sin 18^\circ\right)$

c) $\sqrt[4]{2+2i}$

d) $\sqrt[3]{\frac{12}{3-\sqrt{3}i}}$

ad a) $z = 15\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \Rightarrow \sqrt[2]{z} = \begin{cases} \sqrt{15}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) \\ \sqrt{15}\left(\cos\left(\frac{\pi}{12}+\frac{2\pi}{2}\right) + i\sin\left(\frac{\pi}{12}+\frac{2\pi}{2}\right)\right) = \\ = \sqrt{15}\left(\cos\left(\frac{13}{12}\pi\right) + i\sin\left(\frac{13}{12}\pi\right)\right) \end{cases}$

ad b) $z = 12\left(\cos 18^\circ + i\sin 18^\circ\right) \Rightarrow \sqrt[3]{z} = \begin{cases} = \sqrt[3]{12}\left(\cos\frac{18^\circ}{3} + i\sin\frac{18^\circ}{3}\right) = \sqrt[3]{12}\left(\cos 6^\circ + i\sin 6^\circ\right) \\ \sqrt[3]{12}\left(\cos\left(\frac{18^\circ}{3}+\frac{360^\circ}{3}\right) + i\sin\left(\frac{18^\circ}{3}+\frac{360^\circ}{3}\right)\right) = \sqrt[3]{12}\left(\cos 126^\circ + i\sin 126^\circ\right) \\ \sqrt[3]{12}\left(\cos\left(\frac{18^\circ}{3}+2\,\frac{360^\circ}{2}\right) + i\sin\left(\frac{18^\circ}{3}+2\,\frac{360^\circ}{2}\right)\right) = \sqrt[3]{12}\left(\cos 246^\circ + i\sin 246^\circ\right) \end{cases}$

ad c) $\sqrt[4]{2+2i} = ?$ $\Rightarrow$ označme $2+2i = z$ a určíme goniometrický tvar z :

$|z| = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$

$\left.\begin{aligned} \cos\alpha &= \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ \sin\alpha &= \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \end{aligned}\right\} \Rightarrow \alpha = \frac{\pi}{4}$

$\Rightarrow z = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$\Rightarrow$

$\sqrt[4]{2+2i} = \sqrt[4]{z} = \begin{cases} = \sqrt[4]{2\sqrt{2}}\left(\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}\right) \\ = \sqrt[4]{2\sqrt{2}}\left(\cos\left(\frac{\pi}{16}+\frac{2\pi}{4}\right) + i\sin\left(\frac{\pi}{16}+\frac{2\pi}{4}\right)\right) = \sqrt[4]{2\sqrt{2}}\left(\cos\frac{9}{16}\pi + i\sin\frac{9}{16}\pi\right) \\ = \sqrt[4]{2\sqrt{2}}\left(\cos\left(\frac{\pi}{16}+2\frac{2\pi}{4}\right) + i\sin\left(\frac{\pi}{16}+2\frac{2\pi}{4}\right)\right) = \sqrt[4]{2\sqrt{2}}\left(\cos\frac{17}{16}\pi + i\sin\frac{17}{16}\pi\right) \\ = \sqrt[4]{2\sqrt{2}}\left(\cos\left(\frac{\pi}{16}+3\frac{2\pi}{4}\right) + i\sin\left(\frac{\pi}{16}+3\frac{2\pi}{4}\right)\right) = \sqrt[4]{2\sqrt{2}}\left(\cos\frac{25}{16}\pi + i\sin\frac{25}{16}\pi\right) \end{cases}$

ad d) $\sqrt[3]{\frac{12}{3-\sqrt{3}i}} = ?$ Označme $z = \frac{12}{3-\sqrt{3}i}$. Abychom převedli z na goniometrický tvar, nejprve jej převedeme na algebraický tvar:

$$z = \frac{12}{3-\sqrt{3}i} = \frac{12}{3-\sqrt{3}i} \cdot \frac{(3+\sqrt{3}i)}{(3+\sqrt{3}i)} = \frac{12(3+\sqrt{3}i)}{3^2-(\sqrt{3}i)^2} = \frac{12(3+\sqrt{3}i)}{9+3} = 3+\sqrt{3}i$$

$$\Rightarrow |z| = \sqrt{3^2+(\sqrt{3})^2} = \sqrt{12} = \underline{2\sqrt{3}}$$

$$\left.\begin{array}{l} \cos\alpha = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2} \\ \sin\alpha = \frac{\sqrt{3}}{2\sqrt{3}} = \frac{1}{2} \end{array}\right\} \Rightarrow \underline{\alpha = \frac{\pi}{6}}$$

$$\Rightarrow z = 2\sqrt{3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

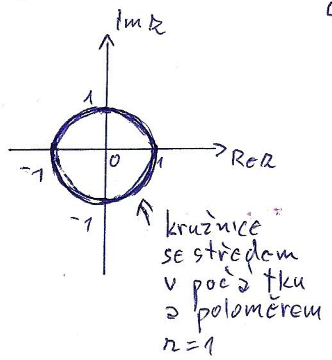
$$\Rightarrow \sqrt[3]{\frac{12}{3-\sqrt{3}i}} = \sqrt[3]{z} = \begin{cases} \underline{\underline{\sqrt[3]{2\sqrt{3}}\left(\cos\frac{\pi}{18} + i\sin\frac{\pi}{18}\right)}} \\ \sqrt[3]{2\sqrt{3}}\left(\cos\left(\frac{\pi}{18}+\frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{18}+\frac{2\pi}{3}\right)\right) = \underline{\underline{\sqrt[3]{2\sqrt{3}}\left(\cos\frac{13}{18}\pi + i\sin\frac{13}{18}\pi\right)}} \\ \sqrt[3]{2\sqrt{3}}\left(\cos\left(\frac{\pi}{18}+2\cdot\frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{18}+2\frac{2\pi}{3}\right)\right) = \underline{\underline{\sqrt[3]{2\sqrt{3}}\left(\cos\frac{25}{18}\pi + i\sin\frac{25}{18}\pi\right)}} \end{cases}$$

Př: V Gaussově rovině znázorněte všechna $z \in \mathbb{C}$ splňující:

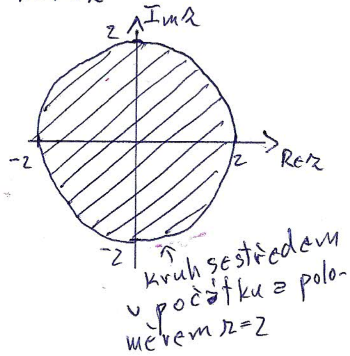
a) $|z| = 1$ b) $|z| \leq 2$ c) $1 \leq |z| < 2$

d) $|z-3| \leq 1$ e) $|z-2i| \leq 2$ f) $|z-(1+i)| \leq 0{,}5$

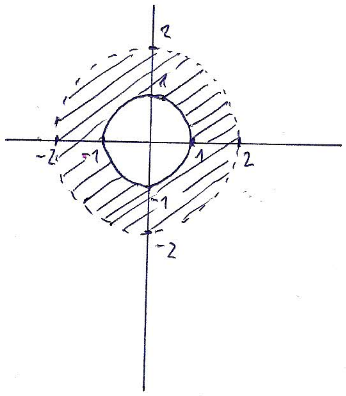
ad a) $|z| = 1$



ad b) $|z| \leq 2$



ad c) $1 \leq |z| < 2$



ad d) $|z-3| \leq 1$