Věta: Na každém otevřeném intervalu, který náleží do definičního oboru integrované funkce platí:

1.) $\int 1\, dx = x$

2.) $\int x^n\, dx = \frac{x^{n+1}}{n+1}$, $n \in \mathbb{R}$, $n \neq -1$

3.) $\int \frac{1}{x}\, dx = \ln|x|$

4.) $\int \sin x\, dx = -\cos x$

5.) $\int \cos x\, dx = \sin x$

6.) $\int \frac{1}{\cos^2 x}\, dx = \operatorname{tg} x$

7.) $\int \frac{1}{\sin^2 x}\, dx = -\operatorname{cotg} x$

8.) $\int \frac{1}{1+x^2}\, dx = \operatorname{arctg} x$

9.) $\int \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin x$

10.) $\int e^x\, dx = e^x$

Věta (O linearitě neurčitého integrálu): Nechť $\alpha, \beta \in \mathbb{R}$ a funkce f a g jsou spojité na $(a,b) \subseteq \mathbb{R}$. Potom na (a,b) platí:

$$\int (\alpha f(x) + \beta g(x))\, dx = \alpha \underbrace{\int f(x)\, dx}_{F(x)} + \beta \underbrace{\int g(x)\, dx}_{G(x)}$$

Důkaz: f a g jsou na (a,b) spojité $\Rightarrow$ existují k nim primitivní funkce $F(x)$ a $G(x)$ na (a,b)

$\Rightarrow \forall x \in (a,b): \ (\alpha F(x) + \beta G(x))' = \alpha F'(x) + \beta G'(x) = \alpha f(x) + \beta g(x)$

$\square$

Př. Určete integrál:

1.) $\int \cos x \, dx = \underline{\underline{\sin x}} + \cancel{C}$ (protože $(\sin x)' = \cos x$)

2.) $\int 3 \cdot \cos x \, dx = 3 \cdot \int \cos x \, dx = \underline{\underline{3 \cdot \sin x}}$ $\left((3 \cdot \sin x)' = 3 \cdot (\sin x)' = 3 \cdot \cos x \right)$

3.) $\int \left(\frac{1}{x} + \sin x\right) dx = \int \frac{1}{x} dx + \int \sin x \, dx = \underline{\underline{\ln|x| - \cos x}}$

4.) $\int (3x^2 + 6e^x) \, dx = 3 \cdot \int x^2 dx + 6 \int e^x dx = 3 \cdot \frac{x^3}{3} + 6e^x = \underline{\underline{x^3 + 6e^x}}$

5.) $\int \frac{2x^3 - x^2 + x + 1}{x^2} dx = \int \left(2x - 1 + \frac{1}{x} + \underset{x^{-2}}{\frac{1}{x^2}}\right) dx = 2 \cdot \frac{x^2}{2} - x + \ln|x| + \frac{x^{-3}}{-3} =$

$= \underline{\underline{x^2 - x + \ln|x| - \frac{1}{3x^3}}}$

6.) $\int \frac{\sqrt[3]{x}\,(x - x^2)}{\sqrt[2]{x}} dx = \int \frac{x^{\frac{1}{3}}(x - x^2)}{x^{\frac{1}{2}}} dx = \int x^{\overset{\frac{2}{6} - \frac{3}{6} = -\frac{1}{6}}{\frac{1}{3} - \frac{1}{2}}} (x - x^2) \, dx =$

$= \int \left(x^{1 - \frac{1}{6}} - x^{2 - \frac{1}{6}}\right) dx = \int \left(x^{\frac{5}{6}} - x^{\frac{11}{6}}\right) dx = \frac{x^{\frac{11}{6}}}{\frac{11}{6}} - \frac{x^{\frac{17}{6}}}{\frac{17}{6}} =$

$= \underline{\underline{\frac{6}{11} \cdot x^{\frac{11}{6}} - \frac{6}{17} x^{\frac{17}{6}}}} = \frac{6}{11} \sqrt[6]{x^{11}} - \frac{6}{17} \sqrt[6]{x^{17}}$

7.) $\int \frac{x^2 + 5x + 6}{x^2 + 3x} dx = \int \frac{(x+3)(x+2)}{x(x+3)} dx = \int \frac{x+2}{x} dx = \int 1 + 2 \cdot \frac{1}{x} = \underline{\underline{x + 2 \ln|x|}}$

Př. Určete integrál :

1.) $\int(\sin x + \cos x)\,dx = \int \sin x\,dx + \int \cos x\,dx = \underline{\underline{-\cos x + \sin x}}$

Zk.: $(-\cos x + \sin x)' = -(-\sin x) + \cos x = \sin x + \cos x$ ✓

2.) $\int(3x^2 - 5e^x)\,dx = 3\cdot\int x^2 dx - 5\int e^x\,dx = 3\cdot\frac{x^3}{3} - 5e^x = \underline{\underline{x^3 - 5e^x}}$

Zk.: $(x^3 - 5e^x)' = 3x^2 - 5e^x$ ✓

3.) $\int \frac{(x-2)^3}{x^2}\,dx = \int \frac{x^3 + 3\cdot x^2(-2) + 3\cdot x\cdot(-2)^2 + (-2)^3}{x^2}\,dx = \int \frac{x^3 - 6x^2 + 12x - 8}{x^2}\,dx =$

$= \int\left(x - 6 + \frac{12}{x} - \frac{8}{x^2}\right)dx = \int(x - 6 + 12\cdot\frac{1}{x} - 8\cdot x^{-2})\,dx =$

$= \frac{x^2}{2} - 6x + 12\ln|x| - 8\cdot\frac{x^{-1}}{-1} = \underline{\underline{\frac{x^2}{2} - 6x + 12\ln|x| + \frac{8}{x}}}$

4.) $\int \frac{3\sqrt[5]{x^2} - 2\sqrt[4]{x^3}}{\sqrt[7]{x^4}}\,dx = \int \frac{3x^{\frac{2}{5}} - 2x^{\frac{3}{4}}}{x^{\frac{4}{7}}}\,dx = \int 3\cdot x^{\frac{2}{5}-\frac{4}{7}} - 2\cdot x^{\frac{3}{4}-\frac{4}{7}}\,dx =$

$= \int 3\cdot x^{\frac{14-20}{35}} - 2x^{\frac{21-16}{28}}\,dx = \int 3\cdot x^{-\frac{6}{35}} - 2\cdot x^{\frac{5}{28}}\,dx =$

$= 3\cdot\frac{x^{\frac{29}{35}}}{\frac{29}{35}} - 2\cdot\frac{x^{\frac{33}{28}}}{\frac{33}{28}} = 3\cdot\frac{35}{29}x^{\frac{29}{35}} - 2\cdot\frac{28}{33}x^{\frac{33}{28}} =$

$= \underline{\underline{\frac{105}{29}\sqrt[35]{x^{29}} - \frac{56}{33}\sqrt[28]{x^{33}}}}$

$\boxed{\text{Př: Určete integrál}}$

1.) $\int (x-1)^3\, dx = \int (x^3-3x^2+3x-1)\, dx = \int x^3 dx - 3\int x^2 dx + 3\int x\, dx - \int 1\, dx =$

$$= \frac{x^4}{4} - 3\frac{x^3}{3} + 3\frac{x^2}{2} - x$$

Zk: $\left(\frac{x^4}{4} - x^3 + \frac{3}{2}x^2 - x\right)' = \frac{4x^3}{4} - 3x^2 + \frac{3\cdot 2}{2}x - 1 =$
$= x^3 - 3x^2 + 3x - 1 = (x-1)^3$ ✓

2.) $\int (6x - 5\sin x)\, dx = 6\int x - 5\int \sin x = 6\frac{x^2}{2} - 5(-\cos x) = 3x^2 + 5\cos x$

Zk: $(3x^2 + 5\cos x)' = 6x - 5\sin x$ ✓

3.) $\int x \cdot e^x\, dx \overset{?}{=} \cancel{\int x\, dx \cdot \int e^x\, dx = \frac{x^2}{2} \cdot e^x}$ NE!

Muselo by platit, že $\left(\frac{x^2}{2}e^x\right)' = x \cdot e^x$, ale $\left(\frac{x^2}{2}e^x\right)' = xe^x + \frac{x^2}{2}e^x$!!!

JAK TEDY INTEGROVAT SOUČIN ???

Věta (Integrace per partes): Nechť funkce f a g mají na $(a,b) \subseteq \mathbb{R}$ spojité první derivace. Potom na (a,b):

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$

Př. Určete integrál

1.) na $(a,b) = \mathbb{R}$: $\int x \cdot e^x\,dx = \begin{vmatrix} f = x & g' = e^x \\ f' = 1 & g = e^x \end{vmatrix} = x \cdot e^x - \int 1 \cdot e^x\,dx = \underline{\underline{x \cdot e^x - e^x}}$

Zk: $(x \cdot e^x - e^x)' = 1 \cdot e^x + x \cdot e^x - e^x = x \cdot e^x$ ✓

2.) na $(a,b) = (0,\infty)$: $\int x^2 \cdot \ln x\,dx = \begin{vmatrix} f = \ln x & g' = x^2 \\ f' = \frac{1}{x} & g = \frac{x^3}{3} \end{vmatrix} = \frac{x^3}{3}\ln x - \int \underbrace{\left(\frac{1}{x} \cdot \frac{x^3}{3}\right)}_{= \frac{x^2}{3}} dx = \underline{\underline{\frac{x^3}{3}\ln x - \frac{x^3}{9}}}$

3.) na $(a,b) = \mathbb{R}$: $\underbrace{\int \sin^2 x\,dx}_{= I} = \begin{vmatrix} f = \sin x & g' = \sin x \\ f' = \cos x & g = -\cos x \end{vmatrix} = -\sin x \cos x + \int \cos^2 x\,dx =$

$= -\sin x \cos x + \int (1 - \sin^2 x)\,dx = -\sin x \cos x + x - \underbrace{\int \sin^2 x}_{= I}$

$\Rightarrow I = -\sin x \cos x + x - I \quad \Rightarrow 2I = x - \sin x \cos x \Rightarrow \underline{\underline{I = \frac{x - \sin x \cos x}{2}}}$

Zk: $\left(\frac{1}{2}(x - \sin x \cos x)\right)' = \frac{1}{2}\left(1 - (\cos x \cdot \cos x + \sin x(-\sin x))\right) =$

$= \frac{1}{2}\left(\underbrace{1 - \cos^2 x}_{\sin^2 x} + \sin^2 x\right) = \sin^2 x$ ✓

Př: Určete integrál:

1.) $\int \underbrace{1}_{u'} \cdot \underbrace{\ln x}_{v} \, dx = \begin{vmatrix} u'=1 & v=\ln x \\ u=x & v'=\frac{1}{x} \end{vmatrix} = \underbrace{x}_{u} \cdot \underbrace{\ln x}_{v} - \int (\underbrace{x}_{u} \cdot \underbrace{\frac{1}{x}}_{v'}) \, dx = \underline{\underline{x \cdot \ln x - x}}$

2.) $\int \underbrace{(x^2-3x+5)}_{u} \underbrace{\sin x}_{v'} \, dx = \begin{vmatrix} u=x^2-3x+5 & v'=\sin x \\ u'=2x-3 & v=-\cos x \end{vmatrix} = \underbrace{(x^2-3x+5)}_{u} \underbrace{(-\cos x)}_{v} - \int \underbrace{(2x-3)}_{u'} \underbrace{(-\cos x)}_{v} \, dx =$

$= \underbrace{-(x^2-3x+5)\cos x}_{\downarrow} + \int \underbrace{(2x-3)}_{u} \underbrace{\cos x}_{v'} \, dx = \begin{vmatrix} u=2x-3 & v'=\cos x \\ u'=2 & v=\sin x \end{vmatrix} =$

$= -(x^2-3x+5)\cos x + \underbrace{(2x-3)}_{u} \underbrace{\sin x}_{v} - \int \underbrace{2}_{u'} \cdot \underbrace{\sin x}_{v} \, dx =$

$= -(x^2-3x+5)\cos x + (2x-3)\sin x - 2(-\cos x) =$

$= \underline{\underline{(-x^2+3x-3)\cos x + (2x-3)\sin x}}$

3.) $\int \underbrace{(x^2+2x+6)}_{u} \underbrace{e^x}_{v'} \, dx = \begin{vmatrix} u=x^2+2x+6 & v'=e^x \\ u'=2x+2 & v=e^x \end{vmatrix} = \underbrace{(x^2+2x+6)}_{u} \underbrace{e^x}_{v} - \int \underbrace{(2x+2)}_{u'} \underbrace{e^x}_{v} \, dx =$

$= \begin{vmatrix} u=2x+2 & v'=e^x \\ u'=2 & v=e^x \end{vmatrix} = (x^2+2x+6)e^x - \left[(2x+2)e^x - \int 2e^x \, dx\right] =$

$= (x^2+2x+6)e^x - (2x+2)e^x + \int 2e^x \, dx = (x^2+2x+6)e^x - (2x+2)e^x + 2e^x =$

$= \underline{\underline{(x^2+6)e^x}}$

4.) $\int x^2 \cdot \cos x \, dx = \begin{vmatrix} u=x^2 & v'=\cos x \\ u'=2x & v=\sin x \end{vmatrix} = x^2 \cdot \sin x - \int 2x \cdot \sin x \, dx = \begin{vmatrix} u=2x & v'=\sin x \\ u'=2 & v=-\cos x \end{vmatrix} =$

$= x^2 \sin x - \left[2x(-\cos x) - \int 2(-\cos x) \, dx\right] =$

$= x^2 \cdot \sin x + 2x \cdot \cos x - \int 2\cos x \, dx = x^2 \sin x + 2x\cos x - 2\sin x =$

$= \underline{\underline{(x^2-2)\sin x + 2x\cos x}}$

Věta (První substituční metoda): Nechť funkce f je spojitá na (α, β), $\forall x \in (a, b)$: $\varphi'(x) \in \mathbb{R}$ a $\varphi(x) \in (\alpha, \beta)$. Jestliže F je primitivní funkce k f na (α, β), pak na (a, b):

$$\int f(\varphi(x))\,\varphi'(x)\,dx = F(\varphi(x))$$

Př.: 1.) $\displaystyle \int \frac{1}{x+2}\,dx = \left| \begin{array}{l} \overset{\varphi(x)}{t = x+2} \\ dt = (x+2)'\,dx = 1 \cdot dx \end{array} \right| = \int \frac{1}{t}\,dt = \ln|t| = \underline{\underline{\ln|x+2|}}$

na $(-\infty, -2)$ i na $(-2, \infty)$

2.) $\displaystyle \int \operatorname{cotg} x\,dx = \int \frac{\cos x}{\sin x}\,dx = \left| \begin{array}{l} \overset{\varphi(x)}{t = \sin x} \\ dt = \cos x\,dx \end{array} \right| = \int \frac{1}{t}\,dt =$

$= \ln|t| = \underline{\underline{\ln|\sin x|}}$, na každém $(k\pi, (k+1)\pi)$, $k \in \mathbb{Z}$

Podrobné zdůvodnění:

$\forall x \in (k\pi, (k+1)\pi)$: $\varphi'(x) = (\sin x)' = \cos x \in \mathbb{R}$ a $\varphi(x) = \sin x \in (\alpha, \beta) = (0, 1)$ — k sudé

$\varphi(x) = \sin x \in (\alpha, \beta) = (-1, 0)$ — k liché

a $f(t) = \frac{1}{t}$ je spojitá na $(0, 1)$ i na $(\alpha, \beta) = (-1, 0)$

$F(t) = \ln|t|$ je primitivní k $f(t) = \frac{1}{t}$ na $(\alpha, \beta) = (0, 1)$ i na $(\alpha, \beta) = (-1, 0)$ $\Rightarrow$

$$\int \underbrace{\frac{1}{\sin x}} \overbrace{\cos x}^{\varphi'(x) = (\sin x)' = \cos x}\,dx = F(\varphi(x)) = \ln|\varphi(x)| = \underline{\underline{\ln|\sin x|}} \text{ na } (k\pi, (k+1)\pi)$$

$\varphi(x) = \sin x \Rightarrow f(\varphi(x)) = \frac{1}{\varphi(x)} = \frac{1}{\sin x}$

3.) $\int \sin(3x)\,dx = \left| \begin{matrix} t = \varphi(x) = 3x \\ dt = \varphi'(x)\,dx = 3\,dx \end{matrix} \right| = \int \frac{1}{3} \sin(3x)\, 3\,dx =$

$= \int \frac{1}{3} \sin t \, dt = \frac{1}{3}(-\cos t) = \underline{\underline{-\frac{1}{3}\cos(3x)}}$

Na jakém intervalu?

a) $F(t) = \frac{1}{3}(-\cos t)$ je primitivní funkce k $\frac{1}{3}\sin t$ na $(\alpha, \beta) = (-\infty, \infty) = \mathbb{R}$. „$f(t)$ je spojitá na (α, β)"

$\forall\, x \in (a, b) = (-\infty, \infty)$: $t = \varphi(x) = 3x \in (\alpha, \beta) = \mathbb{R}$ a $\varphi'(x) = 3 \in \mathbb{R}$.

$\Rightarrow$ $\underline{\underline{\int \sin(3x)\,dx = -\frac{1}{3}\cos(3x) \quad \text{na } (a, b) = (-\infty, \infty) = \mathbb{R}}}$

4.) $\int \frac{1}{x-1}\,dx = \left| \begin{matrix} t = \varphi(x) = x-1 \\ dt = \varphi'(x)\,dx = dx \end{matrix} \right| = \int \frac{1}{t}\,dt = \ln|t| = \underline{\underline{\ln|x-1|}}$

Na jakém intervalu?

$(\ln|t|)' = \frac{1}{t}$ „$f(t)$ je spojitá na (α, β)" na $\underset{a)\ (\alpha,\ \beta)}{(-\infty, 0)}$ i na $\underset{\text{resp. } b)\ (\alpha,\ \beta)}{(0, \infty)}$ (tj. $F(t) = \ln|t|$ je primitivní fce k $\frac{1}{t}$ na $(-\infty,0)$ i na $(0,\infty)$)

ad a) $\forall\, x \in (-\infty, 1) = (a, b)$: $t = \varphi(x) = x - 1 \in (\alpha, \beta) = (-\infty, 0)$ $\wedge$ $\varphi'(x) = 1 \in \mathbb{R}$

ad b) $\forall\, x \in (1, \infty) = (a, b)$: $t = \varphi(x) = x - 1 \in (\alpha, \beta) = (0, \infty)$ $\wedge$ $\varphi'(x) = 1 \in \mathbb{R}$

$\Rightarrow$ $\underline{\underline{\int \frac{1}{x-1}\,dx = \ln|x-1| \quad \text{na } (a_1, b_1) = (-\infty, 1) \text{ i na } (a_2, b_2) = (1, \infty)}}$

Př: Pomocí 1. substituční metody vypočtěte

1.) $\int (1-3x)^{16}\,dx = \left| \begin{array}{l} t = 1-3x \\ dt = -3\,dx \end{array} \right| = \int (1-3x)^{16} \frac{1}{-3} \underbrace{(-3)\,dx}_{=dt} = \int t^{16} \frac{1}{-3}\,dt = \frac{t^{17}}{17} \cdot \frac{1}{-3} = \frac{(1-3x)^{17}}{-51}$

2.) $\int x^2 \sqrt{x^3+1}\,dx = \left| \begin{array}{l} t = x^3+1 \\ dt = 3x^2\,dx \end{array} \right| = \int \frac{1}{3} \underbrace{3x^2 \sqrt{x^3+1}\,dx}_{=dt} = \int \frac{1}{3} \underbrace{\sqrt{t}}_{=t^{\frac{1}{2}}}\,dt = \frac{1}{3} \frac{t^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{9}(x^3+1)^{\frac{3}{2}}$

3.) $\int \frac{7x}{\sqrt{2x^2+3}}\,dx = \left| \begin{array}{l} t = 2x^2+3 \\ dt = 4x\,dx \end{array} \right| = \int \frac{1}{4} \frac{7 \cdot 4x}{\sqrt{2x^2+3}}\,dx = \int \frac{7}{4} \underbrace{\frac{1}{\sqrt{t}}}_{=t^{-\frac{1}{2}}}\,dt = \frac{7}{4} \frac{t^{\frac{1}{2}}}{\frac{1}{2}} = \frac{7}{2}(2x^2+3)^{\frac{1}{2}}$

4.) $\int \frac{x^9}{(1+x^5)^3}\,dx = \left| \begin{array}{l} t = 1+x^5 \\ dt = 5x^4\,dx \end{array} \right| = \int \frac{\overbrace{x^5}^{=t-1}}{(1+x^5)^3} \frac{1}{5} \underbrace{5x^4\,dx}_{=dt} = \int \frac{1}{5} \frac{t-1}{t^3}\,dt = \frac{1}{5} \int \underbrace{\frac{1}{t^2}}_{=t^{-2}} - \underbrace{\frac{1}{t^3}}_{=t^{-3}}\,dt =$

$= \frac{1}{5}\left(\frac{t^{-1}}{-1} - \frac{t^{-2}}{-2}\right) = \frac{1}{5}\left(-(1+x^5)^{-1} + \frac{1}{2}(1+x^5)^{-2}\right) = \frac{-1}{5(1+x^5)} + \frac{1}{10(1+x^5)^2}$

5.) $\int x \cdot e^{x^2}\,dx = \left| \begin{array}{l} t = x^2 \\ dt = 2x\,dx \end{array} \right| = \int \frac{1}{2} e^{x^2} \overbrace{2x\,dx}^{=dt} = \frac{1}{2} \int e^t\,dt = \frac{1}{2} e^t = \frac{1}{2} e^{x^2}$

6.) $\int \frac{1}{x \cdot \ln x}\,dx = \left| \begin{array}{l} t = \ln x \\ dt = \frac{1}{x}\,dx \end{array} \right| = \int \frac{1}{t}\,dt = \ln|t| = \ln|\ln x|$

7.) $\int \frac{(\ln x)^2}{x}\,dx = \left| \begin{array}{l} t = \ln x \\ dt = \frac{1}{x}\,dx \end{array} \right| = \int t^2\,dx = \frac{t^3}{3} = \frac{(\ln x)^3}{3}$

8.) $\int \frac{\ln x^2}{3x}\,dx = \left| \begin{array}{l} t = \ln x^2 \\ dt = \frac{1}{x^2} 2x\,dx \\ dt = \frac{2}{x}\,dx \end{array} \right| = \int \frac{1}{3} \ln x^2 \frac{1}{2} \underbrace{\left(\frac{2}{x}\,dx\right)}_{=dt} = \frac{1}{6} \int t\,dt = \frac{1}{6} \frac{t^2}{2} = \frac{1}{12}(\ln x^2)^2$

9.) $\int \frac{1}{\arcsin x \sqrt{1-x^2}}\,dx = \left| \begin{array}{l} t = \arcsin x \\ dt = \frac{1}{\sqrt{1-x^2}}\,dx \end{array} \right| = \int \frac{1}{t}\,dt = \ln|t| = \ln|\arcsin x|$

10.) $\int \cos(3x) + \sin(4x)\,dx = \frac{1}{3}\int \cos(3x) \cdot 3\,dx + \frac{1}{4}\int \sin(4x) \cdot 4\,dx = \frac{1}{3}\int \cos t\,dt + \frac{1}{4}\int \sin y\,dy =$

$t = 3x$, $dt = 3\,dx$; $y = 4x$, $dy = 4\,dx$

$= \frac{1}{3}\sin t - \frac{1}{4}\cos y = \frac{1}{3}\sin(3x) - \frac{1}{4}\cos(4x)$

11.) $\int \mathrm{tg}\,x\,dx = \int \frac{\sin x}{\cos x}\,dx = \left| \begin{array}{l} t = \cos x \\ dt = -\sin x\,dx \end{array} \right| = \int -1 \frac{1}{\cos x}(-\sin x)\,dx = \int -1 \cdot \frac{1}{t}\,dt = -\ln|t| = -\ln|\cos x|$

12.) $\int \frac{e^x}{e^x+3}\,dx \left| \begin{array}{l} t = e^x+3 \\ dt = e^x\,dx \end{array} \right| = \int \frac{1}{t}\,dt = \ln|t| = \ln|e^x+3|$

Věta (Druhá substituční metoda): Nechť platí podmínky:

1.) fce φ zobrazuje (α, β) na (!) interval (a, b)

2.) φ má na (α, β) spojitou a nenulovou derivaci

3.) fce f je spojitá na (a, b)

Jestliže F je primitivní funkce k funkci $(f \circ \varphi)\varphi'$ na intervalu (α, β). Potom na (a, b) platí:

$$\int f(x)\, dx = F(\varphi^{-1}(x))$$

Př:

$$c \in \mathbb{R}^+ \Rightarrow \int \underbrace{\frac{1}{\sqrt{c^2 - x^2}}}_{D_f = (-c, c)} dx = \left| \begin{matrix} x = c \cdot t \\ dx = c\, dt \end{matrix} \right| = \int \frac{1}{\sqrt{c^2 - c^2 t^2}}\, c\, dt = \int \frac{1}{\cancel{|c|}\sqrt{1-t^2}} \cancel{c}\, dt =$$

$|c| = c$, protože $c \in \mathbb{R}^+$

$$= \int \frac{1}{\sqrt{1-t^2}}\, dt = \arcsin t = \underline{\underline{\arcsin \frac{x}{c}}} \quad \text{na } (-c, c)$$

Podrobné zdůvodnění: Podmínky věty jsou splněny: Chceme integrovat na $(a, b) = \mathcal{D}_f = (-c, c)$ a platí:

1.) funkce $x = \varphi(t) = c \cdot t$ zobrazuje $(\alpha, \beta) = (-1, 1)$ na $(a, b) = (-c, c)$.

2.) $\forall t \in (\overset{-1}{\alpha}, \overset{1}{\beta}) : \varphi'(t) = c \Rightarrow \varphi'(t)$ je na $(-1, 1)$ spojitá a nenulová ($c \in \mathbb{R}^+$).

3.) $f(x) = \dfrac{1}{\sqrt{c^2 - x^2}}$ je spojitá na $(a, b) = (-c, c)$.

1.) $$\int \frac{\sqrt{x}(x^2-1)}{\sqrt[3]{x}}\,dx = \int x^{\frac{1}{2}-\frac{1}{3}}(x^2-1)\,dx = \int (x^{2+\frac{1}{6}} - x^{\frac{1}{6}})dx = \int (x^{\frac{13}{6}} - x^{\frac{1}{6}})dx = \frac{x^{\frac{19}{6}}}{\frac{19}{6}} - \frac{x^{\frac{7}{6}}}{\frac{7}{6}} =$$

$\frac{3}{6}-\frac{2}{6}=\frac{1}{6}$

$$= \underline{\underline{\frac{6}{19}\sqrt[6]{x^{19}} - \frac{6}{7}\sqrt[6]{x^7}}}$$

2.) $$\int \frac{x^3-1}{x^2-x}\,dx = \int \frac{(x-1)(x^2+x+1)}{x(x-1)}\,dx = \int \frac{x^2+x+1}{x} = \int \left(x+1+\frac{1}{x}\right)dx = \underline{\underline{\frac{x^2}{2} + x + \ln|x|}}$$

3.) $$\int (2x+5)\cos x\,dx = \begin{vmatrix} u=2x+5 & v'=\cos x \\ u'=2 & v=\sin x \end{vmatrix} = (2x+5)\sin x - \underbrace{\int 2\sin x\,dx}_{-2\cos x} = \underline{\underline{(2x+5)\sin x - 2\cos x}}$$

4.) $$\int (x^2+x+1)\sin x\,dx = \begin{vmatrix} u=x^2+x+1 & v'=\sin x \\ u'=2x+1 & v=-\cos x \end{vmatrix} = -(x^2+x+1)\cos x - \int (2x+1)(-\cos x)\,dx =$$

$$= -(x^2+x+1)\cos x + \int (2x+1)\cos x\,dx = \begin{vmatrix} u=2x+1 & v'=\cos x \\ u'=2 & v=\sin x \end{vmatrix} =$$

$$= -(x^2+x+1)\cos x + (2x+1)\sin x - \underbrace{\int 2\sin x}_{+2\cos x} =$$

$$= \underline{\underline{(-x^2-x+1)\cos x + (2x+1)\sin x}}$$

5.) $$\int \sqrt{x}\ln x\,dx = \begin{vmatrix} u=\ln x & v'=x^{\frac{1}{2}}=\sqrt{x} \\ u'=\frac{1}{x} & v=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}=\frac{2}{3}x^{\frac{3}{2}} \end{vmatrix} = \frac{2}{3}x^{\frac{3}{2}}\ln x - \int \left(\frac{1}{x}\right)\left(\frac{2}{3}x^{\frac{3}{2}}\right)dx =$$

x^{-1}

$$= \frac{2}{3}x^{\frac{3}{2}}\ln x + \int \frac{2}{3}\cdot x^{\frac{1}{2}}\,dx = \frac{2}{3}x^{\frac{3}{2}}\ln x + \frac{2}{3}\cdot\frac{x^{\frac{3}{2}}}{\frac{3}{2}} =$$

$$= \frac{2}{3}x^{\frac{3}{2}}\ln x + \frac{4}{9}x^{\frac{3}{2}} = \underline{\underline{\sqrt{x^3}\left(\frac{2}{3}\ln x + \frac{4}{9}\right)}}$$

6.) $\int e^{3x+7}dx = \left| \begin{array}{l} t = 3x+7 \\ dt = 3 \cdot dx \end{array} \right| = \int \frac{1}{3} \overbrace{e^{3x+7}}^{=e^t} \cdot \overbrace{3dx}^{=dt} = \frac{1}{3}\int e^t dt =$

$= \frac{1}{3} e^t = \underline{\underline{\frac{1}{3} e^{3x+7}}}$

7.) $\int x^2 \sqrt{x^3+1}\, dx = \left| \begin{array}{l} t = x^3+1 \\ dt = 3x^2 dx \end{array} \right| = \int \frac{1}{3} \overbrace{\sqrt{x^3+1}}^{=\sqrt{t}} \cdot \overbrace{3x^2 dx}^{=dt} = \int \frac{1}{3} t^{\frac{1}{2}} dt =$

$= \frac{1}{3} \frac{t^{\frac{3}{2}}}{\frac{3}{2}} = \underline{\underline{\frac{2}{9}\sqrt{(x^3+1)^3}}}$

8.) $\int \frac{x}{\sqrt[3]{x^2+7}} dx = \left| \begin{array}{l} t = x^2+7 \\ dt = 2x\, dx \end{array} \right| = \int \overbrace{\frac{1}{\sqrt[3]{x^2+7}}}^{=\frac{1}{\sqrt[3]{t}}} \cdot \frac{1}{2} \cdot \overbrace{2x\, dx}^{=dt} = \int \overbrace{\frac{1}{\sqrt[3]{t}}}^{=t^{-\frac{1}{3}}} \cdot \frac{1}{2} dt =$

$= \frac{1}{2} \frac{t^{\frac{2}{3}}}{\frac{2}{3}} = \underline{\underline{\frac{3}{4} \sqrt[3]{(x^2+7)^2}}}$

9.) $\int \frac{x+3}{x^2-1} dx = \int \left(\frac{-1}{x+1} + \frac{2}{x-1} \right) dx = -\int \frac{1}{\underbrace{x+1}_{=t}} dx + 2\int \frac{1}{\underbrace{(x-1)}_{=q}} dx = \underline{\underline{-\ln|x+1| + 2\ln|x-1|}}$

$dt = dx \qquad dq = dx$

$\frac{x+3}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)} \Rightarrow x+3 = A(x-1) + B(x+1)$

$\boxed{x=-1} \Rightarrow 2 = A(-2) \Rightarrow A = -1$

$\boxed{x=1} \Rightarrow 4 = B \cdot 2 \Rightarrow B = 2$

10.) $\displaystyle\int \frac{x^2+x+1}{x^3+x}\,dx = \int \frac{x^2+x+1}{x(x^2+1)}\,dx$

$$\frac{x^2+x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+Bx^2+Cx}{x(x^2+1)} \Rightarrow x^2+x+1 = A(x^2+1)+Bx^2+Cx$$

$\Rightarrow \boxed{x=0} \Rightarrow 1 = A \qquad \boxed{x=1} \Rightarrow 3 = \overset{1}{\overset{\shortparallel}{A}}\cdot 2+B+C \Rightarrow B+C=1$

$\boxed{x=-1} \Rightarrow 1 = \underset{1}{\underset{\shortparallel}{A}}\cdot 2+B-C \Rightarrow B-C=-1$

$\Rightarrow 2B=0 \Rightarrow B=0$

$\Rightarrow \quad C=1$

$$= \int \left(\frac{1}{x} + \frac{1}{x^2+1}\right) dx = \underline{\underline{\ln|x| + \operatorname{arctg} x}}$$

NEBO:

$$\int \frac{x^2+x+1}{x^3+x}\,dx = \int\left(\frac{x^2+1}{x(x^2+1)} + \frac{x}{x(x^2+1)}\right)dx = \int\left(\frac{1}{x}+\frac{1}{x^2+1}\right)dx = \underline{\underline{\ln x + \operatorname{arctg} x}}$$

11.) $\displaystyle\int \sin^3 x\,dx = \int \sin x \cdot \sin^2 x\,dx = \int \sin x\,(1-\cos^2 x)\,dx = \left|\begin{matrix} s = \cos x \\ ds = -\sin x\,dx\end{matrix}\right| =$

$$= \int -(1-s^2)\,ds = \int (s^2-1)\,ds = \frac{s^3}{3} - s = \underline{\underline{\frac{\cos^3 x}{3} - \cos x}}$$

12.) $\displaystyle\int \frac{\sin^3 x}{\cos^2 x}\,dx = \int \frac{\sin x\,(1-\cos^2 x)}{\cos^2 x} = \left|\begin{matrix} s=\cos x \\ ds = -\sin x\end{matrix}\right| = \int -\frac{1-s^2}{s^2}\,ds = \int\left(-\underbrace{\frac{1}{s^2}}_{=s^{-2}}+1\right)ds =$

$$= -\frac{s^{-1}}{-1} + s = \underline{\underline{\frac{1}{\cos x} + \cos x}}$$

13.) $\int \frac{1}{1+\sqrt{x}}\,dx = \left| \begin{array}{l} t = 1+\sqrt{x} \\ (t-1) = \sqrt{x} \\ x = (t-1)^2 \\ dx = 2(t-1)\,dt \end{array} \right| = \int \frac{1}{t}\, 2(t-1)\,dt = \int \left(2 - \frac{1}{t}\right) dt =$

$$= 2t - \ln|t| = \underline{\underline{2(1+\sqrt{x}) - \ln|1+\sqrt{x}|}}$$

14.) $\int x^2 \sqrt{x+2}\,dx = \left| \begin{array}{l} x+2 = t \\ x = t-2 \\ dx = dt \end{array} \right| = \int (t-2)^2 \cdot \sqrt{t}\;dt = \int (t^2 - 4t + 4)\cdot t^{\frac{1}{2}}\,dt =$

$$= \int \left(t^{\frac{5}{2}} - 4t^{\frac{3}{2}} + 4t^{\frac{1}{2}}\right) dt = \frac{2}{7} t^{\frac{7}{2}} - 4\cdot\frac{2}{5} t^{\frac{5}{2}} + 4\cdot\frac{2}{3} t^{\frac{3}{2}} =$$

$$= 2t^{\frac{3}{2}}\left(\frac{1}{7}t^2 - \frac{4}{5}t + \frac{4}{3}\right) = \underline{\underline{2\sqrt{(x+2)^3}\left(\frac{1}{7}(x+2)^2 - \frac{4}{5}(x+2) + \frac{4}{3}\right)}}$$