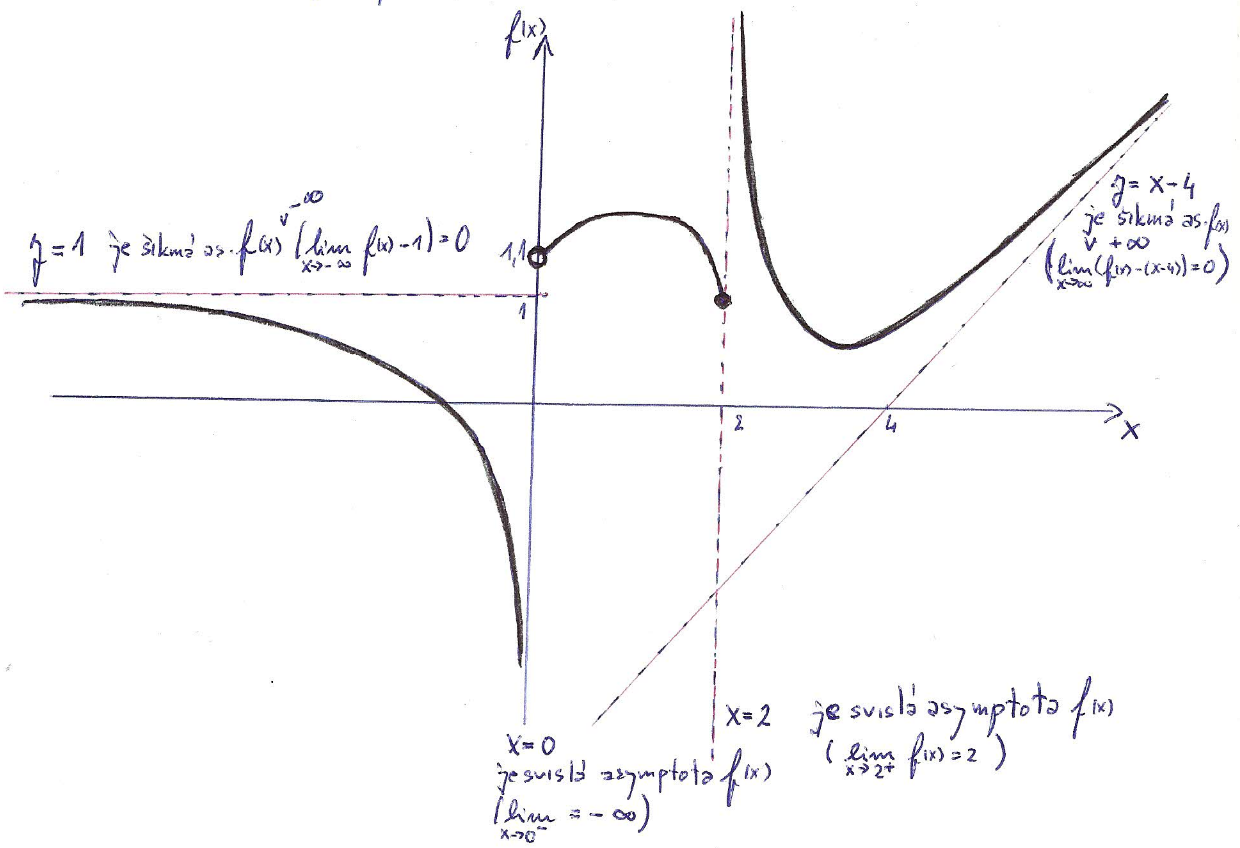
Asymptoty

Motivace



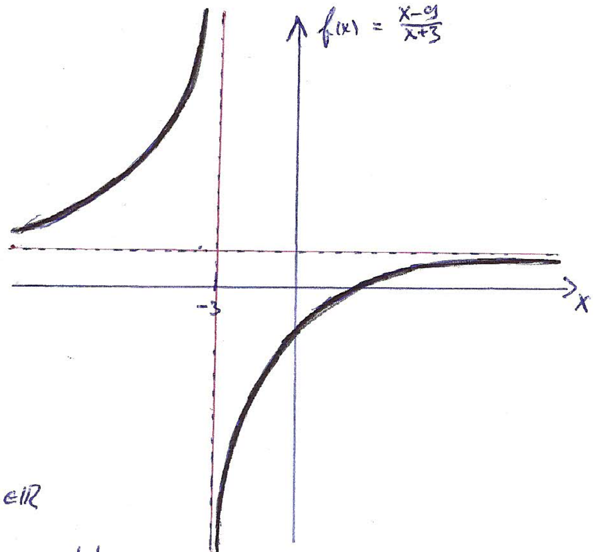
$\Rightarrow$ f(x) má svislou asymptotu v bodě x_0 $\Leftrightarrow \left(\lim_{x\to x_0^-} f(x) = \pm\infty \text{, nebo } \lim_{x\to x_0^+} f(x) = \pm\infty \right)$

$\Rightarrow$ f(x) má šikmou asymptotu v $+\infty$ $y = ax + b$ $\Leftrightarrow \lim_{x\to+\infty} (f(x) - (ax+b)) = 0$
(obdobně v $-\infty$)

$\Leftrightarrow$ 1.) $a = \lim_{x\to+\infty} \frac{f(x)}{x} \in \mathbb{R}$

2.) $b = \lim_{x\to\infty} (f(x) - ax) \in \mathbb{R}$

Př: 1.) $f(x) = \frac{x-9}{x+3}$



a) $\Rightarrow \lim_{x\to-3^+} f(x) = \lim_{x\to-3^+} \frac{\overset{\to -12}{x-9}}{\underset{\to 0^+}{x+3}} = -\infty$

$\Rightarrow$ $x = -3$ je svislá asymptota

b) $\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{1-\frac{9}{x}}{x+3} = 0 = a \in \mathbb{R}$

$\lim_{x\to\infty} (f(x) - ax) = \lim_{x\to\infty} \left(\frac{x-9}{x+3} - 0\cdot x \right) =$

$= \lim_{x\to\infty} \frac{\cancel{x}(1-\frac{9}{x})}{\cancel{x}(1+\frac{3}{x})} = 1 = b \in \mathbb{R}$

$\Rightarrow y = 0x + 1 \Rightarrow$ $y = 1$ je šikmá asymptota v $+\infty$

c) limity $x \to -\infty$ vyjdou stejně $\Rightarrow$ $y = 1$ je šikmá as. i v $-\infty$ (obecně to nemusí vycházet stejně!!!)

2.) $f(x) = \dfrac{x\cdot\sqrt{x^2-1}}{2x^2-1}$

a) svislé asymptoty:

$D_f = \mathbb{R} - (-1,1)$ (tam je definovaná na $\sqrt{x^2-1}$ a také $2x^2-1 \neq 0$)

$D_f = (-\infty,-1\rangle \cup \langle 1,\infty)$

$f(-1) = \dfrac{(-1)\sqrt{(-1)^2-1}}{2(-1)^2-1} = \dfrac{0}{1} = 0$

$f(1) = \dfrac{1\sqrt{1^2-1}}{2\cdot 1^2-1} = \dfrac{0}{1} = 0$

$\Rightarrow$ ~~$\forall x \in D_f : f(x) \in \mathbb{R} \Rightarrow f$ nem~~

$\forall x_0 \in \mathbb{R} : \lim\limits_{x\to x_0} f(x) \neq \pm\infty$ (f je spojitá na $D_f \Rightarrow \lim\limits_{x\to x_0} f(x) = f(x_0) \in \mathbb{R}$)

$\Rightarrow$ **f nemá svislé asymptoty**

b) asymptota v $+\infty$:

$$\lim_{x\to\infty}\frac{f(x)}{x} = \lim_{x\to\infty}\frac{\sqrt{x^2-1}}{2x^2-1} = \lim_{x\to\infty}\frac{\sqrt{x^2(1-\frac{1}{x^2})}}{x^2(2-\frac{1}{x^2})} = \lim_{x\to\infty}\frac{\overset{\sqrt{x^2}}{|x|}\sqrt{1-\frac{1}{x^2}}}{x^2(2-\frac{1}{x^2})} = \lim_{x\to\infty}\frac{x\sqrt{1-\frac{1}{x^2}}}{x^2(2-\frac{1}{x^2})} = 0 = a$$

$\frac{x}{x^2} = \frac{1}{x} \to 0$, $\frac{\sqrt{1-\frac{1}{x^2}}}{2-\frac{1}{x^2}} \to \frac{1}{2}$

(pro dost velká x : $|x| = x$)

$$\lim_{x\to\infty}(f(x)-ax) = \lim_{x\to\infty}\frac{x\sqrt{x^2-1}}{2x^2-1} = \lim_{x\to\infty}\frac{\overset{+x^2 \text{ pro dost velká } x}{x\cdot|x|}\sqrt{1-\frac{1}{x^2}}}{x^2(2-\frac{1}{x^2})} = \lim_{x\to\infty}\frac{+\sqrt{1-\frac{1}{x^2}}}{(2-\frac{1}{x^2})} = \frac{+1}{2} = b$$

$y = ax + b$

$y = 0x + \frac{1}{2}$

$y = \frac{1}{2}$

$\Rightarrow$ **$y = \frac{+1}{2}$ je šikmá asymptota fce f v $+\infty$**

c) asymptota v $-\infty$:

$$\lim_{x\to-\infty}\frac{f(x)}{x} = \lim_{x\to-\infty}\frac{\sqrt{x^2-1}}{2x^2-1} = \lim_{x\to-\infty}\frac{|x|\sqrt{1-\frac{1}{x^2}}}{x^2(2-\frac{1}{x^2})} = 0 = a$$

$\frac{|x|}{x^2} = \frac{1}{|x|} \to 0$

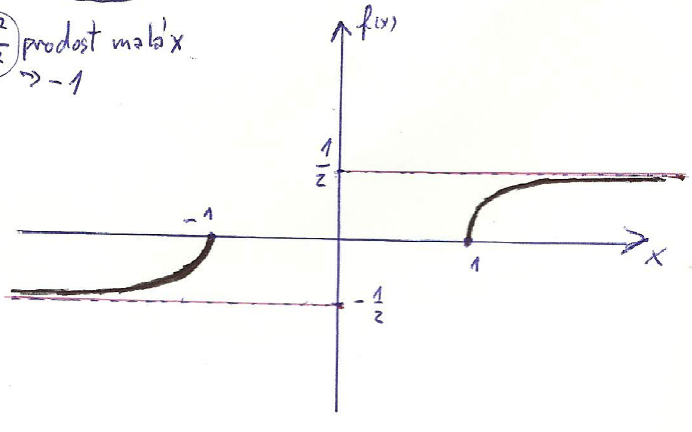
$$\lim_{x\to-\infty}(f(x)-\underbrace{ax}_{0\cdot x=0}) = \lim_{x\to-\infty}\frac{x\cdot\sqrt{x^2-1}}{2x^2-1} = \lim_{x\to-\infty}\frac{x\cdot|x|\sqrt{1-\frac{1}{x^2}}}{x^2(2-\frac{1}{x^2})} = -\frac{1}{2} = b$$

$\frac{\sqrt{1-\frac{1}{x^2}}}{2-\frac{1}{x^2}} \to \frac{1}{2}$

$\frac{x\cdot|x|}{x^2} = \frac{-x^2}{x^2}$ pro dost malá x $\to -1$

$y = 0x - \frac{1}{2}$

$\Rightarrow$ **$y = -\frac{1}{2}$ je šikmá asymptota fce f v $-\infty$**



3) $$f(x) = \frac{4x^4 - 6x^3 + 2x^2 - x + 5}{2x^3 - 6x^2 + 6x - 2}$$

a) svislé asymptoty

$$f(x) = \frac{4x^4 - 6x^3 + 2x^2 - x + 5}{2(x^3 - 3x^2 + 3x - 1)} = \frac{\overbrace{4x^4 - 6x^3 + 2x^2 - x + 5}^{\to 4}}{2\underbrace{(x-1)^3}_{\substack{0^+ \text{ při } x \to 1^+ \\ 0^- \text{ při } x \to 1^-}}} \Rightarrow \lim_{x \to 1^+} f(x) = \infty$$

$$\left(\lim_{x \to 1^-} f(x) = -\infty\right)$$

$\Rightarrow$ $x = 1$ je svislá asymptota fce f

b) šikmá asymptota v $+\infty$

$$\lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{4x^4 - 6x^3 + 2x^2 - x + 5}{2x^4 - 6x^3 + 6x^2 - 2x} = \frac{4}{2} = 2 = a$$

$$\lim_{x \to \infty} (f(x) - ax) = \lim_{x \to \infty} \left(\frac{4x^4 - 6x^3 + 2x^2 - x + 5}{2x^3 - 6x^2 + 6x - 2} - 2x \right) = \lim_{x \to \infty} \frac{4x^4 - 6x^3 + 2x^2 - x + 5 - 2x(2x^3 - 6x^2 + 6x - 2)}{2x^3 - 6x^2 + 6x - 2} =$$

$$= \lim_{x \to \infty} \frac{6x^3 - 10x^2 - 3x + 5}{2x^3 - 6x^2 + 6x - 2} = \frac{6}{2} = 3 = b$$

$\Rightarrow$ $y = 2x + 3$ je šikmá asymptota funkce $f(x)$ v $+\infty$

c) šikmá asymptota v $-\infty$

$$\lim_{x \to -\infty} \frac{f(x)}{x} = \lim_{x \to -\infty} \frac{4x^4 - 6x^3 + 2x^2 - x + 5}{2x^4 - 6x^3 + 6x^2 - 2x} = \frac{4}{2} = 2 = a$$

$$\lim_{x \to -\infty} (f(x) - ax) = \lim_{x \to -\infty} \frac{6x^3 - 10x^2 - 3x + 5}{2x^3 - 6x^2 + 6x - 2} = \frac{6}{2} = 3 = b$$

$\Rightarrow$ $y = 2x + 3$ je šikmá asymptota v $-\infty$

Př: Určete asymptoty funkce

1.) $f(x) = 5x + \dfrac{3\cos x}{x^2}$

svislé: $\lim\limits_{x\to 0^+} f(x) = \lim \underbrace{(5x)}_{\to 0} + \dfrac{\overbrace{3\cos x}^{\to 3}}{\underbrace{(x^2)}_{\to 0^+}} = +\infty \Rightarrow$ Funkce f má v bodě $x_0 = 0$ svislou asymptotu

(Pro jiná $x_0 \in \mathbb{R}$ je fce definovaná a spojitá $\Rightarrow$ jinde asymptotu nemá.)

šikmé: a) $\lim\limits_{x\to\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to\infty} \underbrace{\left(\dfrac{5x}{x}\right)}_{=5} + \dfrac{-3 \le \left(3\cos x\right) \le 3}{\underbrace{(x^3)}_{\to\infty}} = 5 = a$

$\lim\limits_{x\to\infty} (f(x) - ax) = \lim\limits_{x\to\infty} \cancel{5x} + \dfrac{-3 \le (3\cos x) \le 3}{\underbrace{(x^2)}_{\to\infty}} \cancel{-5x} = 0 = b$

$\Rightarrow$ Funkce f má v $+\infty$ šikmou asymptotu $y = 5x + 0$

b) $\lim\limits_{x\to-\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to-\infty} \underbrace{\left(\dfrac{5x}{x}\right)}_{=5} + \dfrac{-3 \le \left(3\cos x\right) \le 3}{\underbrace{(x^3)}_{\to-\infty}} = 5 = a$

$\lim\limits_{x\to-\infty} (f(x) - ax) = \lim\limits_{x\to-\infty} 5x + \dfrac{3\cos x}{x^2} - 5x = 0 = b$

$\Rightarrow$ Funkce má v $-\infty$ šikmou asymptotu $y = 5x + 0$

2.) $f(x) = \dfrac{x^3+2x-1}{x^2-1}$

a) svislé: Hledáme v bodech nespojitosti. Funkce je „poskládána" ze základních elementárních funkcí $\Rightarrow$ je spojitá v každém bodě def. oboru. $D_f = \mathbb{R} - \{+1,-1\}$ $\Rightarrow$ svislé asymptoty mohou být pouze v bodech $x_0 = 1$, nebo $x_1 = -1$.

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{\overbrace{x^3+2x-1}^{\to\, 1+2-1=2}}{\underbrace{x^2-1}_{\to\, 0^+}} = +\infty$$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{\overbrace{x^3+2x-1}^{\to\, -1-2-1=-4}}{\underbrace{x^2-1}_{\to\, 0^+}} = -\infty$$

$\Rightarrow$ **Funkce $f(x)$ má svislé asymptoty v bodech $x_0 = 1$ a $x_1 = -1$**

b) šikmé:

α) $\boxed{v +\infty}$:

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^3+2x^2-1}{x^3-1} = 1 = a$$

$$\lim_{x\to\infty} f(x) - ax = \lim_{x\to\infty} \frac{x^3+2x-1}{x^2-1} - x = \lim_{x\to\infty} \frac{x^3+2x-1-x(x^2-1)}{x^2-1} = \lim_{x\to\infty} \frac{3x-1}{x^2-1} = 0 = b$$

$\Rightarrow$ $y = 1\cdot x + 0$

$\Rightarrow$ **Funkce f má v $+\infty$ šikmou asymptotu $y = x$.**

β) $\boxed{v -\infty}$: vyjde stejně