Věta (Taylorova): Nechť $\forall \alpha \in U(x_0, \delta)\ \exists f^{(n+1)}(\alpha) \in \mathbb{R}$ a $x \in P(x_0, \delta)$. Potom $\exists \xi \in (\min\{x_0, x\}, \max\{x_0, x\})$:

$$f(x) = \underbrace{f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{T_n(x)} + \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}}_{R_{n+1}(x)}$$

$T_n(x)$ ... Taylorův polynom n-tého řádu funkce f v bodě x_0.

$R_{n+1}(x)$ ... zbytek po n-tém členu

Př: Určete $T_n(x)$ v bodě x_0 a odhadněte $f(a)$

1.) $f(x) = \sin x$, $x_0 = 0$, $n = 3$, $a = \frac{\pi}{6}$

$$\left.\begin{array}{ll} f(x) = \sin x & f(x_0) = \sin 0 = 0 \\ f'(x) = \cos x & f'(x_0) = \cos 0 = 1 \\ f''(x) = -\sin x & f''(x_0) = -\sin 0 = 0 \\ f'''(x) = -\cos x & f'''(x_0) = -\cos 0 = -1 \end{array}\right\} \Rightarrow$$

$$T_3(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3$$

$$T_3(x) = 0 + \frac{1}{1}(x-0) + \frac{0}{2}(x-0)^2 + \frac{-1}{6}(x-0)^3$$

$$\underline{\underline{T_3(x) = x - \frac{x^3}{6}}}$$

$0 < \xi < \frac{\pi}{6} \Rightarrow$

$0 < \sin \xi < \frac{1}{2}$

$$f(a) = f\left(\tfrac{\pi}{6}\right) = \sin\frac{\pi}{6} = T_3\left(\tfrac{\pi}{6}\right) + R_4\left(\tfrac{\pi}{6}\right) = \underbrace{\frac{\pi}{6} - \frac{\left(\frac{\pi}{6}\right)^3}{6}}_{=} + \underbrace{\frac{\sin(\xi)}{4!}\left(\frac{\pi}{6}\right)^4}$$

kalkulačka: 0,49967 $\quad \underbrace{\frac{0}{4!}\left(\frac{\pi}{6}\right)^4}_{= 0} < R_4\left(\tfrac{\pi}{6}\right) < \frac{1}{2}\frac{1}{4!}\left(\frac{4}{6}\right)^4 < 0{,}0041$

(Pozn. víme, že $\sin\frac{\pi}{6} = \frac{1}{2}$)

2.) $f(x) = x^3 - 1 \quad ; \quad x_0 = 1 \quad ; \quad n = 2 \quad ; \quad a = 3$

$$\left.\begin{array}{ll} f(x) = x^3-1 & f(1) = 0 \\ f'(x) = 3x^2 & f'(1) = 3 \\ f''(x) = 6x & f''(1) = 6 \end{array}\right\} \Rightarrow$$

$$\left(f'''(x) = 6 \Rightarrow f'''(\xi) = 6 \right)$$

$$T_2(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2$$

$$T_2(x) = 0 + \frac{3}{1!}(x-1) + \frac{6}{2!}(x-1)^2$$

$$\underline{\underline{T_2(x) = 3(x-1) + 3(x-1)^2}}$$

přesně, nejen odhad!

$$\Rightarrow f(a) = f(3) = T_2(3) + R_3(3) = \underbrace{3(3-1)+3(3-1)^2}_{= 18} + \underbrace{\frac{f'''(\xi)}{3!}(3-1)^3}_{= \frac{6}{3!}(3-1)^3 = 2^3 = 8} = \underline{\underline{26}}$$

a opravdu: $f(3) = 3^3 - 1 = 27 - 1 = 26$

3.) $f(x) = x^3 - 1 \quad ; \quad x_0 = 1 \quad ; \quad n = 3 \quad ; \quad a = 2$

$$\left.\begin{array}{ll} f(x) = x^3-1 & f(1) = 0 \\ f'(x) = 3x^2 & f'(1) = 3 \\ f''(x) = 6x & f''(1) = 6 \\ f'''(x) = 6 & f'''(1) = 6 \\ (f^{(4)}(x) = 0 & f^{(4)}(\xi) = 0) \end{array}\right\} \Rightarrow$$

$$T_3(x) = 0 + \frac{3}{1!}(x-1) + \frac{6}{2!}(x-1)^2 + \frac{6}{3!}(x-1)^3$$

$$\underline{\underline{T_3(x) = 3(x-1) + 3(x-1)^2 + (x-1)^3}}$$

$$\underline{R_4(x) = \frac{f^{(4)}(\xi)}{4!}(x-1)^4 = 0}$$

$$\Rightarrow f(x) = T_3(x)$$

$$f(2) = 3(2-1) + 3(2-1)^2 + (2-1)^3 = \underline{\underline{7}}$$

(a opravdu: $f(2) = 2^3 - 1 = 7$)

Poznámka: Aproximujeme-li funkci $f(x)$, která je polynomem n-tého stupně, pak $f(x) = T_n(x)$

například v předchozím př.: $T_3(x) = 3(x-1)+3(x-1)^2+(x-1)^3 = (x-1)\left(3+3(x-1)+(x-1)^2\right) =$

$= (x-1)(3+3x-3+x^2-2x+1) = (x-1)(x^2+x+1) = x^3-1 = f(x)$

4.) Určete Taylorův polynom 3. řádu funkce $f(x) = x^4 - 2x^3 + 4x + 5$ v bodě $x_0 = 1$.

$f(x) = x^4 - 2x^3 + 4x + 5$ $\qquad f(1) = 1 - 2 + 4 + 5 = 8$

$f'(x) = 4x^3 - 6x^2 + 4$ $\qquad f'(1) = 4 - 6 + 4 = 2$

$f''(x) = 12x^2 - 12x$ $\qquad f''(x) = 12 - 12 = 0$

$f'''(x) = 24x - 12$ $\qquad f'''(x) = 24 - 12 = 12$

$$T_3(x) = \frac{f(x_0)}{0!}(x-x_0)^0 + \frac{f'(x_0)}{1!}(x-x_0)^1 + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 = 8 + 2(x-1) + \frac{0}{2}(x-1)^2 + \frac{12}{3!}(x-1)^3 =$$

$$= \underline{\underline{8 + 2(x-1) + 2(x-1)^3}}$$

5.) Určete Taylorův polynom 2. řádu funkce $f(x) = x \cdot \ln x^2$ v bodě $x = \sqrt{e}$

$f(x) = x \cdot \ln x^2$ $\qquad f(\sqrt{e}) = \sqrt{e} \ln(\sqrt{e})^2 = \sqrt{e}$

$f'(x) = \ln x^2 + x \cdot \frac{1}{x^2} 2x = \ln x^2 + 2$ $\qquad f'(\sqrt{e}) = \ln(\sqrt{e})^2 + 2 = 3$

$f''(x) = \frac{1}{x^2} \cdot 2x = \frac{2}{x}$ $\qquad f''(\sqrt{e}) = \frac{2}{\sqrt{e}}$

$$T_2(x) = \sqrt{e} + 3(x - \sqrt{e}) + \frac{1}{2}\frac{2}{\sqrt{e}}(x-\sqrt{e})^2 = \underline{\underline{\sqrt{e} + 3(x-\sqrt{e}) + \frac{1}{\sqrt{e}}(x-\sqrt{e})^2}}$$

6.) Určete Taylorův polynom 2. řádu funkce $f(x) = x \sin(3x)$ v bodě $x_0 = \pi$.

$f(x) = x \cdot \sin(3x)$ $\qquad f(\pi) = \pi \cdot \sin(3\pi) = 0$

$f'(x) = \sin(3x) + 3x\cos(3x)$ $\qquad f'(\pi) = \sin(3\pi) + 3\pi \cdot \underbrace{\cos(3\pi)}_{-1} = -3\pi$

$f''(x) = 3\cos(3x) + 3\cos(3x) - 9x\sin(3x)$ $\qquad f''(\pi) = 6\cos(3\pi) - 9\pi\sin(3\pi) = -6$

$$\Rightarrow \quad T_3(x) = 0 - \frac{3\pi}{1!}(x-\pi)^1 + \frac{6}{2!}(x-\pi)^2 = \underline{\underline{-3\pi(x-\pi) - 3(x-\pi)^2}}$$

Př: Určete hodnotu $\cos 0{,}2$ s přesností 10^{-5}.

Známe hodnotu $\cos 0 = 1$ $\Rightarrow$ můžeme použít Taylorův rozvoj v bodě $x_0 = 0$.

Nejprve určíme n. Chceme, aby platilo:

$$|R_{n+1}(x)| < 10^{-5}$$

$$\left| \frac{f^{(n+1)}(\xi) \cdot (x-x_0)^{n+1}}{(n+1)!} \right| < 10^{-5}$$, víme, že $f^{(n+1)}(\xi) = -\sin\xi$ nebo: $-\cos\xi$ nebo: $\sin\xi$ nebo: $\cos\xi$ $\Rightarrow |f^{(n+1)}(\xi)| \leq 1$,

$$\Rightarrow |R_{n+1}(x)| = \left| \frac{f^{(n+1)}(\xi)\,(x-x_0)^{n+1}}{(n+1)!} \right| = \frac{|f^{(n+1)}(\xi)| \cdot |x - \overset{0}{x_0}|^{n+1}}{(n+1)!} \leq \frac{|x|^{n+1}}{(n+1)!}$$

$$\Rightarrow |R_{n+1}(0{,}2)| \leq \frac{0{,}2^{n+1}}{(n+1)!}$$ $\Rightarrow$ hledám $n \in \mathbb{N}$:

aby platilo: $\frac{0{,}2^{n+1}}{(n+1)!} < 10^{-5}$ $\Rightarrow$

$\boxed{n=1}$ $\Rightarrow \frac{0{,}2^2}{2!} = 0{,}02 > 10^{-5}$

$\boxed{n=2}$ $\Rightarrow \frac{0{,}2^3}{3!} = 0{,}001\overline{3} > 10^{-5}$

$\boxed{n=3}$ $\Rightarrow \frac{0{,}2^4}{4!} = 0{,}0000\overline{6} = 0{,}\overline{6} \cdot 10^{-4} > 10^{-5}$

$\boxed{n=4}$ $\Rightarrow \frac{0{,}2^5}{5!} = 0{,}00000 2\overline{6} = 2{,}\overline{6} \cdot 10^{-6} < 10^{-5}$

$\Rightarrow$ Zvolíme $n = 4$, $x_0 = 0$ $\Rightarrow$

$$\left.\begin{aligned} f(x) &= \cos x \\ f'(x) &= -\sin x \\ f''(x) &= -\cos x \\ f'''(x) &= \sin x \\ f^{(4)}(x) &= \cos x \end{aligned}\right\} \Rightarrow \left.\begin{aligned} f(0) &= 1 \\ f'(0) &= 0 \\ f''(0) &= -1 \\ f'''(0) &= 0 \\ f^{(4)}(0) &= 1 \end{aligned}\right\} \Rightarrow$$

$$T_4(x) = f(0) + f'(0)(x-0) + \tfrac{1}{2} f''(0)(x-0)^2 + \tfrac{1}{6} f'''(0)(x-0)^3 + \tfrac{1}{24} f^{(4)}(0)(x-0)^4$$

$$T_4(x) = 1 + 0\cdot(x-0) + \tfrac{1}{2}(-1)(x-0)^2 + \tfrac{1}{6}\, 0\,(x-0)^3 + \tfrac{1}{24}\, 1\,(x-0)^4$$

$$T_4(x) = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4$$

$$T_4(0{,}2) = 1 - \tfrac{1}{2}\,0{,}2^2 + \tfrac{1}{24}\,0{,}2^4 = 1 - \tfrac{1}{2}\left(\tfrac{1}{5}\right)^2 + \tfrac{1}{24}\left(\tfrac{1}{5}\right)^4 = 1 - \tfrac{1}{50} + \tfrac{1}{15000} = \tfrac{14701}{15000}$$

$$T_4(0{,}2) = 0{,}9800\overline{66} \doteq \cos 0{,}2$$

$$\cos 0{,}2 \in (0{,}9800\overline{66} - 10^{-5};\ 0{,}9800\overline{66} + 10^{-5})$$

kalkulačka: $\cos 0{,}2 = 0{,}98006657$

Př. Určete hodnotu Eulerova čísla e s přesností 10^{-5}.

Uvažujme funkci $f(x)=e^x$. Potřebujeme odhadnout hodnotu $f(1)=e^1=e$. Tu neznáme, ale známe $f(0)=e^0=1$. Zvolíme proto $x_0=0$. Požadovanou přesnost dosáhneme vhodnou volbou n (stupně Taylorova polynomu).

$$\left.\begin{array}{ll} f(x)=e^x & f(0)=e^0=1 \\ f'(x)=e^x & f'(0)=e^0=1 \\ \vdots & \vdots \\ f^{(n)}(x)=e^x & f^{(n)}(0)=e^0=1 \\ f^{(n+1)}(x)=e^x & f^{(n+1)}(\xi)=e^\xi \end{array}\right\} \Rightarrow \quad f(x)=f(x_0)+f'(x_0)(x-x_0)+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

$$e^x = 1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\ldots+\frac{1}{n!}x^n+\underbrace{\frac{e^\xi}{(n+1)!}x^{n+1}}_{R_{n+1}(x)}$$

Chceme, aby $|R_{n+1}(x)|<10^{-5}$ $\Rightarrow$

$$\left|\underbrace{\frac{e^\xi}{(n+1)!}}_{>0}\right|<10^{-5}$$

$$\frac{e^\xi}{(n+1)!}<10^{-5}$$

Odhadujeme $f(x)$ pro $x=1$ a $x_0=0$ $\Rightarrow \xi\in(x_0,x)=(0,1) \Rightarrow$

$$\frac{e^\xi}{(n+1)!}<\underbrace{\frac{e^1}{(n+1)!}<10^{-5}}_{\text{je splněno} \Leftrightarrow}$$

$$(n+1)!>e\cdot 10^5$$

Předpokládejme, že víme, že $\underbrace{3\geq e}_{\text{viz další příklad}}$. Je-li splněno $(n+1)!>3\cdot10^5 \Rightarrow (n+1)!>e\cdot 10^5$ a $(n+1)!>3\cdot 10^5$ je splněno pro $n\geq 8$. Zvolme proto $n=8$ $\Rightarrow$

$$e^1 = 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}+\frac{1}{8!}+R_{n+1}(x) \in (0,10^{-5})$$

$$\underline{\underline{e = 2{,}7182785 + R_{n+1}(x)}}$$

Př. Dokažte, že $\sum_{n=0}^{\infty} \frac{1}{n!} \le 3$.

$\forall k \in \mathbb{N};\ k>1:$

$$X = \sum_{n=0}^{k}\frac{1}{n!} = \frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{k!} =$$

$$= 1+1+\frac{1}{2\cdot1}+\frac{1}{3\cdot2\cdot1}+\cdots+\frac{1}{k(k-1)\cdot\ldots\cdot2\cdot1} \le$$

$$\le 2+\frac{1}{2\cdot1}+\frac{1}{2\cdot2\cdot1}+\cdots+\frac{1}{2(k-1)\cdot\ldots\cdot2\cdot1} =$$

$$= 2+\frac{1}{2}\cdot\left(\frac{1}{1}+\frac{1}{2\cdot1}+\cdots+\frac{1}{(k-1)\cdot\ldots\cdot2\cdot1}\right) =$$

$$= 2+\frac{1}{2}\left(\sum_{n=1}^{k-1}\frac{1}{n!}+\frac{1}{0!}+\frac{1}{k!}-\frac{1}{0!}-\frac{1}{k!}\right) =$$

$$= 2+\frac{1}{2}\left(\sum_{n=0}^{k}\frac{1}{n!}-\overset{=1}{\frac{1}{0!}}-\frac{1}{k!}\right) =$$

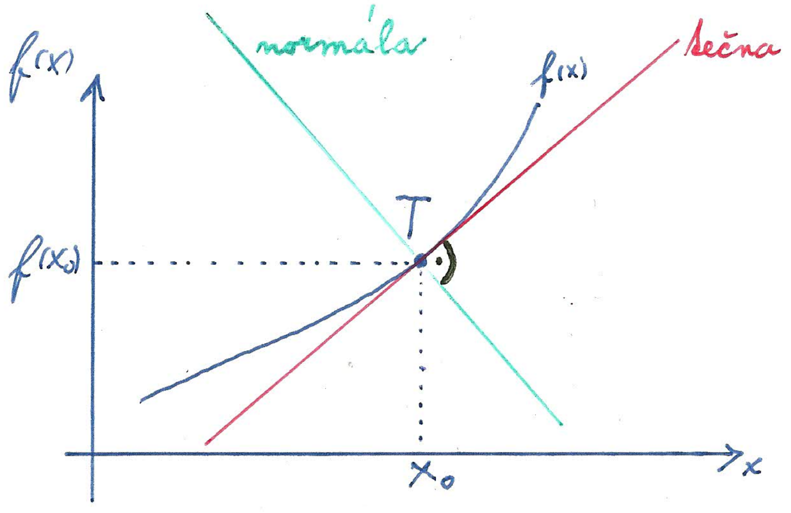
$$= 2-\frac{1}{2}-\frac{1}{2\cdot k!}+\underbrace{\frac{1}{2}\sum_{n=0}^{k}\frac{1}{n!}}_{=\frac{1}{2}X} \Rightarrow$$

$$\frac{1}{2}\sum_{n=0}^{k}\frac{1}{n!} \le 2-\frac{1}{2}-\frac{1}{2k!}$$

$$\sum_{n=0}^{k}\frac{1}{n!} \le 4-1-\frac{1}{k!} \le 3-\frac{1}{k}$$ $\Rightarrow$ při $k\to\infty$:

$$\sum_{n=0}^{\infty}\frac{1}{n!} \le 3$$

Tečna a normála



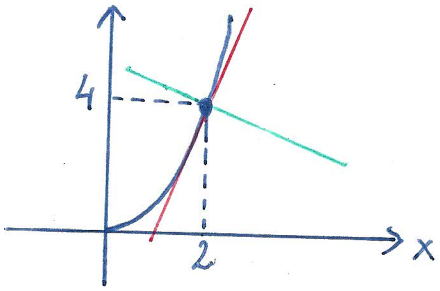
rovnice tečny: $y = kx + c_1$

rovnice normály: $x = -ky + c_2$

kde $k = f'(x_0)$

Př: Určete rovnice normály a tečny grafu funkce f v bodě $T = [x_0, f(x_0)]$

1.) $f(x) = x^2$, $x_0 = 2$



a) tečna

b) normála

$f'(x) = 2x \Rightarrow \underline{k = f'(2) = 4} \Rightarrow$

$y = 4x + c_1$

$x = -4y + c_2$

Bod $T = [2,4]$ leží jak na tečně, tak na normále $\Rightarrow$ $x = 2$ a $y = 4$ musí vyhovovat rovnicím tečny i normály $\Rightarrow$

$\underbrace{4 = 4 \cdot 2 + c_1}$

$\Downarrow$

$c_1 = -4$

$\Downarrow$

$\underbrace{2 = -4 \cdot 4 + c_2}$

$\Downarrow$

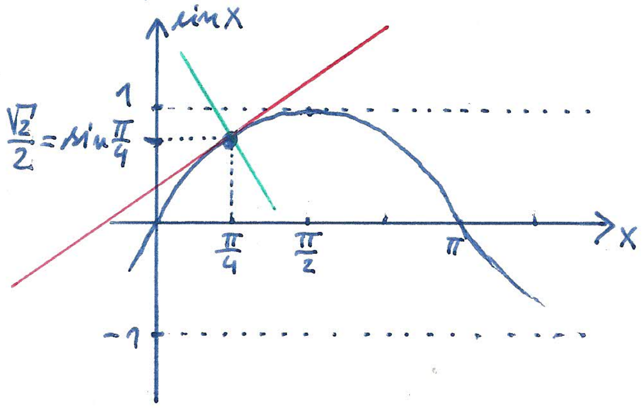
$c_2 = 18$

$\Downarrow$

tečna: $\underline{\underline{y = 4x - 4}}$

normála: $\underline{\underline{x = -4y + 18}}$

2.) $f(x) = \sin x \quad , \quad x_0 = \frac{\pi}{4}$



a) tečna $\qquad\qquad$ b) normála

$f(x) = \sin x \Rightarrow f'(x) = \cos x \Rightarrow k = f'(\frac{\pi}{4}) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$y = \frac{\sqrt{2}}{2}x + C_1 \qquad\qquad x = -\frac{\sqrt{2}}{2}y + C_2$

Dosadíme $x = \frac{\pi}{4}$ a $y = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}\cdot\frac{\pi}{4} + C_1 \qquad\qquad \frac{\pi}{4} = -\underbrace{\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}}_{=\frac{2}{4}=\frac{1}{2}} + C_2$

$C_1 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\frac{\pi}{4} \qquad\qquad C_2 = \frac{\pi}{4} + \frac{1}{2}$

$\Rightarrow$ tečna: $y = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}(1 - \frac{\pi}{4})$ normála: $x = -\frac{\sqrt{2}}{2}y + \frac{\pi}{4} + \frac{1}{2}$

3.) $f(x) = \ln(x^2 - 3x + 5) \quad , \quad x_0 = -1$

$f'(x) = \frac{1}{x^2 - 3x + 5}(2x - 3) \Rightarrow k = f'(-1) = \frac{1}{(-1)^2 - 3(-1) + 5}(2(-1) - 3) = \frac{-5}{9}$

a) tečna: $y = -\frac{5}{9}x + C_1$

dosadíme $x = -1$ a $y = f(-1) = \ln((-1)^2 - 3(-1) + 5) = \ln 9 \Rightarrow$

$\ln 9 = -\frac{5}{9}(-1) + C_1 \Rightarrow C_1 = \ln 9 - \frac{5}{9} \Rightarrow$ tečna: $y = -\frac{5}{9}x + \ln 9 - \frac{5}{9}$

b) normála: $x = \frac{5}{9}y + C_2$

$-1 = \frac{5}{9}\ln 9 + C_2$

$C_2 = -1 - \frac{5}{9}\ln 9 \Rightarrow$ $\qquad$ normála: $x = \frac{5}{9}y - 1 - \frac{5}{9}\ln 9$