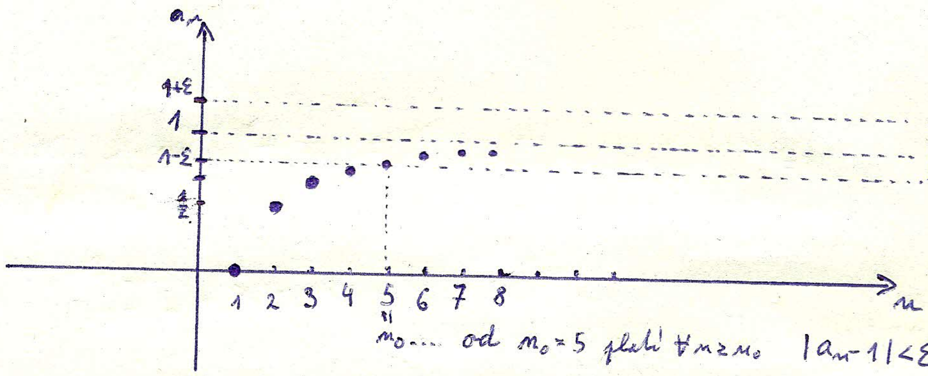
Limita posloupnosti

Motivace: Vezměme posloupnost $a_n = 1 - \frac{1}{n}$:



V tomto pásu jsou hodnoty a_n splňující: $|a_n - 1| < \varepsilon$

n_0 … od $n_0 = 5$ platí $\forall n \geq n_0 \quad |a_n - 1| < \varepsilon$

Pás můžeme libovolně zmenšovat, ale vždy najdeme takové n_0, že pro $n > n_0$ platí $|a_n - 1| < \varepsilon$ (hodnoty jsou v pásu) $\Rightarrow$

$$\forall \varepsilon > 0 \; \exists n_0 \in \mathbb{N} : \; \forall n \geq n_0 : \; |a_n - 1| < \varepsilon$$

Definice: Posloupnost $\{a_n\}_{n=1}^{\infty}$, $a_n \in \mathbb{R}$ má limitu $a \in \mathbb{R}$ právě tehdy, když

$$\forall \varepsilon \in \mathbb{R}^+ \; \exists \, n_0 \in \mathbb{N} : \; \forall \, n \geq n_0 : |a_n - a| < \varepsilon .$$

Značíme $\lim\limits_{n \to \infty} a_n = a$

Př: Podle definice limity dokažte, že

1.) $\lim\limits_{n \to \infty} \frac{1}{n} = 0$

2.) $\lim\limits_{n \to \infty} \frac{3n+1}{n} = 3$

3.) $\lim\limits_{n \to \infty} \frac{5n-2}{3n+1} = \frac{5}{3}$

$\lim a_n, \lim b_n \in \mathbb{R}^*$

Věta: Nechť $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ jsou posloupnosti reálných čísel. Potom

1.) $\lim\limits_{n\to\infty}(a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$

2.) $\lim\limits_{n\to\infty}(c\cdot a_n) = c\cdot \lim\limits_{n\to\infty} a_n \quad ; c\in\mathbb{R}$

3.) $\lim\limits_{n\to\infty}(a_n\cdot b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$

4.) $\lim\limits_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$, je-li $b_n \neq 0 \ \forall n\in\mathbb{N}$

5.) $\lim\limits_{n\to\infty}\sqrt[k]{a_n} = \sqrt[k]{\lim\limits_{n\to\infty} a_n}$; pro $k\in\mathbb{N}-\{1\}$, $a_n\geq 0\ \forall n\in\mathbb{N}$, $\lim\limits_{n\to\infty} a_n \in\mathbb{R}$

mají-li výrazy na pravé straně smysl.

Př: 1.) $\lim\limits_{n\to\infty}\left(3+\frac{1}{n}\right) = \lim\limits_{n\to\infty} 3 + \lim\limits_{n\to\infty}\frac{1}{n} = 3+0 = \underline{3}$

2.) $\lim\limits_{n\to\infty}\dfrac{3n^2+n+1}{4n^2-3n+2} = \lim\limits_{n\to\infty}\dfrac{n^2\left(3+\frac{1}{n}+\frac{1}{n^2}\right)}{n^2\left(4-\frac{3}{n}+\frac{2}{n^2}\right)} = \underline{\underline{\frac{3}{4}}}$

3.) $\lim\limits_{n\to\infty}\dfrac{2n^3-3n}{n^4-2n+2} = \lim\limits_{n\to\infty}\dfrac{n^3}{n^4}\cdot\dfrac{2-\frac{3}{n}}{1-\frac{2}{n^3}+\frac{2}{n^4}} = \underbrace{\lim\limits_{n\to\infty}\frac{1}{n}}_{0}\cdot\underbrace{\lim\limits_{n\to\infty}\frac{2-\frac{3}{n}}{1-\frac{2}{n^3}+\frac{2}{n^4}}}_{2} = 0\cdot 2 = \underline{\underline{0}}$

4.) $\lim\limits_{n\to\infty}\dfrac{-3n^5+6n^4-3n}{2n^3-3n+1} = \lim\limits_{n\to\infty}\dfrac{n^5\left(-3+\frac{6}{n}-\frac{3}{n^4}\right)}{n^3\left(2-\frac{3}{n^2}+\frac{1}{n^3}\right)} = \underbrace{\lim\limits_{n\to\infty} n^2}_{\infty}\cdot\underbrace{\lim\limits_{n\to\infty}\frac{-3+\frac{6}{n}-\frac{3}{n^4}}{2-\frac{3}{n^2}+\frac{1}{n^3}}}_{-\frac{3}{2}} = \infty\cdot\left(-\frac{3}{2}\right) = \underline{\underline{-\infty}}$

5.) $\lim\limits_{n\to\infty} 4n^5-3n^4+6n^3-2n+1 = \lim\limits_{n\to\infty}\underbrace{n^5}_{\infty}\underbrace{\left(4-\frac{3}{n}+\frac{6}{n^2}-\frac{2}{n^4}+\frac{1}{n^5}\right)}_{4} = \infty\cdot 4 = \underline{\underline{\infty}}$

5,5) $\lim\sqrt{3n^4-6n+1} = \lim n^2\sqrt{3-\frac{6}{n^3}+\frac{1}{n^4}}$

6.) $\lim\limits_{n\to\infty}\left(\sqrt{3n^4-4n}-\sqrt{n^3-1}\right) = \lim\limits_{n\to\infty}\sqrt{3n^4-4n}\left(1-\dfrac{\sqrt{n^3-1}}{\sqrt{3n^4-4n}}\right) = \lim\limits_{n\to\infty}\underbrace{\sqrt{n^4\left(3-\frac{4}{n^3}\right)}}_{\infty}\underbrace{\left(1-\sqrt{\overbrace{\frac{n^3-1}{3n^4-4n}}^{\to 0}}\right)}_{1} = \underline{\underline{\infty}}$

7.) $\lim\limits_{n\to\infty}\left(\sqrt{3n-1}-\sqrt{n+3}\right) = \lim\limits_{n\to\infty}\sqrt{3n-1}\left(1-\dfrac{\sqrt{n+3}}{\sqrt{3n-1}}\right) = \lim\limits_{n\to\infty}\underbrace{\sqrt{3n-1}}_{\infty}\underbrace{\left(1-\sqrt{\frac{n+3}{3n-1}}\right)}_{1-\sqrt{\frac{1}{3}}>0} = \infty$

8.) $\lim\limits_{n\to\infty}\left(\sqrt{n-1}-\sqrt{n+3}\right) = \lim\limits_{n\to\infty}\dfrac{\overset{(a\ -\ b)}{\sqrt{n-1}-\sqrt{n+3}}}{\sqrt{n-1}+\sqrt{n+3}}\overset{(a\ +\ b)}{\left(\sqrt{n-1}+\sqrt{n+3}\right)} = \lim\limits_{n\to\infty}\dfrac{\overset{a^2\ -\ b^2}{(n-1)-(n+3)}}{\sqrt{n-1}+\sqrt{n+3}} = \lim\limits_{n\to\infty}\dfrac{-4}{\underbrace{\sqrt{n-1}+\sqrt{n+3}}_{\infty}} = \underline{\underline{0}}$

9.) $\lim\limits_{n\to\infty}\left(\sqrt{n^3-1}-\sqrt{\frac{n^5+4n^3}{n^2-1}}\right) = \lim\limits_{n\to\infty}\dfrac{\sqrt{n^3-1}-\sqrt{\frac{n^5+4n^3}{n^2-1}}}{\sqrt{n^3-1}+\sqrt{\frac{n^5+4n^3}{n^2-1}}}\cdot\left(\sqrt{n^3-1}+\sqrt{\frac{n^5+4n^3}{n^2-1}}\right) =$

$= \lim\limits_{n\to\infty}\dfrac{n^3-1-\frac{n^5+4n^3}{n^2-1}}{\sqrt{n^3-1}+\sqrt{\frac{n^5+4n^3}{n^2-1}}} = \lim\limits_{n\to\infty}\dfrac{\frac{n^5-n^3-n^2+1-(n^5+4n^3)}{n^2-1}}{\sqrt{n^3-1}+\sqrt{\frac{n^5+4n^3}{n^2-1}}} =$

$= \lim\limits_{n\to\infty}\dfrac{\frac{-5n^3-n^2+1}{n^2-1}}{\sqrt{n^3\left(1-\frac{1}{n^3}\right)}+\sqrt{\underbrace{\frac{n^5}{n^2}}_{n^3}\frac{1+\frac{4}{n^2}}{1-\frac{1}{n^2}}}} = \lim\limits_{n\to\infty}\dfrac{\frac{n^3}{n^2}\cdot\overbrace{\frac{-5-\frac{1}{n}+\frac{1}{n^3}}{1-\frac{1}{n^2}}}^{\to -5}}{n^{\frac{3}{2}}\underbrace{\left(\sqrt{1+\frac{1}{n^3}}+\sqrt{\frac{1+\frac{4}{n^2}}{1-\frac{1}{n^2}}}\right)}_{\to 2}} = 0\cdot\frac{-5}{2} = \underline{\underline{0}}$

$n^{3-2-\frac{3}{2}} = n^{-\frac{1}{2}} = \frac{1}{\sqrt{n}} \to 0$

Př. Určete limitu posloupnosti

1.) $\lim \left(3+\frac{1}{n}\right) = \lim 3 + \lim \frac{1}{n} = \underbrace{3+0} = \underline{\underline{3}}$

tento součet je definován, proto jsme mohli „limitit" každé zvlášť

1,5.) $\lim \frac{1}{n^2} = \lim \frac{1}{n} \cdot \lim \frac{1}{n} = 0 \cdot 0 = \underline{\underline{0}}$

2.) $\lim \left(n^2 \right) = \lim n \cdot \lim n = \infty \cdot \infty = \underline{\underline{\infty}}$

3.) $\lim \left(n^3 \right) = \lim n^2 \cdot \lim n = \infty \cdot \infty = \underline{\underline{\infty}}$ $\quad (k \in \mathbb{N} \Rightarrow \lim n^k = \infty)$

4.) $\lim \left(n^2 - n \right) \neq \lim n^2 - \lim n = \infty - \infty$ – to není definováno! Musíme určit limitu jinak!

$$\lim \left(n^2 - n \right) = \lim \underbrace{n^2}_{\infty} \big(\underbrace{1 - \tfrac{1}{n}}_{1} \big) = \lim n^2 \cdot \lim \left(1 - \tfrac{1}{n}\right) = \infty \cdot 1 = \underline{\underline{\infty}}$$

5.) $\lim \left(-3n^4 + n^2 - 5 \right) = \lim \underbrace{n^4}_{\infty} \Big(\underbrace{-3 + \overset{\to 0}{\frac{1}{n^2}} - \overset{\to 0}{\frac{5}{n^4}}}_{(-3)} \Big) = \underline{\underline{-\infty}}$

6.) $\lim \dfrac{n^3 + 2n^2 + n + 1}{5n^3 - 4n + 2} = \lim \dfrac{n^3 \overbrace{\left(1 + 2\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3}\right)}^{\to 1}}{\underbrace{n^3}_{1} \underbrace{\left(5 - \frac{4}{n^2} + \frac{2}{n^3}\right)}_{\to 5}} = \underline{\underline{\dfrac{1}{5}}}$

7.) $\lim \dfrac{4n^5 - 2n^3 + 6}{3n^5 + 4n^4 - n} = \lim \dfrac{\cancel{n^5} \overbrace{\left(4 - \frac{2}{n^2} + \frac{6}{n^5}\right)}^{\to 4}}{\cancel{n^5} \underbrace{\left(3 + \frac{4}{n} - \frac{1}{n^5}\right)}_{\to 3}} = \underline{\underline{\dfrac{4}{3}}}$

8.) $\lim \dfrac{6n^4 - 2n^3 + 1}{7n^5 - 4n + 2} = \lim \underbrace{\dfrac{n^4}{n^5}}_{= \frac{1}{n} \to 0} \underbrace{\dfrac{6 - \frac{2}{n} + \frac{1}{n^4}}{7 - \frac{4}{n^4} + \frac{2}{n^5}}}_{\to \frac{6}{7}} = 0 \cdot \frac{6}{7} = \underline{\underline{0}}$

9.) $\lim \dfrac{7n^8 - 6n^7 + n^3 + 2}{11n^5 - 2n^3 + 1} = \lim \dfrac{n^8 \overbrace{\left(7 - \overset{\to 0}{\frac{6}{n}} + \overset{\to 0}{\frac{1}{n^5}} + \overset{\to 0}{\frac{2}{n^8}}\right)}^{\to 7}}{n^5 \underbrace{\left(11 - \underset{\to 0}{\frac{2}{n^2}} + \underset{\to 0}{\frac{1}{n^5}}\right)}_{\to 11}} = \infty \cdot \frac{7}{11} = \underline{\underline{\infty}}$

$\frac{n^8}{n^5} = n^3 \to \infty$

Př. Určete limitu posloupnosti

1.) $$\lim \sqrt{3n^4-6n+1} = \lim \sqrt{n^4\left(3-\frac{6}{n^3}+\frac{1}{n^4}\right)} = \lim \underbrace{n^2}_{\to\infty}\underbrace{\sqrt{3-\underbrace{\frac{6}{n^3}}_{\to 0}+\underbrace{\frac{1}{n^4}}_{\to 0}}}_{\to\sqrt{3}} = \underline{\underline{\infty}}$$

2.) $$\lim \frac{n\sqrt{8n^4-3n}+n^2}{3n^2-4n\sqrt[3]{n^6+1}} = \lim \frac{n\sqrt{n^4\left(8-\frac{3}{n^3}\right)}+n^2}{3n^2-4n\sqrt[3]{n^6\left(1+\frac{1}{n^6}\right)}} =$$

$$= \lim \frac{n^3\sqrt{8-\frac{3}{n^3}}+n^2}{3n^2-4n^3\sqrt[3]{1+\frac{1}{n^6}}} = \lim \frac{\cancel{n^3}\left(\overbrace{\sqrt{8-\underbrace{\frac{3}{n^3}}_{\to 0}}}^{\to\sqrt{8}}+\overbrace{\left(\frac{1}{n}\right)}^{\to 0}\right)}{\cancel{n^3}\underbrace{\left(\underbrace{\left(\frac{3}{n}\right)}_{\to 0}-4\sqrt[3]{1+\underbrace{\frac{1}{n^6}}_{\to 0}}\right)}_{\to -4}} = \underline{\underline{\frac{\sqrt{8}}{-4}}} = \frac{2\sqrt{2}}{-4} = -\frac{\sqrt{2}}{2}$$

ale to není definováno! => Jinak!

3.) $$\lim \overbrace{\sqrt{3n^2-1}}^{\infty}-\overbrace{\sqrt{5n^2+1}}^{\infty} = \lim \sqrt{n^2\left(3-\frac{1}{n^2}\right)}-\sqrt{n^2\left(5+\frac{1}{n^2}\right)} = \lim \underbrace{n}_{\to\infty}\cdot\underbrace{\left(\sqrt{3-\frac{1}{n^2}}-\sqrt{5+\frac{1}{n^2}}\right)}_{\to\sqrt{3}-\sqrt{5}\,<\,0} = -\infty$$

$\sqrt{n^2}\sqrt{3-\frac{1}{n^2}}$, $\sqrt{n^2}=|n|=n$

Zkusme předcházející postup:

4.) $$\lim \sqrt{n^2-3n}-\sqrt{n^2+1} = \lim \sqrt{n^2\left(1-\frac{3}{n}\right)}-\sqrt{n^2\left(1+\frac{1}{n^2}\right)} = \lim \underbrace{n}_{\to\infty}\cdot\underbrace{\left(\overbrace{\sqrt{1-\frac{3}{n}}}^{\to 1}-\overbrace{\sqrt{1+\frac{1}{n^2}}}^{\to 1}\right)}_{\to 0}$$

$\infty \cdot 0$ není definováno! => Jinak!

$$(a-b)\cdot(a+b) = a^2-b^2$$

$$= \lim \frac{\left(\sqrt{n^2-3n}-\sqrt{n^2+1}\right)}{1}\cdot\frac{\left(\sqrt{n^2-3n}+\sqrt{n^2+1}\right)}{\sqrt{n^2-3n}+\sqrt{n^2+1}} = \lim \frac{(n^2-3n)-(n^2+1)}{\sqrt{n^2-3n}+\sqrt{n^2+1}} =$$

$$= \lim \frac{-3n-1}{\sqrt{n^2\left(1-\frac{3}{n}\right)}+\sqrt{n^2\left(1+\frac{1}{n^2}\right)}} = \lim \frac{\cancel{n}\overbrace{\left(-3-\frac{1}{n}\right)}^{\to -3}}{\cancel{n}\underbrace{\left(\sqrt{1-\frac{3}{n}}+\sqrt{1+\frac{1}{n^2}}\right)}_{\to 2}} = \underline{\underline{-\frac{3}{2}}}$$

Př. Určete limitu posloupnosti

$$(a-b)(a+b)=a^2-b^2$$

1.) $$\lim \sqrt{n^2+n}-\sqrt{n^2-3n+1} = \lim \frac{\sqrt{n^2+n}-\sqrt{n^2-3n+1}}{1}\cdot\frac{\sqrt{n^2+n}+\sqrt{n^2-3n+1}}{\sqrt{n^2+n}+\sqrt{n^2-3n+1}} =$$

$$= \lim \frac{(n^2+n)-(n^2-3n+1)}{\sqrt{n^2+n}+\sqrt{n^2-3n+1}} =$$

$$= \lim \frac{4n-1}{\sqrt{n^2(1+\frac{1}{n})}+\sqrt{n^2(1-\frac{3}{n}+\frac{1}{n^2})}} =$$

$$= \lim \frac{\cancel{n}\overbrace{(4-\frac{1}{n})}^{\to 4}}{\cancel{n}\underbrace{(\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{3}{n}+\frac{1}{n^2}})}_{\to 2}} = \frac{4}{2} = \underline{\underline{2}}$$

2.) $$\lim \sqrt{n^4-2n}-\sqrt{n^4+n} = \lim \frac{\sqrt{n^4-2n}-\sqrt{n^4+n}}{1}\cdot\frac{\sqrt{n^4-2n}+\sqrt{n^4+n}}{\sqrt{n^4-2n}+\sqrt{n^4+n}} =$$

$$= \lim \frac{(n^4-2n)-(n^4+n)}{\sqrt{n^4-2n}+\sqrt{n^4+n}} =$$

$$= \lim \frac{-3n}{\sqrt{n^4(1-\frac{2}{n^3})}+\sqrt{n^4(1+\frac{1}{n^3})}} =$$

$$= \lim \underbrace{\frac{n}{n^2}}_{=\frac{1}{n}\to 0}\cdot\overbrace{\frac{(-3)}{\underbrace{(\sqrt{1-\frac{2}{n^3}}+\sqrt{1+\frac{1}{n^3}})}_{\to 2}}}^{\to -\frac{3}{2}} = 0\cdot\left(-\frac{3}{2}\right) = \underline{\underline{0}}$$

Př. Určete limitu posloupnosti :

1.)
$$\lim \frac{5^{2n+1} - 4^{3n+2}}{2^{3n} - 3^{2n+1} + 4^n} = \lim \frac{5\cdot 5^{2n} - 16\cdot 4^{3n}}{2^{3n} - 3\cdot 3^{2n} + 4^n} =$$

$$= \lim \frac{5\cdot 25^n - 16\cdot 64^n}{8^n - 3\cdot 9^n + 4^n} = \lim \frac{64^n\cdot\left(5\cdot\frac{25^n}{64^n} - 16\right)}{9^n\cdot\left(\frac{8^n}{9^n} - 3 + \frac{4^n}{9^n}\right)} =$$

$$= \lim \underbrace{\left(\frac{64}{9}\right)^n}_{\to\infty} \underbrace{\frac{5\cdot\overbrace{\left(\frac{25}{64}\right)^n}^{\to 0} - 16}{\underbrace{\left(\frac{8}{9}\right)^n}_{\to 0} - 3 + \underbrace{\left(\frac{4}{9}\right)^n}_{\to 0}}}_{\to\ \frac{-16}{-3} = \frac{16}{3}} = \underline{\underline{+\infty}}$$

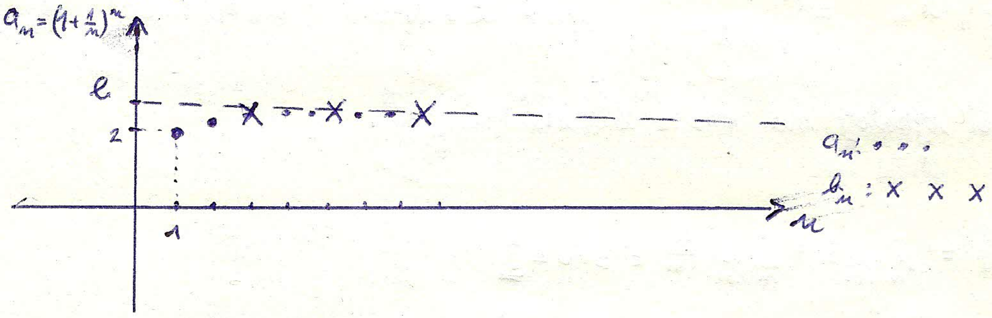
2.)
$$\lim \frac{2^{3n-2} + 3^{2n+1}}{11^{n-1} - 6\cdot 5^{n+8}} = \lim \frac{2^{-2}\cdot(2^3)^n + 3\cdot(3^2)^n}{\frac{1}{11}\cdot 11^n - 6\cdot 5^8\cdot 5^n} =$$

$$= \lim \frac{\frac{1}{4}\cdot 8^n + 3\cdot 9^n}{\frac{1}{11}\cdot 11^n - 6\cdot 5^8\cdot 5^n} = \lim \underbrace{\left(\frac{9}{11}\right)^n}_{\to 0} \cdot \underbrace{\frac{\frac{1}{4}\cdot\overbrace{\left(\frac{8}{9}\right)^n}^{\to 0} + 3}{\frac{1}{11} - 6\cdot 5^8\cdot\underbrace{\left(\frac{5}{11}\right)^n}_{\to 0}}}_{\to\ \frac{3}{\frac{1}{11}} = 33} = \underline{\underline{0}}$$

Věta: Nechť $\{a_{k_n}\}_{n=1}^{\infty}$ je vybraná posloupnost posloupnosti $\{a_n\}_{n=1}^{\infty}$.

Jestliže $\lim\limits_{n\to\infty} a_n = a$, potom $\lim\limits_{n\to\infty} a_{k_n} = a$

Př: Víme, že $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ ... Eulerovo číslo $\Rightarrow$



Označme $b_n = \left(1+\frac{1}{3n}\right)^{3n} \Rightarrow b_n$ je vybraná posloupnost posloupnosti a_n.

$$\Rightarrow \lim\left(1+\frac{1}{3n}\right)^{3n} = e$$

Př: Určete limitu posloupnosti a_n.

1.) $a_n = \left(1+\frac{1}{3n}\right)^n$

$$\lim_{n\to\infty}\left(1+\frac{1}{3n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{3n}\right)^{\frac{3n}{3}} = \lim_{n\to\infty}\left(\left(1+\frac{1}{3n}\right)^{3n}\right)^{\frac{1}{3}} = e^{\frac{1}{3}}$$

2.) $a_n = \left(\frac{n-1}{n-2}\right)^n \Rightarrow$

$$\lim_{n\to\infty}\left(\frac{n-1}{n-2}\right)^n = \lim_{n\to\infty}\left(\frac{n-2+1}{n-2}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n-2}\right)^{n-2+2} = \lim_{n\to\infty}\underbrace{\left(1+\frac{1}{n-2}\right)^{n-2}}_{e}\cdot\underbrace{\left(1+\frac{1}{n-2}\right)^2}_{1} = e$$

3.) $a_n = \left(\frac{2n+4}{2n+3}\right)^{2n}$

$$\lim_{n\to\infty}\left(\frac{2n+4}{2n+3}\right)^{2n} = \lim_{n\to\infty}\left(\frac{2n+3+1}{2n+3}\right)^{2n} = \lim_{n\to\infty}\left(1+\frac{1}{2n+3}\right)^{2n+3-3} = \lim_{n\to\infty}\underbrace{\left(1+\frac{1}{2n+3}\right)^{2n+3}}_{e}\cdot\underbrace{\left(1+\frac{1}{2n+3}\right)^{-3}}_{1} = e$$

4.) $$\lim\left(\frac{5n+3}{5n+6}\right)^{3n+\frac{7}{5}} = \lim\left(\frac{5n+3}{5n+6}\right)^{3n}\cdot\left(\frac{5n+3}{5n+6}\right)^{\frac{7}{5}} =$$

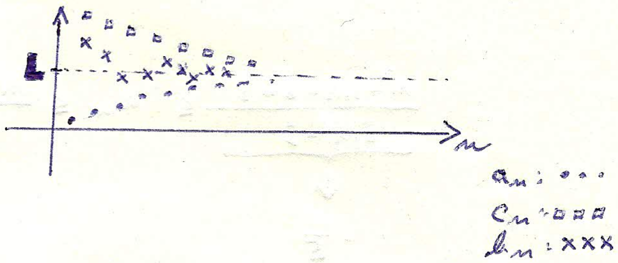
$$= \lim\left(\left(\frac{5n+6-3}{5n+6}\right)^{5n}\right)^{\frac{3}{5}}\cdot\left(\frac{n\left(5+\frac{3}{n}\right)}{n\left(5+\frac{6}{n}\right)}\right)^{\frac{7}{5}} = \lim\left(\underbrace{\left(1+\frac{-3}{5n+6}\right)^{5n+6}}_{e^{-3}}\underbrace{\left(1+\frac{-3}{5n+6}\right)^{-6}}_{1^{-6}=1}\right)^{\frac{3}{5}}\cdot\underbrace{\left(\frac{5+\frac{3}{n}}{5+\frac{6}{n}}\right)^{\frac{7}{5}}}_{1^{\frac{7}{5}}=1}$$

$$= \left(e^{-3}\right)^{\frac{3}{5}} = e^{-\frac{9}{5}}$$

Věta: Nechť $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}, \{c_n\}_{n=1}^{\infty}$ jsou posloupnosti

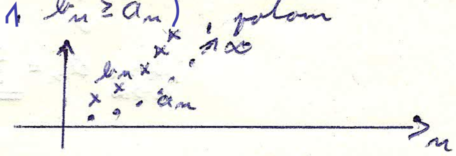
1.) Jestliže $\exists n_0 \in \mathbb{N} : \forall n \geq n_0 : \quad a_n \leq b_n \leq c_n \quad$ a $\quad \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, potom

$$\lim_{n\to\infty} b_n = L$$



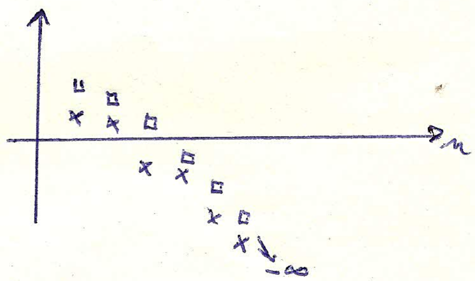
2.) Jestliže $\exists n_0 \in \mathbb{N} : \forall n \geq n_0 : \left(\lim\limits_{n\to\infty} a_n = \infty \wedge b_n \geq a_n\right)$, potom

$$\lim_{n\to\infty} b_n = \infty$$



3.) Jestliže $\exists n_0 \in \mathbb{N} : \forall n \geq n_0 : \left(\lim c_n = -\infty \wedge c_n \geq b_n\right)$, potom

$$\lim_{n\to\infty} b_n = -\infty$$



Př: 1.) $\lim\limits_{n\to\infty} \dfrac{\sin n}{n}$

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$$

$$\underbrace{-\frac{1}{n} \to 0 \qquad \frac{1}{n} \to 0}$$

$$\Downarrow$$

$$\lim_{n\to\infty} \frac{\sin n}{n} = \underline{\underline{0}}$$

2.) $\lim\limits_{n\to\infty} \dfrac{2^n + (-1)^n}{3\cdot 2^n - 2^{n-1}}$

$$\frac{2^n - 1}{3\cdot 2^n - 2^{n-1}} \leq \frac{2^n + (-1)^n}{3\cdot 2^n - 2^{n-1}} \leq \frac{2^n + 1}{3\cdot 2^n - 2^{n-1}}$$

$$\left.\begin{aligned} \lim_{n\to\infty} \frac{2^n - 1}{3\cdot 2^n - 2^{n-1}} &= \lim_{n\to\infty} \frac{2^n\left(1 - \frac{1}{2^n}\right)}{2^n\left(3 - \frac{1}{2}\right)} = \frac{1}{3 - \frac{1}{2}} = \frac{2}{5} \\ \lim_{n\to\infty} \frac{2^n + 1}{3\cdot 2^n - 2^{n-1}} &= \lim_{n\to\infty} \frac{2^n\left(1 + \frac{1}{2^n}\right)}{2^n\left(3 - \frac{1}{2}\right)} = \frac{1}{3 - \frac{1}{2}} = \frac{2}{5} \end{aligned}\right\} \Rightarrow$$

$$\lim_{n\to\infty} \frac{2^n + (-1)^n}{3\cdot 2^n - 2^{n-1}} = \underline{\underline{\frac{2}{5}}}$$

3.) $\lim\limits_{n\to\infty} \underbrace{\frac{n^2-3n+6\cos n-3\sin n}{5n^2-3}}_{a_n}$

$$\underbrace{\frac{n^2-3n-6-3}{5n^2-3}}_{\Downarrow \atop \frac{1}{5}} \leq a_n \leq \underbrace{\frac{n^2-3n+6+3}{5n^2-3}}_{\Downarrow \atop \frac{1}{5}}$$

$$\Downarrow$$

$$\lim_{n\to\infty} a_n = \frac{1}{5}$$

4.) $\lim\limits_{n\to\infty} \underbrace{\frac{3^n-5\cos n}{2^n+\sin n-(2\cos n+1)^2}}_{a_n}$

$-1 \leq 2\cos n+1 \leq 3$

$0 \leq (2\cos n+1)^2 \leq 9$

$$\frac{3^n-5}{2^n+1-0} \leq a_n \leq \frac{3^n+5}{2^n-1-9}$$

$$\frac{3^n\left(1-\frac{5}{3^n}\right)}{2^n\left(1+\frac{1}{2^n}\right)} \leq a_n \leq \frac{3^n\left(1+\frac{5}{3^n}\right)}{2^n\left(1-\frac{10}{2^n}\right)}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\infty \qquad\qquad\qquad \infty$$

$\Rightarrow$ $\lim\limits_{n\to\infty} a_n = \infty$

Příklad: Dokažte, že $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Důkaz: Definujme

$$\sqrt[n]{n} = 1 + h_n$$

pro $n \in \mathbb{N} - \{1\}$. Je jasné, že $h_n \geq 0$ $\forall n \in \mathbb{N} - \{1\}$. Potom platí:

$$n = (1+h_n)^n = \sum_{k=0}^{n} \binom{n}{k} h_n^k \geq \binom{n}{2} h_n^2 = \frac{n!}{2!(n-2)!} h_n^2 = \frac{n(n-1)}{2} h_n^2 \Rightarrow$$

$$n \geq \frac{n(n-1)}{2} h_n^2$$

$$\underset{\downarrow \atop 0}{\frac{2}{n-1}} \geq h_n^2 \geq \underset{\downarrow \atop 0}{0}$$

$$\Rightarrow \lim_{n\to\infty} h_n^2 = 0 \Rightarrow h_n = |h_n| = \sqrt{h_n^2} \to 0 \Rightarrow \lim \sqrt[n]{n} = \lim 1 + h_n = \underline{\underline{1}}$$

$\square$

Př.: $\lim \sqrt[n]{2} = ?$ $\quad \forall n \in \mathbb{N}, n \geq 2$: $\quad \underset{\downarrow \atop 1}{1} \leq \underset{\downarrow \atop 1}{\sqrt[n]{2}} \leq \underset{\downarrow \atop 1}{\sqrt[n]{n}}$

$$\sqrt[n]{\tfrac{1}{2}} = \frac{1}{\sqrt[n]{2}} \to \frac{1}{1}$$

Př.: 1.) $\lim_{n\to\infty} \sqrt[n]{3n^4 - 2n^3 + 4n^2 + 6n - 1}$

$$\underset{\downarrow \atop 1^4}{\lim_{n\to\infty} \sqrt[n]{n^4}} \leq \underset{\downarrow \atop \underline{\underline{1}}}{\lim_{n\to\infty} \sqrt[n]{3n^4 - 2n^3 + 4n^2 + 6n - 1}} \leq \underset{\downarrow \atop 1^5}{\lim_{n\to\infty} \sqrt[n]{n^5}}$$ pro dost velká n

$$\frac{1}{\sqrt[n]{2}} \cdot \sqrt[n]{n} = \sqrt[n]{\frac{n^2}{2n}} \leq$$

2.) $\lim_{n\to\infty} \sqrt[n]{\frac{2n^2 - 2n + 1}{n+3}} =$ $\leq \sqrt[n]{\frac{n^3}{n}}$

3.) $\lim \sqrt[n]{6n^8 - 3n^2 + 5n - 1} = \underline{\underline{1}}$

Pro dost velká $n \in \mathbb{N}$ platí: $\underset{\downarrow \atop 1}{\sqrt[n]{n}} \leq \sqrt[n]{6n^8 - 3n^2 + 5n - 1} \leq \sqrt[n]{n^9} = (\sqrt[n]{n})^9 \to 1$

Platnost těchto nerovností plyne z toho, že: $\lim \frac{6n^8 - 3n^2 + 5n - 1}{n} = \infty$, a také

$$\lim \frac{n^9}{6n^8 - 3n^2 + 5n - 1} = \infty$$

Př: Určete limitu :

1.) $$\lim \frac{\sqrt{5n^5+2} - n^2}{21 + n\sqrt[3]{n^2}} = \lim \frac{\sqrt{n^5\left(5+\frac{2}{n^5}\right)} - n^2}{21 + n \cdot n^{\frac{2}{3}}} = \lim \frac{\overbrace{\sqrt{n^5}}^{= n^{\frac{5}{2}}}\sqrt{5+\frac{2}{n^5}} - n^2}{21 + n^{\frac{5}{3}}} =$$

$$= \lim \frac{n^{\frac{5}{2}}\left(\sqrt{5+\frac{2}{n^5}} - n^{-\frac{1}{2}}\right)}{n^{\frac{5}{3}}\left(\frac{21}{n^{\frac{5}{3}}} + 1\right)} = \lim \underbrace{n^{\overbrace{\frac{5}{2}-\frac{5}{3}}^{= \frac{5}{6}}}}_{= \sqrt[6]{n^5} \to \infty} \cdot \underbrace{\frac{\sqrt{5 - \underbrace{\frac{2}{n^5}}_{\to 0}} - \overbrace{\frac{1}{\sqrt{n}}}^{\to 0}}{\underbrace{\frac{21}{\sqrt[3]{n^5}}}_{\to 0} + 1}}_{\to \frac{\sqrt{5}}{1}} = \underline{\underline{\infty}}$$

2.) $$\lim \left(\frac{5n+3}{5n+6}\right)^{3n+\frac{7}{5}} = \lim \left(\frac{5n+3}{5n+6}\right)^{3n} \cdot \left(\frac{5n+3}{5n+6}\right)^{\frac{7}{5}} =$$

$$= \lim \left(\left(\frac{5n+6-3}{5n+6}\right)^{5n}\right)^{\frac{3}{5}} \cdot \left(\frac{n\left(5+\frac{3}{n}\right)}{n\left(5+\frac{6}{n}\right)}\right)^{\frac{7}{5}} = \lim \left(\underbrace{\left(1+\frac{-3}{5n+6}\right)^{5n+6}}_{\to e^{-3}} \underbrace{\left(1+\frac{-3}{5n+6}\right)^{-6}}_{\to 1^{-6}=1}\right)^{\frac{3}{5}} \cdot \underbrace{\left(\frac{5+\frac{3}{n}}{5+\frac{6}{n}}\right)^{\frac{7}{5}}}_{\to 1^{\frac{7}{5}}=1} =$$

$$= \left(e^{-3}\right)^{\frac{3}{5}} = \underline{\underline{e^{-\frac{9}{5}}}}$$

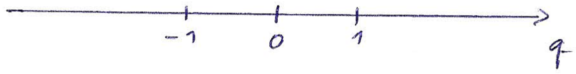
3.) $$\lim \sqrt[n]{6n^8 - 3n^2 + 5n - 1} = \underline{\underline{1}}$$

Pro dost velká $n \in \mathbb{N}$ platí: $\underbrace{\sqrt[n]{n}}_{\to 1} \leq \sqrt[n]{6n^8-3n^2+5n-1} \leq \underbrace{\sqrt[n]{n^9}}_{= \left(\sqrt[n]{n}\right)^9 \to 1}$

Platnost těchto nerovností plyne z toho, že: $\lim \frac{6n^8-3n^2+5n-1}{n} = \infty$, a také

$$\lim \frac{n^9}{6n^8-3n^2+5n-1} = \infty$$

Při $\lim\limits_{n\to\infty} q^n = ?$, $q = \text{konst} \in \mathbb{R}$



a) $q^n \in (1,\infty)$: Podle definice dokážeme, že $\lim\limits_{n\to\infty} q^n = \infty$ to jest, že

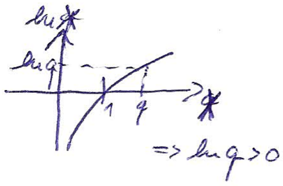
$$\forall k \in \mathbb{R}\ \exists n_0 \in \mathbb{N}\ \forall n > n_0 : \quad q^n > k$$

=> Řešíme, kdy (pro která n) platí:

$$q^n > k \quad / \ln$$

$$\ln q^n > \ln k$$

$$n \ln q > \ln k \quad /$$



$$n > \frac{\ln k}{\ln q}$$

$\Rightarrow \forall\ n > n_0 = \left[\frac{\ln k}{\ln q}\right] + 1$ platí $q^n > k$ =>

$\forall k \in \mathbb{R}\ \exists n_0 = \left[\frac{\ln k}{\ln q}\right]+1\ \forall n > n_0 : \ q^n > k \iff \lim\limits_{n\to\infty} q^n = \infty$

b) $q = 1$: $\forall n \in \mathbb{N}: q^n = 1^n = 1 \Rightarrow q^n$ je konstantní posloupnost $\Rightarrow \lim\limits_{n\to\infty} q^n = 1$.

c) $q \in (0,1)$: $0 < q < 1 \Rightarrow (0 <)\ \frac{1}{q} > 1 \Rightarrow \frac{1}{q^n} = \left(\frac{1}{q}\right)^n \to \infty \Rightarrow \lim\limits_{n\to\infty} q^n = 0$

(neboť $q^n = \frac{1}{\frac{1}{q^n}}$)

d) $q = 0$: $\forall n \in \mathbb{N}: q^n = 0^n = 0 \Rightarrow q^n$ je konstantní posloupnost $\Rightarrow \lim\limits_{n\to\infty} q^n = 0$

e) $q \in (-1,0)$: $q = -|q| \Rightarrow q^n = (-1)^n |q|^n \Rightarrow \underbrace{-|q|^n}_{\downarrow 0} \le q^n \le \underbrace{|q|^n}_{\downarrow 0} \Rightarrow \lim q^n = 0$

f) $q = -1$: $q^n = \begin{cases} -1 & \text{pro } n = 2k-1 \Rightarrow \lim\limits_{k\to\infty} q^{2k-1} = -1 \\ 1 & \text{pro } n = 2k \Rightarrow \lim\limits_{k\to\infty} q^{2k} = 1 \end{cases}$ } dvě vybrané posloupnosti z posl. q^n mají různé limity => $\lim q^n$ neexistuje

g) $q \in (-\infty,-1)$:

$q^n = \begin{cases} -|q|^{2k-1} & \text{pro } n = 2k-1 \\ |q|^{2k} & \text{pro } n = 2k \end{cases} \Rightarrow \begin{cases} -|q|^{2k-1} \to -\infty \\ |q|^{2k} \to \infty \end{cases} \Rightarrow \lim q^n$ neexistuje

Př: Pomocí definice limity dokažte, že $\lim_{n\to\infty} \frac{1}{n} = 0$

$\Rightarrow$ Musíme dokázat: $\forall \varepsilon \in \mathbb{R}^+ \ \exists n_0 \in \mathbb{N} : (\forall n > n_0 : |\frac{1}{n} - 0| < \varepsilon)$

pevně zvolíme (ε), hledáme (n_0)

$$\left|\frac{1}{n} - 0\right| < \varepsilon$$

$$\left|\frac{1}{n}\right| < \varepsilon$$

$$\frac{1}{n} < \varepsilon$$

$n > \frac{1}{\varepsilon}$ zvolíme $n_0 = [\frac{1}{\varepsilon}] + 1$ (celá část) $\Rightarrow$ evidentně $n_0 = [\frac{1}{\varepsilon}] + 1 > \frac{1}{\varepsilon}$

a $\forall n > n_0$ platí $n > n_0 > \frac{1}{\varepsilon} \Rightarrow \forall n > n_0 : \underline{\underline{|\frac{1}{n} - 0| < \varepsilon}}$

$\Rightarrow$ našli jsme takové n_0 !!!

KD!

Př Pomocí definice limity dokažte, že $\lim_{n\to\infty} \frac{3n+1}{n} = 3$

musíme dokázat: $\forall \varepsilon \in \mathbb{R}^+ \ \exists n_0 \in \mathbb{N} : (\forall n \in \mathbb{N}, n > n_0 : |\frac{3n+1}{n} - 3| < \varepsilon)$

$$\left|\frac{3n+1}{n} - 3\right| < \varepsilon$$

$$\left|\frac{3n+1-3n}{n}\right| < \varepsilon$$

$$\left|\frac{1}{n}\right| < \varepsilon$$

$$\frac{1}{n} < \varepsilon$$

$n > \frac{1}{\varepsilon} \Rightarrow$ stejně jako v předchozím $n_0 = [\frac{1}{\varepsilon}] + 1$

Př. Pomocí def. limity dokažte, že $\lim_{n\to\infty} \frac{5n-2}{3n+1} = \frac{5}{3}$

musíme dokázat: $\forall \varepsilon \in \mathbb{R}^+ \ \exists n_0 \in \mathbb{N} : (\forall n \in \mathbb{N}, n > n_0 : |\frac{5n-2}{3n+1} - \frac{5}{3}| < \varepsilon)$

$$\left|\frac{5n-2}{3n+1} - \frac{5}{3}\right| < \varepsilon$$

$$\left|\frac{5}{3}\,\frac{n-\frac{2}{5}}{n+\frac{1}{3}} - \frac{5}{3}\right| < \varepsilon$$

$$\left|\frac{5}{3}\left(\frac{n-\frac{2}{5}}{n+\frac{1}{3}} - 1\right)\right| < \varepsilon$$

$$\left|\frac{\cancel{n} - \frac{2}{5} - \cancel{n} - \frac{1}{3}}{n+\frac{1}{3}}\right| < \frac{3}{5}\varepsilon$$

$$\left|\frac{-\frac{11}{15}}{n+\frac{1}{3}}\right| < \frac{3}{5}\varepsilon$$

$$\frac{1}{n+\frac{1}{3}} < \frac{15}{11}\,\frac{3}{5}\varepsilon$$

$\Rightarrow$

$$\frac{1}{n+\frac{1}{3}} < \frac{9}{11}\varepsilon$$

$$\frac{11}{9}\,\frac{1}{\varepsilon} < n + \frac{1}{3}$$

$n > \frac{11}{9\varepsilon} + \frac{1}{3}$, zvolíme $n_0 = \left[\frac{11}{9\varepsilon} + \frac{1}{3}\right] + 1$ ....