Integrace racionálních funkcí

Každou polynomickou funkci $q(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, kde $a_n, \dots, a_0 \in \mathbb{R}$ lze napsat ve tvaru:

$$\boxed{q(x) = a_n (x-\alpha_1)^{m_1} \cdots (x-\alpha_k)^{m_k} (x^2+\beta_1 x+\gamma_1)^{n_1} \cdots (x^2+\beta_l x+\gamma_l)^{n_l},} \quad (*)$$

kde α_i jsou navzájem různé kořeny (reálné) polynomu $q(x)$; $\beta_i, \gamma_j \in \mathbb{R}$; polynomy $x^2+\beta_i x+\gamma_i$ nemají reálné kořeny; $m_i, n_j \in \mathbb{N} \cup \{0\}$.

Př.: $q(x) = x^6 - x^2 = x^2(x^4-1) = x^2(x^2-1)(x^2+1) = (x-0)^2(x-1)^1(x+1)^1 \underbrace{(x^2+1)}$

$D = 0^2 - 4\cdot1\cdot1 < 0 \Rightarrow$ nemá reálné kořeny

Věta (Rozklad na parciální zlomky): Nechť $p(x)$ a $q(x)$ jsou polynomické funkce, kde stupeň $p(x)$ je menší, než stupeň $q(x)$. Jestliže $q(x)$ má tvar $(*)$, pak existují $a_{ij}, b_{rs}, c_{rs} \in \mathbb{R}$:

v rozkladu: $(x-\alpha_1)^{m_1}$ ... $(x-\alpha_k)^{m_k}$

$$\frac{p(x)}{q(x)} = \left(\frac{a_{11}}{(x-\alpha_1)^1} + \frac{a_{12}}{(x-\alpha_1)^2} + \dots + \frac{a_{1m_1}}{(x-\alpha_1)^{m_1}}\right) + \dots + \left(\frac{a_{k1}}{(x-\alpha_k)^1} + \frac{a_{k2}}{(x-\alpha_k)^2} + \dots + \frac{a_{km_k}}{(x-\alpha_k)^{m_k}}\right) +$$

$(x^2+\beta_1 x+\gamma_1)^{n_1}$

$$+\left(\frac{b_{11}x + c_{11}}{(x^2+\beta_1 x+\gamma_1)^1} + \frac{b_{12}x + c_{12}}{(x^2+\beta_1 x+\gamma_1)^2} + \dots + \frac{b_{1n_1}x + c_{1n_1}}{(x^2+\beta_1 x+\gamma_1)^{n_1}}\right) +$$

$$+ \frac{b_{l1}x + c_{l1}}{(x^2+\beta_l x+c_l)^1} + \frac{b_{l2}x + c_{l2}}{(x^2+\beta_l x+c_l)^2} + \dots - \frac{b_{ln_l}x + c_{ln_l}}{(x^2+\beta_l x+c_l)^{n_l}}$$

Př: Určete integrál

$$\int \frac{5x}{x^2+x-6}\,dx \Rightarrow$$

$$\frac{5x}{x^2+x-6} = \frac{5x}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2} = \frac{A(x-2)+B(x+3)}{(x+3)(x-2)}$$

$$\Rightarrow \quad 5x = A(x-2) + B(x+3)$$

$\boxed{x=2}$ $\quad 10 = 5B \quad \Rightarrow \quad B = 2$

$\boxed{x=-3}$ $\quad -15 = -5A \quad \Rightarrow \quad A = 3$

$$\Rightarrow \int \frac{5x}{x^2+x-6}\,dx = \int \frac{3}{x+3} + \frac{2}{x-2}\,dx = \underline{\underline{3\ln|x+3| + 2\ln|x-2|}}$$

Př: Určete integrál $\int \frac{4x-3}{x^2+6x+8}\,dx$

Rozklad na parciální zlomky:

$$\frac{4x-3}{x^2+6x+8} = \frac{4x-3}{(x+2)(x+4)} = \frac{a}{x+2} + \frac{b}{x+4} = \frac{a(x+4)+b(x+2)}{(x+2)(x+4)}$$

Určíme a, b: $4x-3 = a(x+4)+b(x+2)$ $\Rightarrow$ dosadíme za x:

$\boxed{x=-2}$ $\Rightarrow$ $4(-2)-3 = a(-2+4)+b(-2+2)$

$-11 = 2a$

$a = \frac{-11}{2}$

$\boxed{x=-4}$ $\Rightarrow$ $4(-4)-3 = a(-4+4)+b(-4+2)$

$-19 = -2b$

$b = \frac{19}{2}$

$$\Rightarrow \frac{4x-3}{x^2+6x+8} = \frac{-\frac{11}{2}}{x+2} + \frac{\frac{19}{2}}{x+4}$$

$$\Downarrow$$

$$\int \frac{4x-3}{x^2+6x+8}\,dx = \int \frac{-\frac{11}{2}}{x+2} + \frac{\frac{19}{2}}{x+4}\,dx = -\frac{11}{2}\int \frac{1}{x+2}\,dx + \frac{19}{2}\int\frac{1}{x+4}\,dx =$$

$t = x+2$, $dt = dx$ $\qquad$ $y = x+4$, $dy = dx$

$$= -\frac{11}{2}\int\frac{1}{t}\,dt + \frac{19}{2}\int\frac{1}{y}\,dy = -\frac{11}{2}\ln|t| + \frac{19}{2}\ln|y| =$$

$$= -\frac{11}{2}\ln|x+2| + \frac{19}{2}\ln|x+4|$$

Př. Určete integrál $\int \frac{7x^2+3x+5}{x^3+x}\,dx$.

Rozklad na parciální zlomky:

$$\frac{7x^2+3x+5}{x^3+x} = \frac{7x^2+3x+5}{x\underbrace{(x^2+1)}_{\substack{\text{Již nelze}\\ \text{rozložit}\\ (D<0)}}} = \frac{a}{x} + \frac{b+cx}{x^2+1} = \frac{a(x^2+1)+bx+cx^2}{x(x^2+1)}$$

$\boxed{x=0}$ $\Rightarrow$ $7\cdot 0^2+3\cdot 0+5 = a(0^2+1)+b\cdot 0+c\cdot 0^2$

$5 = a$

$\boxed{x=1}$ $7\cdot 1^2+3\cdot 1+5 = a(1^2+1)+b\cdot 1+c\cdot 1^2$

$15 = 2a+b+c$ $\quad / \; a=5$

$15 = 10+b+c$

$b+c = 5$

$\boxed{x=-1}$ $7(-1)^2+3(-1)+5 = a((-1)^2+1)+b(-1)+c(-1)^2$

$9 = 2a-b+c$ $\quad / \; a=5$

$-1 = -b+c$

$b+c=5$
$-b+c=-1$ } sečteme

$2c=4$

$c=2$ dosadíme do

$b+2=5$

$b=3$

$$\Rightarrow \int \frac{7x^2+3x+5}{x^3+x}\,dx = \int \frac{5}{x} + \frac{3+2x}{x^2+1}\,dx = 5\int\frac{1}{x}\,dx + 3\int\frac{1}{x^2+1}\,dx + \int\frac{2x}{x^2+1}\,dx =$$

$\left| \begin{array}{l} s = x^2+1 \\ ds = 2x\,dx \end{array} \right.$

$$= 5\cdot \ln|x| + 3\,\mathrm{arctg}\,x + \int \frac{1}{s}\,ds = 5\ln|x| + 3\,\mathrm{arctg}\,x + \ln|s| =$$

$$= 5\ln|x| + 3\,\mathrm{arctg}\,x + \ln|x^2+1|$$

Př. Dělte polynom polynomem:

1.) $(x^2+2x+5):(x+1) = x+1$

$$\begin{aligned} &-(x^2+x) \\ &\quad 0 + x + 5 \\ &\qquad -(x+1) \\ &\qquad\quad 4 \end{aligned}$$

$$\Rightarrow \frac{x^2+2x+5}{x+1} = \frac{(x+1)(x+1)+5}{x+1} = \underline{\underline{x+1+\frac{5}{x+1}}}$$

Zk.: $(x+1)(x+1)+4 = x^2+2x+1+4 = x^2+2x+5$ ✓

2.) $(4x^5-2x^3+x+1):(x^3-1) = 4x^2-2$

$$\begin{aligned} &-(4x^5-4x^2) \\ &\quad 0 \quad -2x^3+4x^2+x+1 \\ &\quad -(-2x^3+2) \\ &\qquad 0+4x^2+x-1 \end{aligned}$$

$$\Rightarrow \frac{4x^5-2x^3+x+1}{x^3-1} = \frac{(x^3-1)(4x^2-2)+4x^2+x-1}{x^3-1} =$$

$$= \underline{\underline{4x^2-2+\frac{4x^2+x-1}{x^3-1}}}$$

Zk.: $(x^3-1)(4x^2-2)+4x^2+x-1 = 4x^5-2x^3-4x^2+2+4x^2+x-1 =$

$= 4x^5-2x^3+x+1$ ✓

3.) $(3x^8-2x^5+x^4-5):(x^5+x+1) = 3x^3-2$

$$\begin{aligned} &-(3x^8+3x^4+3x^3) \\ &\quad 0-2x^5-2x^4-3x^3-5 \\ &\quad -(-2x^5-2x-2) \\ &\qquad -2x^4-3x^3+2x-3 \end{aligned}$$

$$\frac{3x^8-2x^5+x^4-5}{x^5+x+1} = \underline{\underline{3x^3-2+\frac{-2x^4-3x^3+2x-3}{x^5+x+1}}}$$

Zk.: $(x^5+x+1)(3x^3-2)-2x^4-3x^3+2x-3 =$

$= 3x^8-2x^5+3x^4-2x+3x^3-2-2x^4-3x^3+2x-3 =$

$= 3x^8-2x^5+x^4-5$ ✓

Př: Určete $\int \frac{x^4+4x^3-4x^2-11x+14}{x^2+6x+8}\,dx$

Pozor! Není splněna podmínka $st(p(x)) < st(q(x))$! => nejprve podělíme:

$$\left.\begin{aligned}&(x^4+4x^3-4x^2-11x+14):(x^2+6x+8) = x^2-2x\\ &-(x^4+6x^3+8x^2)\\ &\quad -2x^3-12x^2-11x+14\\ &\quad -(-2x^3-12x^2-16x)\\ &\qquad zb: 5x+14\end{aligned}\right\} \Rightarrow$$

$$\int \frac{x^4+4x^3-4x^2-11x+14}{x^2+6x+8}\,dx = \int x^2-2x+\frac{5x+14}{x^2+6x+8}\,dx$$

α) $\int x^2\,dx = \frac{x^3}{3}$ β) $\int -2x\,dx = -x^2$

γ) $\int \frac{5x+14}{x^2+6x+8}\,dx$ => parciální zlomky

$$\frac{5x+14}{x^2+6x+8} = \frac{5x+14}{(x+2)(x+4)} = \frac{a}{x+2}+\frac{b}{x+4}$$

$$\frac{5x+14}{x^2+6x+8} = \frac{a(x+4)+b(x+2)}{(x+2)(x+4)} \Rightarrow 5x+14 = a(x+4)+b(x+2)$$

$[x=-4] \Rightarrow -6 = 0-2b \Rightarrow b=3$

$[x=-2] \Rightarrow 4 = 2a+0 \Rightarrow a=2$

$$\Rightarrow \int \frac{5x+14}{x^2+6x-8}\,dx = \int \frac{2}{x+2}+\frac{3}{x+4}\,dx = 2\ln|x+2|+3\ln|x+4|$$

$$\Rightarrow \int \frac{x^4+4x^3-4x^2-11x+14}{x^2+6x+8}\,dx = \frac{x^3}{3}-x^2+2\ln|x+2|+3\ln|x+4|$$

Integrály typu $\int R\left(x, \sqrt[s]{\frac{ax+b}{cx+d}}\right) dx$

$a, b, c, d \in \mathbb{R}$, $s \in \mathbb{N} - \{1\}$, $ad \neq bc$ $\Rightarrow$ volíme substituci $t = \sqrt[s]{\frac{ax+b}{cx+d}}$

Př. 1. $\int \frac{x}{\sqrt{x+1}} dx = \left| \begin{array}{l} t = \sqrt{x+1} \Rightarrow t^2 - 1 = x \\ 2t\,dt = dx \end{array} \right| = \int \frac{t^2-1}{t} 2t\,dt = \int (2t^2 - 2)\,dt =$

$= \frac{2}{3}t^3 - 2t = \underline{\underline{\frac{2}{3}\left(\sqrt{x+1}\right)^3 - 2\sqrt{x+1}}}$

Př. 2. $\cancel{\int \frac{4\left(\sqrt[3]{x+1}\right)^2 - 11\sqrt[3]{x+1} - 18}{\left(\sqrt[3]{x+1}\right)^2 (x+1) - \left(\sqrt[3]{x+1}\right)^4 - 6(x+1)}}$

$$\int \frac{4\left(\sqrt[3]{x+1}\right)^2 - 11\sqrt[3]{x+1} - 18}{(x+1)\left[\left(\sqrt[3]{x+1}\right)^2 - \sqrt[3]{x+1} - 6\right]} dx = \left| \begin{array}{l} t = \sqrt[3]{x+1} \Rightarrow t^3 - 1 = x \\ 3t^2 dt = dx \end{array} \right| =$$

$$= \int \frac{4t^2 - 11t - 18}{t^3\left[t^2 - t - 6\right]} 3t^2\,dt = 3\int \frac{4t^2 - 11t - 18}{t(t-3)(t+2)} dt$$

$$\Rightarrow \frac{4t^2 - 11t - 18}{t(t-3)(t+2)} = \frac{a}{t} + \frac{b}{t-3} + \frac{c}{t+2} =$$

$$= \frac{a(t-3)(t+2) + b(t+2)t + c(t-3)t}{t(t-3)(t+2)}$$

$\Rightarrow$ dosadíme kořeny do rovnice: $4t^2 - 11t - 18 = a(t-3)(t+2) + b(t+2)t + c(t-3)t$

$\Rightarrow \boxed{t_1 = 0} \Rightarrow \quad -18 = -6a$

$\boxed{t_2 = 3} \Rightarrow \quad 36 - 33 - 18 = 15b$

$\boxed{t_3 = -2} \Rightarrow \quad 16 + 22 - 18 = 10c$

$$\left.\begin{array}{l} a = 3 \\ b = -1 \\ c = 2 \end{array}\right\} \Rightarrow \frac{4t^2 - 11t - 18}{t(t-3)(t+2)} = \frac{3}{t} - \frac{1}{t-3} + \frac{2}{t+2}$$

$$\Rightarrow 3\int \left(\frac{3}{t} - \frac{1}{t-3} + \frac{2}{t+2}\right) dt = 3 \cdot \left[3\ln|t| - \ln|t-3| + 2\ln|t+2|\right] = 9\ln\left|\sqrt[3]{x+1}\right| - 3\ln\left|\sqrt[3]{x+1} - 3\right| +$$

$$\underline{\underline{+ 6\ln\left|\sqrt[3]{x+1} + 2\right|}}$$

Př.: Určitý integrál $I = \int \frac{\sqrt{x+1}}{2x}\, dx$

$$I = \int \frac{\sqrt{x+1}}{2x}\, dx = \left| \begin{array}{l} t = \sqrt{x+1} \Rightarrow t^2 = x+1 \Rightarrow x = t^2-1 \\ \quad dx = 2t\, dt \end{array} \right| = \int \frac{t}{2(t^2-1)}\, 2t\, dt = \int \frac{t^2}{t^2-1}\, dt =$$

$$= \int \frac{t^2-1+1}{t^2-1}\, dt = \int \left(1 + \frac{1}{t^2-1}\right) dt = \underbrace{\int 1\, dt}_{I_1} + \underbrace{\int \frac{1}{t^2-1}\, dt}_{I_2}$$

$$I_1 = \int 1 \cdot dt = t = \sqrt{x+1}$$

$$I_2 = \int \frac{1}{t^2-1}\, dt = \int \frac{1}{(t-1)(t+1)}\, dt = \int \frac{A}{t-1} + \frac{B}{t+1}\, dt =$$

$$\frac{A}{t-1} + \frac{B}{t+1} = \frac{A(t+1) + B(t-1)}{(t-1)(t+1)} = \frac{1}{(t-1)(t+1)} \Rightarrow A(t+1) + B(t-1) = 1$$

$\boxed{t=1} \Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2}$

$\boxed{t=-1} \Rightarrow -2B = 1 \Rightarrow B = -\frac{1}{2}$

$$I_2 = \int \frac{\frac{1}{2}}{t-1}\, dt + \int \frac{-\frac{1}{2}}{t+1}\, dt = \frac{1}{2}\int \frac{1}{t-1}\, dt - \frac{1}{2}\int \frac{1}{t+1}\, dt =$$

$u = t-1$, $du = dt$ $\qquad$ $v = t+1$, $dv = dt$

$$= \frac{1}{2}\int \frac{1}{u}\, du - \frac{1}{2}\int \frac{1}{v}\, dv = \frac{1}{2} \ln|u| - \frac{1}{2} \ln|v| =$$

$$= \frac{1}{2} \ln|t-1| - \frac{1}{2} \ln|t+1| = \frac{1}{2} \ln|\sqrt{x+1} - 1| - \frac{1}{2} \ln|\sqrt{x+1} + 1|$$

$$\Rightarrow \int \frac{\sqrt{x+1}}{2x}\, dx = \underline{\underline{\sqrt{x+1} + \frac{1}{2} \ln|\sqrt{x+1} - 1| - \frac{1}{2} \ln|\sqrt{x+1} + 1|}}$$

Př. 3 : $\int \frac{\sqrt[3]{2x+6}}{x-1}\,dx = \left| \begin{array}{l} t=\sqrt[3]{2x+6} \Rightarrow \frac{t^3-6}{2}=x \\ \frac{3}{2}t^2\,dt = dx \end{array} \right| = \int \frac{t}{\frac{t^3-6}{2}-1}\,\frac{3}{2}t^2\,dt =$

$$= \int \frac{3t^3}{t^3-8}\,dt = \int \frac{3(t^3-8)+24}{t^3-8}\,dt = \int 3 + \frac{24}{t^3-8}\,dt$$

$$\frac{24}{\underset{\overset{\shortparallel}{2^3}}{t^3-8}} = \frac{24}{(t-2)\underset{\uparrow}{(t^2+2t+4)}} = \frac{a}{(t-2)} + \frac{bt+c}{t^2+2t+4} =$$

už nelze rozložit, protože $D = 4-16 = -12 < 0$

$$= \frac{a(t^2+2t+4)+b(t^2-2t)+c(t-2)}{(t-2)(t^2+2t+4)} =$$

$$= \frac{\overbrace{t^2(a+b)+t(2a-2b+c)+(4a-2c)}^{=24}}{(t-2)(t^2+2t+4)}$$

$\Rightarrow$ $a+b=0 \Rightarrow a=-b$

$2a-2b+c=0$

$4a-2c=24$

$-4b+c=0 \Rightarrow c=4b$

$-4b-2c=24$

$-12b=24 \Rightarrow \underline{\underline{b=-2}} \Rightarrow \underline{\underline{c=-8}} \Rightarrow \underline{\underline{a=8}}$

$\Rightarrow \frac{24}{t^3-8} = \frac{8}{t-2} + \frac{-2t-8}{t^2+2t+4}$

$$\Rightarrow = \int \left(3 + \frac{8}{t-2} - 2\,\frac{t+4}{t^2+2t+4}\right) dt$$

a) $\int 3\,dt = 3t = 3\sqrt[3]{2x+6}$

b) $\int \frac{8}{t-2}\,dt = 8\ln|t-2| = 8\ln\left|\sqrt[3]{2x+6}-2\right|$

c) $-2\int \frac{t+4}{t^2+2t+4}\,dt = -\int \frac{2t+8}{t^2+2t+4}\,dt = -\int \frac{2t+2}{t^2+2t+4}\,dt - \int \frac{6}{t^2+2t+4}\,dt =$

$\left.\begin{array}{l} r=t^2+2t+4 \\ dr=(2t+2)\,dt \end{array}\right\}$

$$= -\int \frac{2t+2}{r}\,\frac{dr}{(2t+2)} - \int \frac{6}{(t+1)^2+3}\,dt =$$

$$= -\ln|r| - \int \frac{2}{\left(\frac{t+1}{\sqrt3}\right)^2+1}\,dt = -\ln|t^2+2t+4| - \frac{2}{\sqrt3}\int \frac{1}{y^2+1}\,dy =$$

$\left| y = \frac{t+1}{\sqrt3} \Rightarrow dy = \frac{1}{\sqrt3}\,dt \right|$

$$= -\ln\left|\left(\sqrt[3]{2x+6}\right)^2 + 2\sqrt[3]{2x+6}+4\right| - \frac{2}{\sqrt3}\,\mathrm{arctg}\,y = -\ln|\text{—}\shortparallel\text{—}| - \frac{2}{\sqrt3}\,\mathrm{arctg}\,\frac{t+1}{\sqrt3}$$

$$\Rightarrow \int \frac{\sqrt[3]{2x+6}}{x-1}\,dx = 3\sqrt[3]{2x+6} + 8\ln\left|\sqrt[3]{2x-6}-2\right| - \ln\left|\left(\sqrt[3]{2x+6}\right)^2 + 2\sqrt[3]{2x+6}+4\right| -$$

$$-\frac{2}{\sqrt{3}}\,\operatorname{arctg}\frac{\sqrt[3]{2x-6}+1}{\sqrt{3}}$$

Př. 4.:

$$\int \sqrt{\frac{x+1}{x+2}}\,dx = \left|\begin{array}{l|l} ad \neq b\cdot c & \Rightarrow t^2 = \frac{x+1}{x+2} \\ t = \sqrt{\frac{x+1}{x+2}} & t^2 = 1-\frac{1}{x+2} \\ & t^2-1 = -\frac{1}{x+2} \\ & x+2 = \frac{1}{1-t^2} \\ 2t\,dt = & \\ dt = & x = \frac{1}{1-t^2}-2 \end{array}\right| = \int t\,\frac{2t}{(1-t^2)^2}\,dt =$$

$$\Rightarrow dx = -1(1-t^2)^{-2}\cdot(-2t)\,dt$$

$$= \int \frac{2t^2}{(1-t)^2(1+t)^2}\,dt$$

$\Rightarrow$ parciální zlomky

$$\frac{2t^2}{(1-t)^2(1+t)^2} = \frac{a}{1-t} + \frac{b}{(1-t)^2} + \frac{c}{(1+t)} + \frac{d}{(1+t)^2} =$$

$$= \frac{a(t-1)(1+2t+t^2) + b(1+2t+t^2) + c(1-2t+t^2)(1+t) + d(1-2t+t^2)}{(1-t)^2(1+t)^2} =$$

$$= \frac{a(t^3+t^2-t-1) + b(1+2t+t^2) + c(t^3-t^2-t+1) + d(1-2t+t^2)}{(1-t)^2(1+t)^2} =$$

$$= \frac{t^3(a+c) + t^2(a+b-c+d) + t(-a+2b-c-2d) - a+b+c+d}{(1-t)^2(1+t)^2}$$

$$\Rightarrow \left.\begin{array}{r} a+c=0 \checkmark \\ a+b-c+d=2 \\ -a+2b-c-2d=0 \\ -a+b+c+d=0 \end{array}\right\} \left.\begin{array}{c} a+c=0 \\ 2b+2d=2 \\ -a+2b-c-2d=2 \\ -a+b+c+d=0 \end{array}\right\} \Rightarrow \;....\; \text{atd.}$$

Integrály typu $\int \sin^n x \cdot \cos^m x \, dx$

m, nebo n liché

1.) $$\int (\sin x)^5 \, dx = \int (\sin x)^4 \cdot \sin x \, dx = \int \left((\sin x)^2\right)^2 \cdot \sin x \, dx =$$

$$= \int (1-\cos^2 x)^2 \cdot \underbrace{(\sin x \, dx)}_{(-1)\cdot dt} = \left| \begin{array}{l} t = \cos x \\ dt = -\sin x \cdot dx \end{array} \right| =$$

$$= \int (1-t^2)^2 \cdot (-1) \, dt = -\int 1 - 2t^2 + t^4 \, dt =$$

$$= \int -1 + 2t^2 - t^4 \, dt = -t + \frac{2}{3} t^3 - \frac{1}{5} t^5 =$$

$$= -\cos x + \frac{2}{3} (\cos x)^3 - \frac{1}{5} (\cos x)^5$$

Zk: $$\left(-\cos x + \frac{2}{3}(\cos x)^3 - \frac{1}{5}(\cos x)^5\right)' = \sin x + 2(\cos x)^2(-\sin x) - (\cos x)^4(-\sin x) =$$

$$= \sin x \left((\cos x)^4 - 2(\cos x)^2 + 1\right) = \sin x \left(\cos^2 x - 1\right)^2 =$$

$$= \sin x \, (\sin^2 x)^2 = (\sin x)^5 \quad \checkmark$$

2.) $$\int (\cos x)^3 (\sin x)^4 \, dx = \int (\cos x)^2 (\sin x)^4 \cos x \, dx = \int (1 - \sin^2 x)(\sin x)^4 \underbrace{(\cos x \, dx)}_{dt} =$$

$$= \left| \begin{array}{l} t = \sin x \\ dt = \cos x \, dx \end{array} \right| = \int (1-t^2) \, t^4 \, dt = \int t^4 - t^6 \, dt =$$

$$= \frac{1}{5} t^5 - \frac{1}{7} t^7 = \frac{1}{5} (\sin x)^5 - \frac{1}{7} (\sin x)^7$$

$\underset{\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim}{m \text{ i } n \text{ sudé}}$: Můžeme využít : $(\sin x)^2 = \frac{1-\cos(2x)}{2}$; $(\cos x)^2 = \frac{1+\cos(2x)}{2}$

$$\int (\sin x)^4 (\cos x)^6 \, dx = \int \left((\sin x)^2\right)^2 \left((\cos x)^2\right)^3 dx = \int \left(\frac{1-\cos(2x)}{2}\right)^2 \left(\frac{1+\cos(2x)}{2}\right)^3 dx =$$

$$= \frac{1}{2^5} \int (1-\cos(2x))^2 (1+\cos(2x))^3 \, dx =$$

$$= \frac{1}{2^5} \int \Big(\underbrace{(1-\cos(2x))}_{(a-b)} \underbrace{(1+\cos(2x))}_{(a+b)}\Big)^2 (1+\cos(2x)) \, dx =$$

$$= \frac{1}{2^5} \int (1-\cos^2(2x))^2 (1+\cos(2x)) \, dx =$$

$$= \frac{1}{2^5} \int (\sin^2(2x))^2 (1+\cos(2x)) \, dx =$$

$$= \frac{1}{2^5} \int (\sin(2x))^4 (1+\cos(2X)) \, dx =$$

$$= \frac{1}{2^5} \underbrace{\int (\sin(2x))^4 dx}_{\overset{\shortparallel}{I_1}} + \frac{1}{2^5} \underbrace{\int (\sin(2x))^4 \cos(2x) \, dx}_{\overset{\shortparallel}{I_2}}$$

$$I_1 = \int (\sin(2x))^4 \, dx = \int \left((\sin(2x))^2\right)^2 dx = \int \left(\frac{1-\cos(4x)}{2}\right)^2 dx = \frac{1}{4} \int 1 - 2\cos(4x) + \cos^2(4x) \, dx =$$

$$= \frac{1}{4} \int 1 - 2\cos(4x) + \frac{1+\cos(8x)}{2} \, dx = \frac{1}{4} \int \frac{3}{2} - 2\cos(4x) + \frac{1}{2}\cos(8X) \, dx =$$

$$= \underline{\frac{1}{4}\left(\frac{3}{2}X - \frac{2}{4}\sin(4x) + \frac{1}{16}\sin(8X)\right)}$$

$$I_2 = \int \underbrace{(\sin(2x))^4}_{t^4} \underbrace{\cos(2x) \, dx}_{\frac{1}{2} dt} = \left| \begin{array}{l} t = \sin(2x) \\ dt = \cos(2x) \cdot 2 \, dx \end{array} \right| = \int \frac{1}{2} t^4 \, dt = \frac{t^6}{12} = \underline{\frac{(\sin(2x))^6}{12}}$$

$$\Rightarrow \int (\sin x)^4 (\cos x)^6 \, dx = \underline{\underline{\frac{1}{2^5} \cdot \frac{1}{4}\left(\frac{3}{2}X - \frac{2}{4}\sin(4x) + \frac{1}{16}\sin(8X)\right) + \frac{1}{2^5} \frac{(\sin(2x))^6}{12}}}$$

Integrály typu $\int R(\sin x, \cos x)\, dx$

substituce $t = \operatorname{tg}\frac{x}{2}$ $\Rightarrow$ $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$

$$dx = \frac{2}{1+t^2}\,dt$$

1.) $$\int \frac{\sin x}{\cos x + 1}\,dx = \int \frac{\frac{2t}{1+t^2}}{\frac{1-t^2}{1+t^2}+1}\,\frac{2}{1+t^2}\,dt = \int \frac{2t}{(1+t^2)\left(\frac{1-t^2}{1+t^2}+1\right)}\,\frac{2}{1+t^2}\,dt =$$

$$= \int \frac{4t}{1-t^2+1+t^2}\,\frac{1}{1+t^2}\,dt = \int \frac{4t}{2(1+t^2)}\,dt = \int \frac{1}{1+t^2}\,\underbrace{2t\,dt}_{dz} = \left| \begin{array}{l} z = 1+t^2 \\ dz = 2t\,dt \end{array} \right|$$

$$= \int \frac{1}{z}\,dz = \ln|z| = \ln|1+t^2| = \ln\left|1+\left(\operatorname{tg}\frac{x}{2}\right)^2\right|$$

Pozor! kratší způsob (lze i jinak): $$\int \frac{\sin x\,dx}{\cos x+1} = \left| \begin{array}{l} t = \cos x + 1 \\ dt = -\sin x\,dx \end{array} \right| = \int -\frac{1}{t}\,dt = -\ln|t| = -\ln|\cos x + 1|$$

2.) $$\int \frac{1}{5\sin x + 2\cos x + 2}\,dx = \left| t = \operatorname{tg}\frac{x}{2} \right| = \int \frac{1}{5\frac{2t}{1+t^2} + 2\frac{1-t^2}{1+t^2} + 2}\,\frac{2}{1+t^2}\,dt =$$

$$= \int \frac{2}{\frac{10t + 2 \cancel{-2t^2} + 2 \cancel{+2t^2}}{\cancel{1+t^2}}}\,\frac{1}{\cancel{1+t^2}}\,dt = \int \frac{2}{10t+4}\,dt =$$

$$= \int \frac{1}{5t+2}\,dt = \left| \begin{array}{l} z = 5t+2 \\ dz = 5\,dt \\ dt = \frac{1}{5}dz \end{array} \right| = \int \frac{1}{z}\,\frac{1}{5}\,dz = \frac{1}{5}\ln|z| =$$

$$= \frac{1}{5}\ln|5t+2| = \frac{1}{5}\ln\left|5\operatorname{tg}\frac{x}{2} + 2\right|$$

Př. Určete $\int \frac{5x-12}{x^2-5x+6}\,dx$

$$\frac{5x-12}{x^2-5x+6} = \frac{5x-12}{(x-2)(x-3)} = \frac{a}{x-2} + \frac{b}{x-3} = \frac{a(x-3)+b(x-2)}{(x-2)(x-3)}$$

$\underline{x=3} \Rightarrow$ $5\cdot 3-12 = a\cdot 0 + b\cdot 1$

$\underline{3 = b}$

$\boxed{x=2} \Rightarrow$ $5\cdot 2-12 = a(-1) + b\cdot 0$

$-2 = -a$

$\underline{a = 2}$

$$\Rightarrow \frac{5x-12}{x^2-5x+6} = \frac{3}{x-2} + \frac{2}{x-3}$$

$$\Rightarrow \int \frac{5x-12}{x^2-5x+6}\,dx = \int \left(\frac{3}{x-2} + \frac{2}{x-3}\right)dx$$

$$\int \frac{3}{x-2}\,dx = \left|\begin{matrix} t = x-2 \\ dt = dx \end{matrix}\right| = \int \frac{3}{t}\,dt = 3\ln|t| = 3\ln|x-2|$$

$$\int \frac{2}{x-3}\,dx = \left|\begin{matrix} t = x-3 \\ dt = dx \end{matrix}\right| = \int \frac{2}{t}\,dt = 2\ln|t| = 2\ln|x-3|$$

$$\Rightarrow \int \frac{5x-12}{x^2-5x+6}\,dx = \underline{\underline{2\ln|x-2| + 3\ln|x-3|}}$$

Př: Určete integrál

$$\int \frac{8x^2+4x-6}{x^3+x^2-2x}\,dx$$

$$\frac{8x^2+4x-6}{x^3+x^2-2x} = \frac{8x^2+4x-6}{x(x^2+x-2)} = \frac{8x^2+4x-6}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2} = \frac{A(x-1)(x+2)+B(x+2)x+Cx(x-1)}{x(x-1)(x+2)}$$

$$\Rightarrow \quad 8x^2+4x-6 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$$

$\boxed{x=0}$ $\quad -6 = -2A \quad \Rightarrow A = 3$

$\boxed{x=1}$ $\quad 6 = 3B \quad \Rightarrow B = 2$

$\boxed{x=-2}$ $\quad 18 = 6C \quad \Rightarrow C = 3$

$$\Rightarrow \int \frac{8x^2+4x-6}{x^3+x^2-2x}\,dx = \int \frac{3}{x} + \frac{2}{x-1} + \frac{3}{x+2}\,dx$$

$$\int \frac{3}{x}\,dx = 3\ln|x|$$

$$\int \frac{2}{x-1}\,dx = \left|\begin{matrix} t = x-1 \\ dt = dx \end{matrix}\right| = \int \frac{2}{t}\,dt = 2\ln|t| = 2\ln|x-1|$$

$$\int \frac{3}{x+2}\,dx = \left|\begin{matrix} t = x+2 \\ dt = dx \end{matrix}\right| = \int \frac{3}{t}\,dt = 3\ln|t| = 3\ln|x+2|$$

$$\Rightarrow \int \frac{8x^2+4x-6}{x^3+x^2-2x}\,dx = \underline{\underline{3\ln|x| + 2\ln|x-1| + 3\ln|x+2|}}$$

Př: Určete $\int \frac{x^2+x+1}{x^4-1}\,dx$

$$\frac{x^2+x+1}{x^4-1} = \frac{x^2+x+1}{(x^2-1)(x^2+1)} = \frac{x^2+x+1}{(x-1)(x+1)(x^2+1)} = \frac{a}{x-1} + \frac{b}{x+1} + \frac{cx+d}{x^2+1} \Rightarrow$$

$\Rightarrow a=?,\ b=?,\ c=?,\ d=? \qquad \Rightarrow \qquad \forall x \in \mathbb{R} - \{1,-1\}:$

$$\frac{x^2+x+1}{x^4-1} = \frac{a(x+1)(x^2+1) + b(x-1)(x^2+1) + cx(x-1)(x+1) + d(x-1)(x+1)}{(x-1)(x+1)(x^2+1)}$$

$\Rightarrow \forall x \in \mathbb{R}$ (ze spojitosti polynomů) : $\quad x^2+x+1 = a(x+1)(x^2+1) + b(x-1)(x^2+1) + cx(x-1)(x+1) + d(x-1)(x+1)$

zvolíme:

$$\left.\begin{aligned}
&\boxed{x=1} \Rightarrow 3 = a\cdot 2\cdot 2 + 0 + 0 + 0 &&\Rightarrow a = \tfrac{3}{4}\\
&\boxed{x=-1} \Rightarrow 1 = 0 - 4b + 0 + 0 &&\Rightarrow b = -\tfrac{1}{4}\\
&\boxed{x=0} \Rightarrow 1 = \tfrac{3}{4} + \tfrac{1}{4} + 0 - d &&\Rightarrow d = 0\\
&\boxed{x=2} \Rightarrow 7 = \tfrac{45}{4} - \tfrac{5}{4} + 6c + 0 &&\Rightarrow c = -\tfrac{1}{2}
\end{aligned}\right\} \Rightarrow$$

$$\int \frac{x^2+x+1}{x^4-1}\,dx = \int \frac{\frac{3}{4}}{x-1} + \frac{-\frac{1}{4}}{x+1} + \frac{-\frac{1}{2}x}{x^2+1}\,dx =$$

$$\left|\begin{aligned}
&\int \tfrac{3}{4}\tfrac{1}{x-1}\,dx = \tfrac{3}{4}\ln|x-1|\\
&\int -\tfrac{1}{4}\tfrac{1}{x+1}\,dx = -\tfrac{1}{4}\ln|x+1|\\
&-\tfrac{1}{2}\int \frac{x}{x^2+1}\,dx = \left|\begin{aligned}t &= x^2+1\\ dt &= 2x\,dx\end{aligned}\right| = -\tfrac{1}{2}\int \frac{x}{t}\frac{dt}{2x} =\\
&\qquad dx = \tfrac{dt}{2x}\\
&= -\tfrac{1}{4}\ln|t| = -\tfrac{1}{4}\ln|x^2+1|.
\end{aligned}\right|$$

$$= \underline{\underline{\frac{3}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{4}\ln|x^2+1|}}$$

Př: Určete $\int \frac{x^2-x+2}{x^4+3x^3+2x^2}\,dx$

$$\frac{x^2-x+2}{x^4+3x^3+2x^2} = \frac{x^2-x+2}{x^2(x^2+3x+2)} = \frac{x^2-x+2}{x^2(x+1)(x+2)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x+1} + \frac{d}{x+2} \Rightarrow$$

$$\frac{x^2-x+2}{x^4+3x^3+2x^2} = \frac{ax(x+1)(x+2) + b(x+1)(x+2) + cx^2(x+2) + dx^2(x+1)}{x^2(x+1)(x+2)}$$

$\Rightarrow \forall x \in \mathbb{R}: \quad x^2-x+2 = ax(x+1)(x+2) + b(x+1)(x+2) + cx^2(x+2) + dx^2(x+1)$

$$\left.\begin{array}{llll} \boxed{x=0} \Rightarrow & 2 = 0+2b+0+0 & \Rightarrow & b=1 \\ \boxed{x=-1} \Rightarrow & 4 = 0+0+c+0 & \Rightarrow & c=4 \\ \boxed{x=-2} \Rightarrow & 8 = 0+0+0-4d & \Rightarrow & d=-2 \\ \boxed{x=1} \Rightarrow & 2 = 6a+6+12-4 & \Rightarrow & a=-2 \end{array}\right\} \Rightarrow$$

$$\int \frac{x^2-x+2}{x^4+3x^3+2x^2}\,dx = \int \frac{-2}{x} + \frac{1}{x^2} + \frac{4}{x+1} + \frac{2}{x+2}\,dx =$$

$$= -2\ln x - x^{-1} + 4\ln|x+1| + 2\ln|x+2|$$

Př: