Vlastní čísla a vlastní vektory

- Jestliže A je komplexní ~~m~~ čtvercová matice, pak vlastním vektorem příslušným k vlastnímu číslu $\lambda \in \mathbb{C}$ nazveme vektor $\bar{x} \in \mathbb{C}^n - \{\bar{o}\}$ splňující

$$\boxed{A \cdot \bar{x} = \lambda \cdot \bar{x}}$$

Př:

a) $\underbrace{\begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 0 \\ 2 \end{pmatrix}}_{\bar{x}} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} = \underset{\lambda}{3} \cdot \underset{\bar{x}}{\begin{pmatrix} 0 \\ 2 \end{pmatrix}}$ $\Rightarrow$ $\bar{x} = (0,2)$ je vlastní vektor matice A

b) $\begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \neq \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\Rightarrow$ $\bar{x} = (1,2)$ není vlastní vektor matice A

c) $\begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix} \underbrace{\begin{pmatrix} 0 \\ 2\lambda \end{pmatrix}}_{\lambda(0,2)} = \lambda \begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \lambda \cdot 3 \begin{pmatrix} 0 \\ 2 \end{pmatrix} = 3 \cdot \begin{pmatrix} 0 \\ 2\lambda \end{pmatrix}$ $\Rightarrow$ $\bar{x} = (0,2)$ je vl. vektor $\Rightarrow$ $\lambda(0,2)$ je také vlastní vektor matice A pro $\lambda \in \mathbb{C} - \{0\}$

Př: Nalezněte vlastní čísla a vlastní vektory dané matice A (komplexní čtvercové)

$$A \cdot \bar{x} = \lambda \cdot \bar{x}$$
$$A \cdot \bar{x} = \lambda \cdot E \cdot \bar{x}$$
$$A \cdot \bar{x} - \lambda E \bar{x} = \bar{o}$$
$$\underbrace{(A - \lambda E)}_{A^*} \bar{x} = \bar{o}$$
$$A^* \bar{x} = \bar{o}$$

$\Rightarrow$ řešíme soustavu lin. rovnic $A^* \bar{x} = \bar{o}$, ta určitě řešení má, a to $\bar{x} = \bar{o}$, ale vlastní vektor musí být nenulový
$\Rightarrow$ chceme, aby tato soustava měla i jiné řešení
$\Leftrightarrow$ má nekonečně mnoho řešení $\Leftrightarrow$
některý z řádků se při GEM vynuluje $\left(A^* \sim \begin{pmatrix} \cdots \\ 0 \ldots 0 \end{pmatrix}\right) \Leftrightarrow$
$|A^*| = 0$

$\Rightarrow$ 1.) $|A - \lambda E| = 0$ $\Rightarrow$ určíme λ

2.) $(A - \lambda E)\bar{x} = \bar{o}$ $\Rightarrow$ určíme $\bar{x}$

Př.: Nalezněte vlastní čísla a vlastní vektory matice $A = \begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix}$

1.) $|A - \lambda \cdot E| = 0 \quad \Rightarrow \quad \begin{vmatrix} (1-\lambda) & 0 \\ -1 & (3-\lambda) \end{vmatrix} = 0$

$$(1-\lambda)(3-\lambda) = 0$$

$$\lambda_{1,2} = \begin{cases} \underline{\underline{1}} \\ \underline{\underline{3}} \end{cases}$$

2.) $(A - \lambda E) \cdot \bar{v} = \bar{o} \quad \Rightarrow \quad$ řešme Gaussovou eliminační metodou

α) $\boxed{\lambda_1 = 1}$

$$\Rightarrow \left(\begin{array}{cc|c} (1-1) & 0 & 0 \\ -1 & (3-1) & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow$$

$v_2 = t \in \mathbb{R} \Rightarrow v_1 = 2t \quad \Rightarrow$

$\underline{\underline{\bar{v} = (2t, t) = t\,(2,1)\,,\ t \in \mathbb{R} - \{0\}}}$ $\qquad$ Zk: $\begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ ✓

β) $\boxed{\lambda_2 = 3}$

$$\Rightarrow \left(\begin{array}{cc|c} (1-3) & 0 & 0 \\ -1 & (3-3) & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -2 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right) \begin{array}{c} :(-2) \\ \end{array} \sim \left(\begin{array}{cc|c} 1 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow$$

$v_1 = 0\ ,\quad v_2 = t \in \mathbb{R} \quad \Rightarrow$

$\underline{\underline{\bar{v} = (0, t) = t\,(0,1)\,,\ t \in \mathbb{R} - \{0\}}}$ $\qquad$ Zk: $\begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ✓

Př: Určete vlastní čísla a vlastní vektory matice A

a) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

1.) Určíme λ:

$$0 = \begin{vmatrix} (2-\lambda) & 1 \\ 1 & (2-\lambda) \end{vmatrix} = (2-\lambda)(2-\lambda) - 1 = \lambda^2 - 4\lambda + 4 - 1 = \lambda^2 - 4\lambda + 3 = 0 \Rightarrow \lambda_{1,2} = \frac{4 \pm \sqrt{16-12}}{2} = \frac{4 \pm 2}{2} = \begin{cases} \underline{\underline{3}} \\ \underline{\underline{1}} \end{cases}$$

2.) Určíme vlastní vektory $\bar{x}$:

$\boxed{\lambda = 3}$

$$\Rightarrow \left(\begin{array}{cc|c} (2-3) & 1 & 0 \\ 1 & (2-3) & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right)_{+r_1} \sim \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow -x_1 + x_2 = 0 \Rightarrow \begin{array}{l} x_2 = t \\ x_1 = t \end{array}$$

$$\Rightarrow \bar{x} = (t, t) = \underline{\underline{t(1,1),\ t \in \mathbb{C} - \{0\}}}$$

$\boxed{\lambda = 1}$

$$\Rightarrow \left(\begin{array}{cc|c} (2-1) & 1 & 0 \\ 1 & (2-1) & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow x_1 + x_2 = 0 \Rightarrow \begin{array}{l} x_2 = t \\ x_1 = -t \end{array}$$

$$\Rightarrow \bar{x} = (-t, t) = \underline{\underline{t(-1,1),\ t \in \mathbb{C} - \{0\}}}$$

zk: $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ ✓

b) $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

1.) Určíme λ:

$$0 = \begin{vmatrix} (1-\lambda) & 2 \\ 1 & (0-\lambda) \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0 \Rightarrow \lambda = \begin{cases} \underline{\underline{2}} \\ \underline{\underline{-1}} \end{cases}$$

2.) Určíme vlastní vektory $\bar{x}$:

$\boxed{\lambda = 2}$

$$\left(\begin{array}{cc|c} (1-2) & 2 & 0 \\ 1 & 0-2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -1 & 2 & 0 \\ 1 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow -x_1 + 2x_2 = 0 \Rightarrow \begin{array}{l} x_2 = t \\ x_1 = 2t \end{array} \Rightarrow \bar{x} = (2t, t) = \underline{\underline{t(2,1),\ t \in \mathbb{C} - \{0\}}}$$

$\boxed{\lambda = -1}$

$$\left(\begin{array}{cc|c} (1+1) & 2 & 0 \\ 1 & +1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & 2 & 0 \\ 1 & 1 & 0 \end{array}\right)\begin{array}{l} :2 \\ -\frac{1}{2}r_1 \end{array} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow x_1 + x_2 = 0 \Rightarrow \begin{array}{l} x_2 = t \\ x_1 = -t \end{array} \Rightarrow \bar{x} = (-t, t) = \underline{\underline{t(-1,1),\ t \in \mathbb{C} - \{0\}}}$$

zk: $\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 1 \end{pmatrix}$; $\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -1\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ ✓

c) $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 4 & 2 \end{pmatrix}$

1.) Určíme λ:

$$0 = \begin{vmatrix} (2-\lambda) & 0 & 0 \\ 0 & (3-\lambda) & 3 \\ 0 & 4 & (2-\lambda) \end{vmatrix} = (2-\lambda)(-1)^2 \begin{vmatrix} (3-\lambda) & 3 \\ 4 & (2-\lambda) \end{vmatrix} = (2-\lambda)\left[(3-\lambda)(2-\lambda)-12\right] = (2-\lambda)\left[\lambda^2-5\lambda-6\right] = (2-\lambda)(\lambda-6)(\lambda+1) =$$

rozvojem

$$\Rightarrow \lambda = \begin{cases} \underline{\underline{2}} \\ \underline{\underline{6}} \\ \underline{\underline{-1}} \end{cases}$$

2.) Určíme vlastní vektory $\bar{x}$:

$\boxed{\lambda = 2}$

$$\left(\begin{array}{ccc|c} (2-2) & 0 & 0 & 0 \\ 0 & (3-2) & 3 & 0 \\ 0 & 4 & (2-2) & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 4 & 2 & 0 \end{array}\right) -4r_1 \sim \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -10 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_3 = 0 \\ x_2 + 3\cdot 0 = 0 \Rightarrow x_2 = 0 \\ x_1 = k \end{array}$$

$$\Rightarrow \bar{x} = (k, 0, 0) = \underline{\underline{k(1,0,0),\ k \in \mathbb{C}-\{0\}}}$$

$\boxed{\lambda = 6}$

$$\left(\begin{array}{ccc|c} (2-6) & 0 & 0 & 0 \\ 0 & (3-6) & 3 & 0 \\ 0 & 4 & (2-6) & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -4 & 0 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 4 & -4 & 0 \end{array}\right) \begin{array}{l} :(-4) \\ :(-3) \\ :4 \end{array} \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) -r_2 \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = k \\ x_3 = k \end{array}$$

$$\Rightarrow \bar{x} = (0, k, k) = \underline{\underline{k(0,1,1),\ k \in \mathbb{C}-\{0\}}}$$

$\boxed{\lambda = -1}$

$$\left(\begin{array}{ccc|c} (2+1) & 0 & 0 & 0 \\ 0 & (3+1) & 3 & 0 \\ 0 & 4 & (2+1) & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 0 & 4 & 3 & 0 \\ 0 & 4 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 0 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad 4x_2 + 3x_3 = 0 \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = -3k \\ x_3 = 4k \end{array}$$

$$\bar{x} = (0, -3k, 4k) = \underline{\underline{k(0,-3,4),\ k \in \mathbb{C}-\{0\}}}$$

Př: Nalezněte vlastní čísla a vlastní vektory matice $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

1.) $\begin{vmatrix} (0-\lambda) & -1 \\ 1 & (0-\lambda) \end{vmatrix} = 0 \iff \lambda^2+1=0 \iff \lambda = \begin{cases} \underline{\underline{i}} \\ \underline{\underline{-i}} \end{cases}$

2.) α) $\boxed{\lambda = i}$ ⇒ řešíme soustavu

$$\left(\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right)_{\cdot i} \sim \left(\begin{array}{cc|c} -i & -1 & 0 \\ i & 1 & 0 \end{array}\right)_{+r_1} \sim \left(\begin{array}{cc|c} -i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow -ix_1 - x_2 = 0 \Rightarrow \begin{array}{l} x_1 = t \in \mathbb{C} \\ x_2 = -it \end{array}$$

$$\Rightarrow \underline{\underline{\bar{x} = (t, -it) = t\,(1, -i)\,,\ t \in \mathbb{C} - \{0\}}}$$

Zk: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix} = i\begin{pmatrix} 1 \\ -i \end{pmatrix}$ ✓

β) $\boxed{\lambda = -i}$ ⇒ řešíme soustavu

$$\left(\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right)_{\cdot i} \left(\begin{array}{cc|c} i & -1 & 0 \\ i & -1 & 0 \end{array}\right)_{-r_1} \sim \left(\begin{array}{cc|c} i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow ix_1 - x_2 = 0 \Rightarrow \begin{array}{l} x_1 = t \in \mathbb{C} \\ x_2 = it \end{array}$$

$$\Rightarrow \underline{\underline{\bar{x} = (t, it) = t\,(1, i)\,,\ t \in \mathbb{C} - \{0\}}}$$

Zk: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix} = -i\begin{pmatrix} 1 \\ i \end{pmatrix}$ ✓

Př. Nalezněte vlastní čísla a vlastní vektory matice $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & 2 \end{pmatrix}$

1.) Nalezneme vlastní čísla λ: $\quad |A - \lambda I| = 0$

$$\begin{vmatrix} (2-\lambda) & 0 & 1 \\ 0 & (2-\lambda) & 2 \\ 1 & 0 & (2-\lambda) \end{vmatrix} = 0 \quad \Rightarrow \quad (2-\lambda)(-1)^{1+1} \begin{vmatrix} (2-\lambda) & 1 \\ 1 & (2-\lambda) \end{vmatrix} = (2-\lambda)\left[(2-\lambda)^2 - 1\right] = (2-\lambda)\left[\lambda^2 - 4\lambda + 4 - 1\right] =$$

$$= (2-\lambda)(\lambda^2 - 4\lambda + 3) = (2-\lambda)(\lambda-3)(\lambda-1) = 0$$

$$\Rightarrow \quad \lambda_{1,2,3} = \begin{cases} \underline{\underline{2}} \\ \underline{\underline{3}} \\ \underline{\underline{1}} \end{cases}$$

2.) Určíme vlastní vektory: $\quad (A - \lambda I) \cdot \bar{v} = \bar{o}$

a) $\boxed{\lambda_1 = 2}$

GEM: $\left(\begin{array}{ccc|c} (2-2) & 0 & 1 & 0 \\ 0 & (2-2) & 2 & 0 \\ 1 & 0 & (2-2) & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) \underset{-2r_1}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} v_1 = 0 \\ v_3 = 0 \\ v_2 = t \in \mathbb{C} \end{array}$

(neznámé v_1, v_2, v_3)

$\Rightarrow \bar{v}_1 = (0, t, 0) = \underline{\underline{t(0,1,0), \ t \in \mathbb{C} - \{0\}}}$ $\qquad$ Zk: $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = 2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ ✓

b) $\boxed{\lambda_2 = 3}$

GEM: $\left(\begin{array}{ccc|c} (2-3) & 0 & 1 & 0 \\ 0 & (2-3) & 2 & 0 \\ 1 & 0 & (2-3) & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right)_{+r_1} \sim \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} v_2 = 2t \\ v_3 = t \end{array} \quad v_1 = t$

$\Rightarrow \bar{v}_2 = (t, 2t, t) = \underline{\underline{t(1,2,1), \ t \in \mathbb{C} - \{0\}}}$ $\qquad$ Zk: $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ ✓

c) $\boxed{\lambda_3 = 1}$

GEM: $\left(\begin{array}{ccc|c} (2-1) & 0 & 1 & 0 \\ 0 & (2-1) & 2 & 0 \\ 1 & 0 & (2-1) & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)_{-r_1} \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} v_1 = -t \\ v_2 = -2t \\ v_3 = t \in \mathbb{C} \end{array}$

$\Rightarrow \bar{v}_3 = (-t, -2t, t) = \underline{\underline{t(-1,-2,1), \ t \in \mathbb{C} - \{0\}}}$ $\qquad$ Zk: $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ ✓

Spektrální rozklad

Věta: Nechť A je reálná symetrická matice a λ je vlastní vektor matice A, pak $\lambda \in \mathbb{R}$.

Věta: Nechť A je reál. symetrická matice, pak vl. vekt. přísl. různým vl. číslům jsou na sebe kolmé.

Věta: Nechť A je reálná symetrická matice potom existuje ortogonální matice Q (tzn $Q \cdot Q^T = E$) a diagonální matice D taková, že

$$\boxed{A = Q^T D Q}$$

(Řádky Q tvoří ortonormální vlastní vektory matice A, diag. prvky D jsou vlastní čísla A)

Př: Nalezněte spektrální rozklad matice $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

1.) Určíme λ:

$$\begin{vmatrix} (3-\lambda) & 1 \\ 1 & (3-\lambda) \end{vmatrix} = 0 \Rightarrow (3-\lambda)^2 - 1 = 0 \Rightarrow \lambda^2 - 6\lambda + 8 = 0 \Rightarrow (\lambda-2)(\lambda-4) = 0 \Rightarrow \lambda = \left\langle \begin{matrix} \underline{\underline{2}} \\ \underline{\underline{4}} \end{matrix} \right.$$

2.) Určíme vlastní vektory:

α) $\left(\begin{matrix} (3-2) & 1 \\ 1 & (3-2) \end{matrix} \middle| \begin{matrix} 0 \\ 0 \end{matrix}\right) \sim \left(\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \end{matrix}\right) \sim \left(\begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \end{matrix}\right) \Rightarrow \begin{matrix} v_1 = -t \\ v_2 = t \end{matrix} \Rightarrow \bar{v}_1 = t(-1, 1), \ t \in \mathbb{R} - \{0\}$

β) $\left(\begin{matrix} (3-4) & 1 \\ 1 & (3-4) \end{matrix} \middle| \begin{matrix} 0 \\ 0 \end{matrix}\right) \sim \left(\begin{matrix} -1 & 1 \\ 1 & -1 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \end{matrix}\right) \sim \left(\begin{matrix} -1 & 1 \\ 0 & 0 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \end{matrix}\right) \Rightarrow \begin{matrix} v_1 = t \\ v_2 = t \end{matrix} \Rightarrow \bar{v}_2 = t(1, 1), \ t \in \mathbb{R} - \{0\}$

3.) Normalizujeme vlastní vektory:

$$\bar{q}_1 = \frac{1}{\sqrt{\bar{v}_1 \bar{v}_1}} \bar{v}_1 = \frac{1}{\sqrt{2}}(-1, 1) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\bar{q}_2 = \frac{1}{\sqrt{\bar{v}_2 \bar{v}_2}} \bar{v}_2 = \frac{1}{\sqrt{2}}(1, 1) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

4.) $\Rightarrow$

$$\underline{\underline{A = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}}$$

5.) Zkouška:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -2 & 4 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}}} \checkmark$$

Př: Nalezněte spektrální rozklad matice $A = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & 0 \\ 4 & 0 & 8 \end{pmatrix}$

1) Nalezneme vlastní čísla a vlastní vektory:

$$\begin{vmatrix} (2-\lambda) & 0 & 4 \\ 0 & (1-\lambda) & 0 \\ 4 & 0 & (8-\lambda) \end{vmatrix} = (1-\lambda) \begin{vmatrix} (2-\lambda) & 4 \\ 4 & (8-\lambda) \end{vmatrix} = (1-\lambda)\left[(2-\lambda)(8-\lambda)-16\right] = (1-\lambda)\left[\lambda^2 - 10\lambda\right] = \lambda(1-\lambda)(\lambda-10) = 0$$

$$\Rightarrow \quad \lambda_{1,2,3} = \begin{cases} \underline{\underline{0}} \\ \underline{\underline{1}} \\ \underline{\underline{10}} \end{cases}$$

$$\boxed{\lambda_1 = 0} \Rightarrow \left(\begin{array}{ccc|c} 2 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 0 & 8 & 0 \end{array}\right) \begin{array}{c} \\ \\ -2r_1 \end{array} \sim \left(\begin{array}{ccc|c} 2 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 = -2t \\ x_2 = 0 \\ x_3 = t \end{array} \Rightarrow \underline{\underline{\bar{X}_1 = t(-2,0,1), \; t \in \mathbb{C}-\{0\}}}$$

$$\boxed{\lambda_2 = 1} \quad \left(\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & 7 & 0 \end{array}\right) \begin{array}{c} \\ \\ -4r_1 \end{array} \sim \left(\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 0 & -9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = t \\ x_3 = 0 \end{array} \Rightarrow \underline{\underline{\bar{X}_2 = t(0,1,0), \; t \in \mathbb{C}-\{0\}}}$$

$$\boxed{\lambda_3 = 10} \quad \left(\begin{array}{ccc|c} -8 & 0 & 4 & 0 \\ 0 & -9 & 0 & 0 \\ 4 & 0 & -2 & 0 \end{array}\right) \begin{array}{c} \\ :-9 \\ \cdot 2 \end{array} \sim \left(\begin{array}{ccc|c} -8 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 8 & 0 & -4 & 0 \end{array}\right) \begin{array}{c} \\ \\ +r_1 \end{array} \sim \left(\begin{array}{ccc|c} -8 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 = t \\ x_2 = 0 \\ x_3 = 2t \end{array} \Rightarrow \underline{\underline{\bar{X} = t(1,0,2), \; t \in \mathbb{C}-\{0\}}}$$

~~2) Vlastní~~

2.) Vlastní vektory podělíme jejich velikostí:

$$\bar{e}_1 = \frac{1}{|\bar{X}_1|} \cdot \bar{X}_1 = \frac{1}{\sqrt{(-2)^2+0^2+1^2}}(-2,0,1) = \frac{1}{\sqrt{5}}(-2,0,1)$$

$$\bar{e}_2 = \frac{1}{|\bar{X}_2|} \cdot \bar{X}_2 = \frac{1}{\sqrt{0^2+1^2+0^2}}(0,1,0) = (0,1,0)$$

$$\bar{e}_3 = \frac{1}{|\bar{X}_3|} \cdot \bar{X}_3 = \frac{1}{\sqrt{1^2+0^2+2^2}} = \frac{1}{\sqrt{5}}(1,0,2)$$

3) Rozklad:

$$A = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & 0 \\ 4 & 0 & 8 \end{pmatrix} = \underbrace{\begin{pmatrix} -\frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} -\frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \end{pmatrix}}_{Q^T}$$

Př: Naleznète spektrální rozklad matice $A=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$

$$\begin{vmatrix}(1-\lambda)&1&1\\1&(1-\lambda)&1\\1&1&(1-\lambda)\end{vmatrix} = (1-\lambda)^3+1+1-[(1-\lambda)+(1-\lambda)+(1-\lambda)] =$$

$$= (1-\lambda)^3+2-3(1-\lambda) = 1-3\lambda+3\lambda^2-\lambda^3+2-3+3\lambda =$$

$$= -\lambda^3+3\lambda^2 = -\lambda^2(\lambda-3)=0 \Rightarrow \lambda = \begin{cases}0\\3\end{cases}$$

$\boxed{\lambda=0} \Rightarrow \left(\begin{array}{ccc|c}1&1&1&0\\1&1&1&0\\1&1&1&0\end{array}\right)\begin{array}{l}\\-r_1\\-r_1\end{array} \sim \left(\begin{array}{ccc|c}1&1&1&0\\0&0&0&0\\0&0&0&0\end{array}\right) \Rightarrow$ $x_1+x_2+x_3=0 \Rightarrow x_3=t,\ x_2=s.$ $x_1+s+t=0 \Rightarrow x_1=-s-t$

$$\Rightarrow \bar{x} = \begin{pmatrix}-s-t\\s\\t\end{pmatrix} = s\begin{pmatrix}-1\\1\\0\end{pmatrix}+t\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

$\boxed{\lambda=3}$ $\left(\begin{array}{ccc|c}-2&1&1&0\\1&-2&1&0\\1&1&-2&0\end{array}\right) \sim \left(\begin{array}{ccc|c}-2&1&1&0\\2&-4&2&0\\2&2&-4&0\end{array}\right) \sim \left(\begin{array}{ccc|c}-2&1&1&0\\0&-3&3&0\\0&3&-3&0\end{array}\right) \sim \left(\begin{array}{ccc|c}-2&1&1&0\\0&-3&3&0\\0&0&0&0\end{array}\right)$ $-2x_1+t+t=0 \Rightarrow x_1=t$; $-3x_2+3x_3=0 \Rightarrow x_3=t \Rightarrow x_2=t$

$$\Rightarrow \bar{x} = \begin{pmatrix}t\\t\\t\end{pmatrix} = t\begin{pmatrix}1\\1\\1\end{pmatrix}$$

Problém: $\begin{pmatrix}-1\\1\\0\end{pmatrix}$ není kolmé na $\begin{pmatrix}-1\\0\\1\end{pmatrix}$ (neboť $(-1,1,0)\cdot(-1,0,1)=1\neq 0$)

$\Rightarrow$ musíme z nich vytvořit dva navzájem kolmé vektory:

$\bar{f}_1 = (-1,1,0)$

$\bar{f}_2 = (-1,0,1)+\alpha(-1,1,0)$ tak, aby $\bar{f}_1\cdot\bar{f}_2 = (-1,1,0)\,[(-1,0,1)+\alpha(-1,1,0)] = 0$

$1 + \alpha\cdot 2 = 0$

$\alpha = -\frac{1}{2}$

$\bar{f}_2 = (-1,0,1)-\frac{1}{2}(-1,1,0) = (-\frac{1}{2},-\frac{1}{2},1) = -\frac{1}{2}(1,1,-2)$

$\Rightarrow$ Vektory $(-1,1,0)$; $(1,1,-2)$ a $(1,1,1)$ jsou navzájem kolmé a vlastní vektory matice A

$\Rightarrow$ normujeme je:

$(\lambda=0)\ \bar{e}_1 = \frac{1}{\sqrt{(-1)^2+1^2+0^2}}(-1,1,0) = \frac{1}{\sqrt{2}}(-1,1,0)$

$(\lambda=0)\ \bar{e}_2 = \frac{1}{\sqrt{1^2+1^2+(-2)^2}}\cdot(1,1,-2) = \frac{1}{\sqrt{6}}(1,1,-2)$

$(\lambda=3)\ \bar{e}_3 = \frac{1}{\sqrt{1^2+1^2+1^2}}\cdot(1,1,1) = \frac{1}{\sqrt{3}}(1,1,1)$

$$\Rightarrow A = \begin{pmatrix}-\frac{1}{\sqrt2}&\frac{1}{\sqrt6}&\frac{1}{\sqrt3}\\\frac{1}{\sqrt2}&\frac{1}{\sqrt6}&\frac{1}{\sqrt3}\\0&-\frac{2}{\sqrt6}&\frac{1}{\sqrt3}\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&3\end{pmatrix}\begin{pmatrix}-\frac{1}{\sqrt2}&\frac{1}{\sqrt2}&0\\\frac{1}{\sqrt6}&\frac{1}{\sqrt6}&-\frac{2}{\sqrt6}\\\frac{1}{\sqrt3}&\frac{1}{\sqrt3}&\frac{1}{\sqrt3}\end{pmatrix}$$

zk: $\begin{pmatrix}-\frac{1}{\sqrt2}&\frac{1}{\sqrt6}&\frac{1}{\sqrt3}\\\frac{1}{\sqrt2}&\frac{1}{\sqrt6}&\frac{1}{\sqrt3}\\0&-\frac{2}{\sqrt6}&\frac{1}{\sqrt3}\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\\frac{3}{\sqrt3}&\frac{3}{\sqrt3}&\frac{3}{\sqrt3}\end{pmatrix} = \begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} = A\ \checkmark$

Př. Určete spektrální rozklad matice A

a) $A = \begin{pmatrix} -3 & 0 & 4 \\ 0 & -3 & 0 \\ 4 & 0 & 3 \end{pmatrix}$

1.) Určíme λ:

$$0 = \begin{vmatrix} (-3-\lambda) & 0 & 4 \\ 0 & (-3-\lambda) & 0 \\ 4 & 0 & (3-\lambda) \end{vmatrix} = (-3-\lambda) \begin{vmatrix} (-3-\lambda) & 4 \\ 4 & (3-\lambda) \end{vmatrix} = (-3-\lambda)\left[(-3-\lambda)(3-\lambda) - \cancel{25}\, 16\right] =$$

rozvoj podle 2. ř.

$$= (-3-\lambda)\left[\lambda^2 \underbrace{-9-16}_{-25}\right] = (-3-\lambda)(\lambda-5)(\lambda+5) = 0$$

$$\Rightarrow \quad \lambda = \begin{cases} -3 \\ 5 \\ -5 \end{cases}$$

2) Určíme vl. vektory $\bar{x}$ a znormujeme je $(\bar{q}_i)$

$\boxed{\lambda = -3}$

$$\left(\begin{array}{ccc|c} (-3+3) & 0 & 4 & 0 \\ 0 & (-3+3) & 0 & 0 \\ 4 & 0 & (3+3) & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & 6 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 4 & 0 & 6 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \Rightarrow x_1 = 0 \\ \Rightarrow x_3 = 0 \end{array} \qquad x_2 = 1$$

$$\bar{x} = 1(0,1,0) \quad \Rightarrow \bar{q}_1 = \frac{1}{\sqrt{(0,1,0)(0,1,0)}}(0,1,0) = \frac{1}{1}(0,1,0) = (0,1,0)$$

$\boxed{\lambda = 5}$

$$\left(\begin{array}{ccc|c} -3-5 & 0 & 4 & 0 \\ 0 & (-3-5) & 0 & 0 \\ 4 & 0 & (3-5) & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -8 & 0 & 4 & 0 \\ 0 & -8 & 0 & 0 \\ 4 & 0 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow x_2 = 0 \qquad x_1 = 1 \Rightarrow x_3 = 2 \cdot 1$$

$$\bar{x} = 1(1,0,2) \quad \Rightarrow \bar{q}_2 = \frac{1}{\sqrt{(1,0,2)(1,0,2)}}(1,0,2) = \frac{1}{\sqrt{5}}(1,0,2)$$

$\boxed{\lambda = -5}$

$$\left(\begin{array}{ccc|c} (-3+5) & 0 & 4 & 0 \\ 0 & (-3+5) & 0 & 0 \\ 4 & 0 & (3+5) & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & 0 & 4 & 0 \\ 0 & -8 & 0 & 0 \\ 2 & 0 & 8 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} x_3 = 1 \Rightarrow x_1 = -2 \cdot 1 \\ \Rightarrow x_2 = 0 \end{array}$$

$$\bar{x} = 1(-2,0,1) \quad \Rightarrow \bar{q}_3 = \frac{1}{\sqrt{(-2,0,1)(-2,0,1)}}(-2,0,1) = \frac{1}{\sqrt{5}}(-2,0,1)$$

$$\Rightarrow A = \begin{pmatrix} 0 & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} -3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{pmatrix}$$

Geršgorinova věta – lokalizace spektra

Spektrum matice: $\sigma(A)$ = Množina vlastních čísel matice A

Věta: (Geršgorinova): Nechť $A = (a_{ij})$ je čtvercová matice řádu n.
Nechť

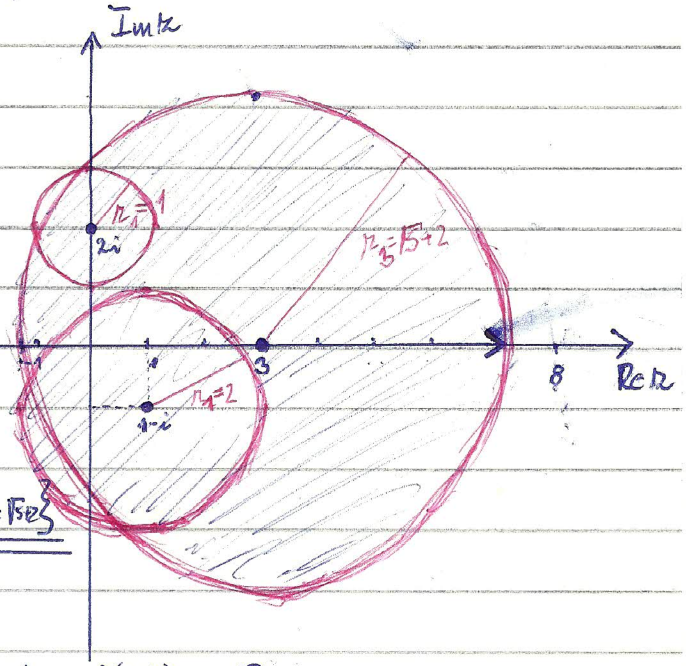
$$r_i = \left(\sum_{j=1}^{n} |a_{ij}|\right) - |a_{ii}| \quad ; \quad S_i = \{ z \in \mathbb{C} \mid |z - a_{ii}| \le r_i \}$$

pro $i = 1, 2, \ldots, n$.

Potom $\sigma(A) \subseteq S_1 \cup S_2 \cup \ldots \cup S_n$

Př Pomocí Geršgorinovy věty lokalizujte spektrum matice

$$A = \begin{pmatrix} 1-i & i & 1 \\ 0 & 2i & -i \\ 2+i & -2 & 3 \end{pmatrix}$$



$$r_1 = |i| + |1| = \sqrt{0^2+1^2} + \sqrt{1^2+0^2} = 2$$

$$r_2 = |0| + |-i| = \sqrt{0^2+0^2} + \sqrt{0^2+(-1)^2} = 1$$

$$r_3 = \sqrt{2^2+1^2} + \sqrt{(-2)^2+0^2} = \sqrt{5}+2 \doteq 4{,}24$$

$$\Rightarrow \sigma(A) \subseteq \{ z \in \mathbb{C} \mid |z-(1-i)| \le 2 \} \cup \{ z \in \mathbb{C} \mid |z-2i| \le 1 \} \cup \{ z \in \mathbb{C} \mid |z-3| \le \sqrt{5}+2 \}$$

Věta: Nechť A je reálná symetrická matice, pak $\sigma(A) \subset \mathbb{R}$.

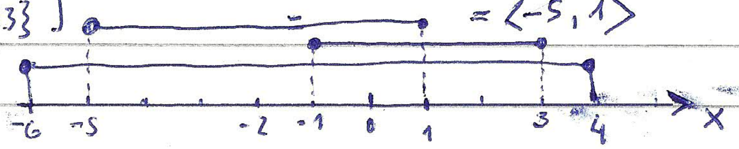
Př: Lokalizujte spektrum matice $A = \begin{pmatrix} -1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & -2 \end{pmatrix}$

(Místo modulů kompl. čísel stačí určovat abs. hodnoty a hledat jen na reálné ose)

$$\Rightarrow r_1 = |2| + |3| = 5 \Rightarrow S_1 = \{ x \in \mathbb{R} \mid |x+1| \le 5 \} = \langle -6, 4 \rangle$$

$$r_2 = |2| + |0| = 2 \Rightarrow S_2 = \{ x \in \mathbb{R} \mid |x-1| \le 2 \} = \langle -1, 3 \rangle$$

$$r_3 = |3| + |0| = 3 \Rightarrow S_3 = \{ x \in \mathbb{R} \mid |x+2| \le 3 \} = \langle -5, 1 \rangle$$



$$\Rightarrow \sigma(A) \subseteq \langle -6, 4 \rangle \cup \langle -1, 3 \rangle \cup \langle -5, 1 \rangle$$

$$\sigma(A) \subseteq \langle -6, 4 \rangle$$

Ověřte, že matice A je pozitivně definitní.

Př: $A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} \Rightarrow \left. \begin{array}{l} S_1 = \{ x \in \mathbb{R} \mid |x-4| \le 1 \} = \langle 3, 5 \rangle \\ S_2 = \{ x \in \mathbb{R} \mid |x-3| \le 2 \} = \langle 1, 5 \rangle \\ S_3 = \{ x \in \mathbb{R} \mid |x-2| \le 1 \} = \langle 1, 3 \rangle \end{array} \right\} \Rightarrow \sigma(A) \subseteq \langle 1, 5 \rangle \Rightarrow$ Všechna vl. čísla jsou kladná $\Rightarrow$ matice je pozitivně definitní