Věta (l'Hospitalovo pravidlo): Nechť $x_0, a \in \mathbb{R}^*$. Platí:

1.) Jestliže $\left(\lim\limits_{x\to x_0^+} f(x) = \lim\limits_{x\to x_0^+} g(x) = 0\right.$, nebo $\left.\lim\limits_{x\to x_0^+} |g(x)| = +\infty\right)$ a $\lim\limits_{x\to x_0^+} \frac{f'(x)}{g'(x)} = a$,

pak $\lim\limits_{x\to x_0^+} \frac{f(x)}{g(x)} = a$.

2.) Totéž, ale limity jsou zleva ($x \to x_0^-$).

3.) Jestliže $\left(\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = 0\right.$, nebo $\left.\lim\limits_{x\to x_0} |g(x)| = +\infty\right)$ a $\lim\limits_{x\to x_0} \frac{f'(x)}{g'(x)} = a$,

pak $\lim\limits_{x\to x_0} \frac{f(x)}{g(x)} = a$. (Platí i analogická tvrzení pro limitu z leva a z prava)

Př.:

1.) $$\lim_{x\to 0} \frac{\overset{\to 0}{3x^2}}{\underset{\to 0}{x}} \overset{l'H}{=} \underbrace{\lim_{x\to 0} \frac{6x}{1}}_{\text{existuje}} = \underline{\underline{0}}$$

2.) $$\lim_{x\to\infty} \frac{\overset{\to\infty}{6x^2-4x+5}}{\underset{\to\infty}{2x^2+6x-3}} \overset{l'H}{=} \underbrace{\lim_{x\to\infty} \frac{\overset{\to\infty}{12x-4}}{\underset{\to\infty}{4x+6}}}_{\text{existuje}} \overset{l'H}{=} \underbrace{\lim_{x\to\infty} \frac{12}{4}}_{\text{existuje}} = \underline{\underline{3}}$$

3.) $$\lim_{x\to\infty} \frac{\sin x}{x} \overset{l'H}{\neq} \lim_{x\to\infty} \frac{\cos x}{1}$$

neexistuje!!! => neznamená to, že neexistuje zadaná limita, ale že ji musíme určit jinak. Např.:

$$\underset{\downarrow\, 0}{\frac{-1}{x}} \le \frac{\sin x}{x} \le \underset{\downarrow\, 0}{\frac{1}{x}}$$

$$\Downarrow$$

$$\lim_{x\to\infty} \frac{\sin x}{x} = \underline{\underline{0}}$$

4.) $\lim\limits_{x\to 2} \dfrac{\overbrace{x^3-4x}^{\to 0}}{\underbrace{x^4+x-18}_{0}} \overset{l'H}{=} \lim\limits_{x\to 2} \dfrac{3x^2-4}{4x^3+1} = \dfrac{1}{4} \quad \dfrac{}{1} = \underline{\underline{\dfrac{8}{33}}}$

5.) $\lim\limits_{x\to 3} \dfrac{x^2-7x+12}{\underbrace{x^2-x-6}_{\to 0}} \overset{l'H}{=} \lim\limits_{x\to 3} \dfrac{\overbrace{2x-7}^{\to -1}}{\underbrace{2x-1}_{\to 5}} = \underline{\underline{-\dfrac{1}{5}}}$

6.) $\lim\limits_{x\to -\infty} \dfrac{\overbrace{3x^3-2x^2+3x-1}^{\to -\infty}}{\underbrace{4x^3+531x^2+6}_{\to -\infty}} \overset{l'H}{=} \lim\limits_{x\to \infty} \dfrac{\overbrace{9x^2-4x+3}^{\to\infty}}{\underbrace{12x^2+1062x}_{\infty}} \overset{l'H}{=} \lim\limits_{x\to -\infty} \dfrac{\overbrace{18x-4}^{\infty}}{\underbrace{24x+1062}_{\infty}} \overset{l'H}{=} \lim \dfrac{18}{24} = \dfrac{18}{24} = \underline{\underline{\dfrac{3}{4}}}$

7.) $\lim\limits_{x\to 0} \dfrac{x}{\sin x} \overset{l'H}{=} \lim\limits_{x\to 0} \dfrac{1}{\cos x} = \dfrac{1}{1} = \underline{\underline{1}}$

8.) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x} \overset{l'H}{=} \lim\limits_{x\to 0} \dfrac{0-(-\sin x)}{1} = \underline{\underline{0}}$

9.) $\lim\limits_{x\to 0} \dfrac{\overbrace{1-\cos x}^{\to 0}}{\underbrace{x^2}_{\to 0}} \overset{l'H}{=} \lim\limits_{x\to 0} \dfrac{\overbrace{\sin x}^{\to 0}}{\underbrace{2x}_{\to 0}} \overset{l'H}{=} \lim\limits_{x\to 0} \dfrac{\overbrace{\cos x}^{\to 1}}{2} = \underline{\underline{\dfrac{1}{2}}}$

10.) $\lim\limits_{x\to 0} \dfrac{\overbrace{\sin x}^{\to 0}}{\underbrace{x}_{\to 0}} \overset{l'H}{=} \lim\limits_{x\to 0} \dfrac{\overbrace{\cos x}^{\to 1}}{1} = \underline{\underline{1}}$

11.) $\lim\limits_{x\to 0} \dfrac{\overbrace{x}^{\to 0}}{\underbrace{\cos x}_{\to 1}}$!!! není příklad na l'Hospitalovo pravidlo !!! $= \dfrac{0}{1} = \underline{\underline{0}}$

12.) $\lim\limits_{x\to 0} \dfrac{2x^{3} \to 0}{\underbrace{\sin x+\sin 3x}_{0}} \overset{l'H}{=} \lim\limits_{x\to 0} \dfrac{2}{\cos x + 3\cos 3x} = \dfrac{2}{1+3} = \underline{\underline{\dfrac{1}{2}}}$

13.) $\lim\limits_{x\to 0^+} \dfrac{\overbrace{\ln x}^{\to\infty}}{\underbrace{1+2\ln\sin x}_{\to\infty}} \overset{l'H}{=} \lim\limits_{x\to 0^+} \dfrac{\frac{1}{x}}{2\frac{1}{\sin x}\cos x} = \lim\limits_{x\to 0^+} \dfrac{1}{2\underbrace{\left(\frac{x}{\sin x}\right)}_{1}} \cdot \dfrac{1}{\underbrace{\cos x}_{\to 1}} = \dfrac{1}{2}$

14.) $\lim\limits_{x\to 1} \dfrac{\overbrace{x^2-1-2\ln x}^{\to 0}}{\underbrace{2(\ln x)(x^2-1)}_{0}} \overset{l'H}{=} \lim\limits_{x\to 1} \dfrac{\overbrace{2x-\frac{2}{x}}^{\to 0}}{\frac{2}{x}\underbrace{(x^2-1)}_{0} + \underbrace{2(\ln x)}_{0}2x} \overset{l'H}{=} \lim\limits_{x\to 1} \dfrac{2-2(-x^{-2})}{2(-x^{-2})(x^2-1)+\frac{2}{x}2x+\frac{4}{x}\cdot x+4\ln x} =$

$= \lim\limits_{x\to 1} \dfrac{1+\frac{1}{x^2}}{\underbrace{-\frac{(x^2-1)}{x^2}}_{0} + 2 + 2 + \underbrace{2\ln x}_{0}} = \dfrac{1+\frac{1}{1}}{4} = \underline{\underline{\dfrac{1}{2}}}$

15.) $$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = \lim_{x\to\infty} e^{\ln\left(1+\frac{1}{x}\right)^x} = \lim_{x\to\infty} e^{x\cdot\ln\left(1+\frac{1}{x}\right)} = \lim_{x\to\infty} e^{\left(\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}\right)} = \underline{\underline{e}}$$

$$\Big/ \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} = 1 \Big/$$

(e^x je spojitá v 1)

16.) $$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{x} = \lim_{x\to\infty} e^{\ln\left(1+\frac{a}{x}\right)^x} = \lim_{x\to\infty} e^{x\cdot\ln\left(1+\frac{a}{x}\right)} = \underline{\underline{e^{a}}}$$

$$\Big/ \lim_{x\to\infty} \underbrace{x}_{\infty}\cdot\underbrace{\ln\left(1+\frac{a}{x}\right)}_{0} = \lim_{x\to\infty}\frac{\ln\left(1+\frac{a}{x}\right)}{\frac{1}{x}} \overset{l'H}{=} \lim_{x\to\infty}\frac{\frac{1}{1+\frac{a}{x}}\cdot\left(1+\frac{a}{x}\right)'}{(x^{-1})'} =$$

není def.

x^{-1}

$$= \lim_{x\to\infty}\frac{\frac{1}{\left(1+\frac{a}{x}\right)}\cdot a\left(-x^{-2}\right)}{-x^{-2}} = a \Big/$$

exponenciální funkce je v bodě a spojitá

17.) $$\lim_{x\to\infty}\left(\frac{x-1}{x+1}\right)^{2x+3} = \lim_{x\to\infty} e^{\ln\left(\frac{x-1}{x+1}\right)^{2x+3}} = \lim_{x\to\infty} e^{(2x+3)\ln\left(\frac{x-1}{x+1}\right)} = \lim_{x\to\infty} e^{\frac{\ln\left(\frac{x-1}{x+1}\right)}{\frac{1}{2x+3}}} = \underline{\underline{e^{-4}}}$$

$$\Big/ \lim_{x\to\infty}\frac{\ln\left(\frac{x-1}{x+1}\right)}{\frac{1}{2x+3}} \overset{l'H}{=} \lim_{x\to\infty}\frac{\frac{1}{\frac{x-1}{x+1}}\left(\frac{x-1}{x+1}\right)'}{-(2x+3)^{-2}\cdot 2} = \lim_{x\to\infty} -(2x+3)^2\cdot\frac{x+1}{x-1}\cdot\frac{1(x+1)-(x-1)\cdot 1}{(x+1)^2\cdot 2} =$$

$\frac{1}{2x+3} = (2x+3)^{-1}$

$$= \lim_{x\to\infty}\frac{-(4x^2+6x+9)\cdot \cancel{2}}{(x-1)(x+1)\cdot\cancel{2}} = -4 \Big/$$

18.) $$\lim_{x\to\infty}\left(\frac{2x-1}{2x+2}\right)^{4x+3} = \lim_{x\to\infty} e^{(4x+3)\ln\left(\frac{2x-1}{2x+2}\right)} = \lim_{x\to\infty} e^{\frac{\ln\left(\frac{2x-1}{2x+2}\right)}{\frac{1}{4x+3}}} = \underline{\underline{e^{-6}}}$$

$$\Big/ \lim_{x\to\infty}\frac{\ln\left(\frac{2x-1}{2x+2}\right)}{\frac{1}{4x+3}} = \lim_{x\to\infty}\frac{\frac{2x+2}{2x-1}\cdot\frac{2(2x+2)-(2x-1)\cdot 2}{(2x+2)^2}}{-\frac{1}{(4x+3)^2}\cdot 4} = \lim_{x\to\infty}\frac{\frac{2x+2}{2x-1}\cdot\frac{4x+4-4x+2}{(2x+2)^2}}{-\frac{4}{(4x+3)^2}} =$$

$$= \lim_{x\to\infty}\frac{\frac{1}{2x-1}\cdot\frac{6}{2x+2}}{\frac{-4}{(4x+3)^2}} = \lim_{x\to\infty} -\frac{6}{4}\cdot\frac{16x^2+24x+9}{\underbrace{4x^2+4x-2x-2}_{\frac{16}{4}=4}} = -6 \Big/$$

Př:

1.) $$\lim_{x\to\infty} \underbrace{x^3}_{\to\infty} \cdot \underbrace{e^{-x}}_{\to 0} = \lim_{x\to\infty} \frac{x^3}{e^x} \overset{l'H}{=} \lim_{x\to\infty} \frac{3x^2}{e^x} \overset{l'H}{=} \lim_{x\to\infty} \frac{6x}{e^x} \overset{l'H}{=} \lim_{x\to\infty} \frac{6}{\underbrace{e^x}_{\to\infty}} = \underline{\underline{0}}$$

2.) $$\lim_{x\to 0^+} \underbrace{x}_{0} \cdot \underbrace{\ln x}_{-\infty} = \lim_{x\to 0^+} \frac{\ln x}{\underbrace{\frac{1}{x}}_{x^{-1}}} \overset{l'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} -\frac{x^2}{x} = \lim_{x\to 0^+} -x = \underline{\underline{0}}$$

3.) $$\lim_{x\to 0^+} \underbrace{\sqrt{x}}_{0}\, \underbrace{\ln x^2}_{-\infty} = \lim_{x\to 0^+} \frac{\ln x^2}{x^{-\frac{1}{2}}} \overset{l'H}{=} \lim_{x\to 0^+} \frac{2x\,\frac{1}{x^2}}{-\frac{1}{2}x^{-\frac{3}{2}}} = \lim_{x\to 0^+} -4\,\underbrace{\frac{x^{\frac{3}{2}}}{x^1}}_{= x^{\frac{1}{2}} = \sqrt{x} \to 0} = \underline{\underline{0}}$$

4.) $$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln x^x} = \lim_{x\to 0^+} e^{\overbrace{x\cdot\ln x}^{\to 0 \text{ viz 2.)}}} = e^0 = \underline{\underline{1}}$$
(protože e^x je spojitá)

$$\lim_{x\to 1^+} \Big(\underbrace{\frac{x}{x-1}}_{\infty} - \underbrace{\frac{1}{\ln x}}_{\infty}\Big) = \lim_{x\to 1^+} \frac{\overbrace{x\cdot\ln x}^{\to 0} - \overbrace{(x-1)}^{\to 0}}{\underbrace{(x-1)}_{0}\,\underbrace{\ln x}_{0}} \overset{l'H}{=} \lim_{x\to 1^+} \frac{\ln x + \cancel{\tfrac{x}{x}} \cancel{-1}}{\underbrace{\frac{x-1}{x}}_{0} + \underbrace{\ln x}_{0}} =$$

$$= \lim_{x\to 1^+} \frac{\overbrace{x\cdot\ln x}^{\to 0}}{\underbrace{(x-1)}_{0} + \underbrace{x\ln x}_{0}} \overset{l'H}{=} \lim_{x\to 1^+} \frac{\ln x + 1}{1 + \ln x + 1} = \frac{0+1}{1+0+1} = \underline{\underline{\frac{1}{2}}}$$

$$\lim_{x\to 0^+} \underbrace{x^3}_{0} \cdot \underbrace{\ln x}_{-\infty} = \lim_{x\to 0^+} \frac{\overset{\to -\infty}{\ln x}}{\underbrace{x^{-3}}_{= \frac{1}{x^3} \to \infty}} \overset{l'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x} = x^{-1}}{-3x^{-4}} = \lim_{x\to 0^+} -\frac{1}{3}\,\underbrace{x^{-1}\cdot x^4}_{= x^3 \to 0} = \underline{\underline{0}} = \underline{\underline{0}}$$

$$\lim_{x\to 0^+} \Big(\underbrace{\frac{1}{\underbrace{\sin x}_{\to 0^+}}}_{(+\infty)} - \underbrace{\frac{1}{x}}_{(+\infty)}\Big) = \lim_{x\to 0^+} \frac{\overbrace{x - \sin x}^{\to 0}}{\underbrace{x\cdot\sin x}_{\to 0}} \overset{l'H}{=} \lim_{x\to 0^+} \frac{\overbrace{1-\cos x}^{\to 0}}{\underbrace{\sin x + x\cdot\cos x}_{\to 0}} \overset{l'H}{=}$$

$(+\infty) - (+\infty)$ neurč.

$$\overset{l'H}{=} \lim_{x\to 0^+} \frac{\overbrace{\sin x}^{\to 0}}{\underbrace{\cos x}_{1} + \underbrace{\cos x}_{1} + \underbrace{x(-\sin x)}_{0}} = \frac{0}{2} = \underline{\underline{0}}$$

Př. Určete limitu

1.) $$\lim_{x\to\frac{\pi}{2}} \frac{\overbrace{\ln(\overbrace{\sin(5x)}^{\to 1})}^{\to 0}}{\underbrace{(x-\frac{\pi}{2})^2}_{\to 0}} \overset{l'H}{=} \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{\sin(5x)}\cdot 5(\cos(5x))}{2(x-\frac{\pi}{2})} = \lim_{x\to\frac{\pi}{2}} \frac{5\cdot \overbrace{\cos(5x)}^{\to 0}}{2\underbrace{(x-\frac{\pi}{2})}_{\to 0}\underbrace{(\sin(5x))}_{\to 1}} =$$

$$\overset{l'H}{=} \lim_{x\to\frac{\pi}{2}} \frac{-5\overbrace{\sin(5x)}^{\to 1}\cdot 5}{2\underbrace{\sin(5x)}_{\to 1} + 2(x-\frac{\pi}{2})\underbrace{\cos(5x)}_{\to 0}\cdot 5} = \underline{\underline{\frac{-25}{2}}}$$

2.) $$\lim_{x\to 0^+} \overbrace{\ln(\underbrace{x^2}_{\to 0^+})}^{\to -\infty}\cdot \overbrace{\mathrm{tg}\, x}^{\to 0} = \lim_{x\to 0^+} \frac{\ln(x^2)}{\frac{1}{\mathrm{tg}\, x}} = \lim_{x\to 0^+} \frac{\overbrace{\ln(x^2)}^{\to -\infty}}{\underbrace{\frac{\cos x}{\sin x}}_{\to +\infty}} \overset{l'H}{=}$$

$1 \leftarrow \cos x$, $0^+ \leftarrow \sin x$, $\frac{\cos x}{\sin x} = \mathrm{cotg}\, x$

$$\lim_{x\to 0^+} \frac{\frac{1}{x^2}\cdot 2x}{\frac{-1}{\sin^2 x}} = \lim_{x\to 0^+} \frac{-2\overbrace{\sin^2 x}^{\to 0^+}}{\underbrace{x}_{\to 0^+}} \overset{l'H}{=} \lim \frac{-4\overbrace{\sin x}^{\to 0}\cdot \cos x}{1} = \underline{\underline{0}}$$

Poznámka: $\lim_{x\to 0} \ln(x^2)\cdot \mathrm{tg}\, x$ mohu řešit stejně! je splněn požadavek $\left|\frac{\cos x}{\sin x}\right| \to +\infty$ proto mohu použít l'Hospitalovo pravidlo! nevadí, že $\lim_{x\to 0} \frac{\cos x}{\sin x}$ neexistuje.

3.) $$\lim_{x\to\infty} \frac{\overbrace{x}^{\to\infty}}{\underbrace{\sqrt{x^2+1}}_{\to\infty}} \overset{l'H}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot 2x} = \lim_{x\to\infty} \frac{(x^2+1)^{\frac{1}{2}}}{x} \overset{l'H}{=} \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$$

$\sqrt{x^2+1} = (x^2+1)^{\frac{1}{2}}$

$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$ = zadaná limita!
$\Rightarrow$ l'Hospitalovo pravidlo tady nepomůže!

ale: $$\lim_{x\to\infty} \frac{x}{\sqrt{x^2(1+\frac{1}{x^2})}} = \lim_{x\to\infty} \frac{x}{x\cdot\sqrt{1+\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\underbrace{\frac{1}{x^2}}_{\to 0}}} = \underline{\underline{1}}$$

$\sqrt{x^2(1+\frac{1}{x^2})} = \sqrt{x^2}\cdot\sqrt{1+\frac{1}{x^2}} = |x|\cdot\sqrt{1+\frac{1}{x^2}} = x\cdot\sqrt{1+\frac{1}{x^2}}$

x pro dost velká x a $x\to\infty$, proto mohu $|x|$ nahradit symbolem x

Př: Určete limitu funkce :

3.) $$\lim_{x\to\frac{\pi}{4}} \frac{\overbrace{2e^{2x-\frac{\pi}{2}}-4x+\pi-2}^{\to 0}}{\underbrace{3\cdot\cos^2(2x)}_{\to 0}} \overset{l'H}{=} \lim_{x\to\frac{\pi}{4}} \frac{\overbrace{2\cdot e^{2x-\frac{\pi}{2}}(2)-4}^{\to 0}}{6\underbrace{\cos(2x)}_{\to 0}\cdot \underbrace{2}_{(-\sin(2x))}} \overset{l'H}{=} \lim_{x\to\frac{\pi}{4}} \frac{4e^{2x-\frac{\pi}{2}}\cdot(2)}{-12\cdot(\cos(2x)\cdot\sin(2x))'} =$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{8e^{2x-\frac{\pi}{2}}}{\underbrace{-12\left(-\sin(2x)\cdot 2\cdot \underset{\sin\frac{\pi}{2}=1}{\sin(2x)} + \underset{\cos\frac{\pi}{2}=0}{\cos(2x)}\cdot 2\cos(2x)\right)}_{-12(-1\cdot 2\cdot 1+0\cdot 2\cdot 0)=24}} = \frac{8\cdot e^0}{24} = \frac{1}{3}$$