LU rozklad

Permutační matice P vznikne z jednotkové matice nějakou výměnou (permutací) sloupců (řádků).

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\quad\curvearrowleft\ \text{sloupce } 1,2\quad} P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Př. Vynásobte:

$$\underset{A}{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}} \cdot \underset{P}{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} = \underset{AP}{\begin{pmatrix} 2 & 1 & 3 \\ 2 & 1 & 3 \\ 2 & 1 & 3 \end{pmatrix}}$$

Vynásobíme-li matici A maticí P zprava, dojde ke stejné výměně sloupců v A, jako při vzniku matice P z matice A.

Věta: Nechť P je permutační matice. Potom $P^{-1} = P^T$.

Důkaz: Všimněme si, že

$$P = \begin{pmatrix} s_1 & s_2 & \cdots & s_n \end{pmatrix}$$; s_i je nějaký sloupec jednotkové matice $\overset{(n,n)}{E}$, $s_i \neq s_j$ pro $i \neq j$.

$$\Rightarrow \quad P^T = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} \quad \Rightarrow$$

$$P^T \cdot P = \overset{(n,n)}{\begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}} \overset{(n,n)}{\begin{pmatrix} s_1 & s_2 & \cdots & s_n \end{pmatrix}} = \begin{pmatrix} \overset{=1}{s_1 s_1} & \overset{=0}{s_1 s_2} & \cdots & \overset{=0}{s_1 s_n} \\ \underset{=0}{s_2 s_1} & \underset{=1}{s_2 s_2} & \cdots & \underset{=0}{s_2 s_n} \\ \vdots & & & \\ \underset{=0}{s_n s_1} & \underset{=0}{s_n s_2} & \cdots & \underset{=1}{s_n s_n} \end{pmatrix} = E$$

$s_i s_i = 1$; $s_i s_j = 0$ pro $i \neq j$

jednička na stejné pozici

jedničky na jiných místech

Věta: Nechť A je regulární čtvercová matice. Potom existuje permutační matice P, dolní trojúhelníková matice L (pod diagonálou nuly) a horní trojúhelníková matice U (nad diag. nuly) takové, že:

$$A\cdot P = L\cdot U$$

Př: Nalezněte LU rozklad matice $A=\begin{pmatrix}2&0&1\\4&1&1\\-2&3&2\end{pmatrix}$.

Plán: 1.) $(A\,|\,E)\sim\dots\sim(\underbrace{U}_{\text{nuly pod diag.}}\,|\,\underbrace{\tilde{L}}_{\text{nuly nad diag.}})$

Budeme přičítat násobky řádků v A jen směrem dolů! Nesmíme vyměňovat řádky!

$\Rightarrow \tilde{L}\cdot A = U \quad /\tilde{L}^{-1}\cdot$

2.) $(\tilde{L}\,|\,E)\sim\dots\sim(E\,|\,L)$

$\tilde{L}$ má nuly nad diag. $\Rightarrow \tilde{L}^{-1}=L$ také!

$$\Rightarrow 1.)\ \left(\begin{array}{ccc|ccc}2&0&1&1&0&0\\4&1&1&0&1&0\\-2&3&2&0&0&1\end{array}\right)\begin{array}{l}\\-2r_1\\+r_1\end{array}\sim\left(\begin{array}{ccc|ccc}2&0&1&1&0&0\\0&1&-1&-2&1&0\\0&3&3&1&0&1\end{array}\right)\begin{array}{l}\\\\-3r_2\end{array}\sim\left(\begin{array}{ccc|ccc}2&0&1&1&0&0\\0&1&-1&-2&1&0\\0&0&6&7&-3&1\end{array}\right)$$

$$\qquad\quad A \qquad\quad E \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underbrace{}_{U}\ \ \underbrace{}_{\tilde{L}}$$

$$2.)\ \left(\begin{array}{ccc|ccc}1&0&0&1&0&0\\-2&1&0&0&1&0\\7&-3&1&0&0&1\end{array}\right)\begin{array}{l}\\+2r_1\\-7r_1\end{array}\sim\left(\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&2&1&0\\0&-3&1&-7&0&1\end{array}\right)\begin{array}{l}\\\\+3r_2\end{array}\sim\left(\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&2&1&0\\0&0&1&-1&3&1\end{array}\right)$$

$$\qquad\quad \tilde{L} \qquad\quad E \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad L=\tilde{L}^{-1}$$

$$\Rightarrow \underline{\underline{A = L\cdot U = \begin{pmatrix}1&0&0\\2&1&0\\-1&3&1\end{pmatrix}\begin{pmatrix}2&0&1\\0&1&-1\\0&0&6\end{pmatrix}}}$$

Zk: $\underset{L}{\begin{pmatrix}1&0&0\\2&1&0\\-1&3&1\end{pmatrix}}\underset{\cdot\ \ U}{\begin{pmatrix}2&0&1\\0&1&-1\\0&0&6\end{pmatrix}}\underset{=}{=}\underset{A}{\begin{pmatrix}2&0&1\\4&1&1\\-2&3&2\end{pmatrix}}$ ✓

Poznámka: V kroku 1.) se někdy neobejdeme bez výměny sloupců. Například pro $A=\begin{pmatrix}0&2&-1\\\enclose{circle}{1}&-1&1\\\enclose{circle}{1}&-3&4\end{pmatrix}$ – chceme upravit na schodový tvar, ale nesmíme vyměňovat řádky! Proto vyměníme sloupce pomocí P a rozkládáme matici AP.

Př. Nalezněte LU rozklad matice $A = \begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 1 & -3 & 4 \end{pmatrix}$.

1.)
$$\left(\underbrace{\overset{\longleftrightarrow}{\begin{matrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 1 & -3 & 4 \end{matrix}}}_{A} \middle| \underbrace{\overset{\text{zde ne!}}{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}}_{E}\right) \sim \left(\underbrace{\begin{matrix} -1 & 2 & 0 \\ 1 & -1 & 1 \\ 4 & -3 & 1 \end{matrix}}_{A\cdot P} \middle| \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{E}\right) \begin{matrix} \\ +r_1 \\ +4r_1 \end{matrix} \sim \left(\underbrace{\begin{matrix} -1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 5 & 1 \end{matrix}}_{T_1AP} \middle| \underbrace{\begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 0 & 1 \end{matrix}}_{T_1}\right) \begin{matrix} \\ \\ -5r_2 \end{matrix} \sim \left(\underbrace{\begin{matrix} -1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{matrix}}_{\underbrace{\underbrace{T_2T_1}_{\tilde{L}}AP}_{= U}} \middle| \underbrace{\begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -5 & 1 \end{matrix}}_{\underbrace{T_2T_1}_{\tilde{L}}}\right)$$

chceme upravit na schodový tvar, nesmíme vyměňovat řádky! ⇒ vyměníme sloupce!

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

2.)
$$\left(\underbrace{\begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -5 & 1 \end{matrix}}_{\tilde{L}} \middle| \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{E}\right) \begin{matrix} \\ -r_1 \\ +r_1 \end{matrix} \sim \left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{matrix} \middle| \begin{matrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{matrix}\right) \begin{matrix} \\ \\ +5r_2 \end{matrix} \sim \left(\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{E} \middle| \underbrace{\begin{matrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -4 & 5 & 1 \end{matrix}}_{\tilde{L}^{-1}}\right) = L$$

$$\Rightarrow \quad \underbrace{\begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 1 & -3 & 4 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{P} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -4 & 5 & 1 \end{pmatrix}}_{L} \cdot \underbrace{\begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{pmatrix}}_{U}$$

Zk: $\begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 1 & -3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 0 \\ 1 & -1 & 1 \\ 4 & -3 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -4 & 5 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 0 \\ 1 & & \\ & & \end{pmatrix}$

Př. Nalezněte LU rozklad matice $A = \begin{pmatrix} 0 & 1 & 5 \\ 2 & 3 & 2 \\ 4 & -1 & 1 \end{pmatrix}$

1.)
$$\left(\underbrace{\overset{\leftrightarrow}{\begin{matrix} 0 & 1 & 5 \\ 2 & 3 & 2 \\ 4 & -1 & 1 \end{matrix}}}_{A} \middle| \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{E}\right) \sim \left(\underbrace{\begin{matrix} 1 & 0 & 5 \\ 3 & 2 & 2 \\ -1 & 4 & 1 \end{matrix}}_{AP} \middle| \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{E}\right) \begin{matrix} \\ -3r_1 \\ +r_1 \end{matrix} \sim \left(\begin{matrix} 1 & 0 & 5 \\ 0 & 2 & -13 \\ 0 & 4 & 6 \end{matrix} \middle| \begin{matrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & 0 & 1 \end{matrix}\right) \begin{matrix} \\ \\ -2r_2 \end{matrix} \sim \left(\underbrace{\begin{matrix} 1 & 0 & 5 \\ 0 & 2 & -13 \\ 0 & 0 & 32 \end{matrix}}_{U} \middle| \underbrace{\begin{matrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 7 & -2 & 1 \end{matrix}}_{\tilde{L} = L^{-1}}\right)$$

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.)
$$\left(\underbrace{\begin{matrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 7 & -2 & 1 \end{matrix}}_{\tilde{L}} \middle| \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{E}\right) \begin{matrix} \\ +3r_1 \\ -7r_1 \end{matrix} \sim \left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{matrix} \middle| \begin{matrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -7 & 0 & 1 \end{matrix}\right) \begin{matrix} \\ \\ +2r_2 \end{matrix} \sim \left(\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{E} \middle| \underbrace{\begin{matrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 2 & 1 \end{matrix}}_{L}\right)$$

$$\Rightarrow \quad \underbrace{\overset{A}{\begin{pmatrix} 0 & 1 & 5 \\ 2 & 3 & 2 \\ 4 & -1 & 1 \end{pmatrix}} \overset{P}{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}}_{} = \overset{L}{\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix}} \overset{U}{\begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & -13 \\ 0 & 0 & 32 \end{pmatrix}}$$

Zk: $\begin{pmatrix} 1 & 0 & 5 \\ 3 & 2 & 2 \\ -1 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 5 \\ 3 & 2 & 2 \\ -1 & 4 & 1 \end{pmatrix}$ V

Poznámka: Někdy není od začátku zřejmé, jakou permutační matici P bude třeba použít! Nevadí! Viz příklad:

Př. Nalezněte LU rozklad matice $A=\begin{pmatrix}0&1&1\\1&1&1\\2&3&2\end{pmatrix}$

1.)
$$\overset{\leftrightarrow}{\left(\begin{array}{ccc|ccc}0&1&1&1&0&0\\1&1&1&0&1&0\\2&3&2&0&0&1\end{array}\right)}_{A\quad E}\sim\left(\begin{array}{ccc|ccc}1&1&0&1&0&0\\1&1&1&0&1&0\\2&3&2&0&0&1\end{array}\right)_{AP_1\quad E}\begin{matrix}\\-r_1\\-2r_1\end{matrix}\sim\overset{\leftrightarrow}{\left(\begin{array}{ccc|ccc}1&1&0&1&0&0\\0&0&1&-1&1&0\\0&1&2&-2&0&1\end{array}\right)}_{T_1AP_1\quad T_1}\sim\left(\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&0&-1&1&0\\0&2&1&-2&0&1\end{array}\right)_{T_1AP_1P_2\quad T_1}\begin{matrix}\\ \\-2r_2\end{matrix}\sim\left(\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&0&-1&1&0\\0&0&1&0&-2&1\end{array}\right)$$

$$\underbrace{\underbrace{T_2T_1}_{\tilde{L}}A\underbrace{P_1P_2}_{P}}_{U}\quad\underbrace{T_2T_1}_{\tilde{L}}$$

$P_1=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$ $\quad T_1=\begin{pmatrix}1&0&0\\-1&1&0\\-2&0&1\end{pmatrix}$ $\quad P_2=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$ $\quad T_2=\begin{pmatrix}1&0&0\\0&1&0\\0&-2&1\end{pmatrix}$

$\Rightarrow P=P_1\cdot P_2=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$ nebo : $\underset{E}{\overset{\leftrightarrow}{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}}}\sim\underset{P_1}{\overset{\leftrightarrow}{\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}}}\sim\underset{P_1P_2=P}{\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}}$

2.)
$$\left(\begin{array}{ccc|ccc}1&0&0&1&0&0\\-1&1&0&0&1&0\\0&-2&1&0&0&1\end{array}\right)_{\tilde{L}\quad E}\begin{matrix}\\+r_1\\ \end{matrix}\sim\left(\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&1&1&0\\0&-2&1&0&0&1\end{array}\right)\begin{matrix}\\ \\+2r_2\end{matrix}\sim\left(\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&1&1&0\\0&0&1&2&2&1\end{array}\right)_{E\quad L}$$

$$\Rightarrow\begin{pmatrix}0&1&1\\1&1&1\\2&3&2\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}=\begin{pmatrix}1&0&0\\1&1&0\\2&2&1\end{pmatrix}\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}$$

Zk: $\begin{pmatrix}1&0&1\\1&1&1\\2&2&3\end{pmatrix}=\begin{pmatrix}1&0&1\\1&1&1\\2&2&3\end{pmatrix}$ ✓

Poznámka: Díky asociativitě násobení matic je jedno v jakém pořadí upravujeme matici A - zda nejprve provedeme řádkové úpravy, nebo výměny sloupců:

$T_2(T_1(AP_1)P_2)=U$ (tak jsme obdrželi U, ale také platí:)

$U=(T_2T_1)A(P_1P_2)=\tilde{L}\cdot A\cdot P$, kde $\tilde{L}=T_2T_1$, $P=P_1P_2$

$\Rightarrow$ Nemusíme dopředu vědět, které sloupce bude třeba v A vyměnit. Zjistíme to během úprav při hledání matice $\tilde{L}$.

Př: Nalezněte LU rozklad matice $A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & -2 & 0 & 1 \\ 0 & 1 & -1 & -2 \\ 2 & 3 & 0 & 1 \end{pmatrix}$

1.) $$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ \enclose{circle}{-1} & -2 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 0 & 0 & 1 & 0 \\ \enclose{circle}{2} & 3 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)\begin{matrix} \\ +r_1 \\ \\ -2r_1 \end{matrix} \sim \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & \enclose{circle}{1} & -1 & -2 & 0 & 0 & 1 & 0 \\ 0 & \enclose{circle}{1} & -2 & 1 & -2 & 0 & 0 & 1 \end{array}\right)\begin{matrix} \\ \\ +r_2 \\ +r_2 \end{matrix} \sim \left(\begin{array}{cccc|cccc} 1 & 1 & \overset{\curvearrowright}{1} & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & 1 & 0 \\ 0 & 0 & \enclose{circle}{-1} & 2 & -1 & 1 & 0 & 1 \end{array}\right) \sim$$

$$\Rightarrow P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & \enclose{circle}{2} & -1 & -1 & 1 & 0 & 1 \end{array}\right)\begin{matrix} \\ \\ \\ +2r_3 \end{matrix} \sim \left(\underbrace{\begin{matrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{matrix}}_{U}\middle|\underbrace{\begin{matrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 3 & 2 & 1 \end{matrix}}_{\tilde{L}}\right)$$

2.) $$\left(\underbrace{\begin{matrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 3 & 2 & 1 \end{matrix}}_{\tilde{L}}\middle|\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}\right)\begin{matrix} \\ -r_1 \\ -r_1 \\ -r_1 \end{matrix} \sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 3 & 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right)\begin{matrix} \\ \\ -r_2 \\ -3r_2 \end{matrix} \sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & 2 & -3 & 0 & 1 \end{array}\right)\begin{matrix} \\ \\ \\ -2r_3 \end{matrix} \sim$$

$$\sim \left(\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}\middle|\underbrace{\begin{matrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & -1 & -2 & 1 \end{matrix}}_{L}\right)$$

$$\underline{\underline{\begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & -2 & 0 & 1 \\ 0 & 1 & -1 & -2 \\ 2 & 3 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & -1 & -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}}$$

$$\underbrace{A \qquad\qquad P}_{} \qquad\qquad\qquad \underbrace{L \qquad\qquad U}_{}$$

Zk: $$\begin{pmatrix} 1 & 1 & 0 & 1 \\ -1 & -2 & 1 & 0 \\ 0 & 1 & -2 & -1 \\ 2 & 3 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ -1 & -2 & 1 & 0 \\ 0 & 1 & -2 & -1 \\ 2 & 3 & 1 & 0 \end{pmatrix}$$

Řešení soustav lin. rovnic LU rozkladem

Je třeba vyřešit soustavu $\underline{A \cdot \bar{x} = \bar{b}}$

$$A \cdot P \cdot P^{-1} \cdot \bar{x} = \bar{b} \qquad / \; A \cdot P = L \cdot U$$

$$L U P^{-1} \cdot \bar{x} = \bar{b} \qquad / \; P^{-1} \cdot \bar{x} = \bar{y}$$

$$L U \bar{y} = \bar{b} \qquad / \; U \bar{y} = \bar{z}$$

$$L \bar{z} = \bar{b}$$

=> Postup :

1) Nalezneme LU rozklad matice A : určíme $\tilde{L}, U, P$

2) Vyřešíme soustavu $L\bar{z} = \bar{b}$: např. pomocí $\hat{L} = \tilde{L}^{-1}$ =>

$$\underline{\bar{z} = \tilde{L} \cdot \bar{b}}$$

3.) Vyřešíme soustavu :

$$\underline{U \bar{y} = \bar{z}} \qquad (\Rightarrow \text{určíme } \bar{y})$$

4.) Určíme $\bar{x}$:

$$\underline{\bar{x} = P \cdot \bar{y}}$$

Př: Vyřešte soustavu rovnic

$$\left.\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ -2x_1 + 3x_2 - x_3 &= -3 \\ 3x_1 + 6x_2 - 2x_3 &= 1 \end{aligned}\right\} =$$

1.) LU rozklad : $A = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 3 & -1 \\ 3 & 6 & -2 \end{pmatrix} \Rightarrow$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -2 & 3 & -1 & 0 & 1 & 0 \\ 3 & 6 & -2 & 0 & 0 & 1 \end{array}\right) \overset{+2r_1}{\underset{-3r_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 2 & 1 & 0 \\ \underbrace{0 \quad 0 \quad -5}_{U} & & & \underbrace{-3 \quad 0 \quad 1}_{\hat{L}} & & \end{array}\right) \qquad P = E$$

2.) $$\bar{z} = \hat{L} \cdot \bar{b} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}$$

3.) Vyřešíme $U\bar{y} = \bar{z}$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 7 & 1 & 1 \\ 0 & 0 & -5 & -5 \end{array}\right) \Leftrightarrow \left.\begin{aligned} & y_1 + 2 \cdot 0 + 1 \cdot 1 = 2 \Rightarrow \underline{y_1 = 1} \\ & 7y_2 + 1 = 1 \Rightarrow \underline{y_2 = 0} \\ & -5y_3 = -5 \Rightarrow \underline{y_3 = 1} \end{aligned}\right\} \Rightarrow \bar{y} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

4.) Určíme $\bar{x}$:

$$\bar{x} = P \cdot \bar{y} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}}$$

Př. Vyřešte soustavu rovnic LU rozkladem

$$\left.\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ 3x_1 - 2x_2 + 8x_3 &= 13 \\ x_1 + x_2 + 7x_3 &= 6 \end{aligned}\right\} \Rightarrow$$

1.) $$\left(\underbrace{\begin{matrix} 1 & -1 & 2 \\ 3 & -2 & 8 \\ 1 & 1 & 7 \end{matrix}}_{A\cdot P}\middle|\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{E}\right)\begin{matrix} \\ -3r_1 \\ \sim \\ -r_1 \end{matrix} \sim \left(\begin{matrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{matrix}\middle|\begin{matrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{matrix}\right) \sim \left(\underbrace{\begin{matrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{matrix}}_{U}\middle|\underbrace{\begin{matrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 5 & -2 & 1 \end{matrix}}_{\tilde{L}}\right)$$

$P = E$

2.) $$\bar{r} = \tilde{L}\cdot\bar{b} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 5 & -2 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 13 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$$

3.) $$\left(\underbrace{\begin{matrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{matrix}}_{U}\middle|\underbrace{\begin{matrix} 4 \\ 1 \\ 0 \end{matrix}}_{\bar{r}}\right) \Rightarrow \begin{aligned} y_1 &= 5 \\ y_2 &= 1 \\ y_3 &= 0 \end{aligned}$$

4.) $$\bar{x} = P\cdot\bar{y} = E\bar{y} = \bar{y} = \underline{\underline{\begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}}}$$

Př Vyřešte soustavu rovnic LU rozkladem

$$\begin{aligned} x_2 - 2x_3 &= -4 \\ x_1 - x_2 + 2x_3 &= 5 \\ -3x_1 + 2x_2 - x_3 &= -2 \end{aligned}$$

1.) $$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ (vyměníme 1. a 2. sloupec)

$$\left(\underbrace{\begin{matrix} 0 & 1 & -2 \\ 1 & -1 & 2 \\ -3 & 2 & -1 \end{matrix}}_{A}\middle|\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{E}\right) \sim \left(\underbrace{\begin{matrix} 1 & 0 & -2 \\ -1 & 1 & 2 \\ 2 & -3 & -1 \end{matrix}}_{A\cdot P}\middle|\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{E}\right) \sim \left(\begin{matrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & -3 & 3 \end{matrix}\middle|\begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{matrix}\right) \sim \left(\underbrace{\begin{matrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{matrix}}_{U}\middle|\underbrace{\begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{matrix}}_{\hat{L}}\right)$$

2.) $$\bar{r} = \hat{L}\cdot\bar{b} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{pmatrix}\begin{pmatrix} -4 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ 9 \end{pmatrix}$$

3.) $$\left(\begin{matrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{matrix}\middle|\begin{matrix} -4 \\ 1 \\ 9 \end{matrix}\right) \Rightarrow \begin{aligned} &y_1 - 6 = -4 \Rightarrow y_1 = 2 \\ &y_2 = 1 \\ &y_3 = 3 \end{aligned} \qquad \Rightarrow \bar{y} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

4.) $$\bar{x} = P\bar{y} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}}$$

Př: Pomocí LU rozkladu vyřešte soustavu lin. rovnic:

$$\begin{aligned} 2x_2 + x_3 &= 7 \\ x_1 + 2x_2 + x_3 &= 8 \\ x_1 + x_3 &= 4 \end{aligned}$$

1.) $$\overset{P_1}{\left(\begin{array}{ccc|ccc}0&2&1&1&0&0\\1&2&1&0&1&0\\1&1&0&0&0&1\end{array}\right)} \sim \left(\begin{array}{ccc|ccc}1&2&0&1&0&0\\1&2&1&0&1&0\\0&1&1&0&0&1\end{array}\right)\begin{array}{c}\\-r_1\\ \\ \end{array} \sim \overset{P_2}{\left(\begin{array}{ccc|ccc}1&2&0&1&0&0\\0&0&1&-1&1&0\\0&1&1&0&0&1\end{array}\right)} \sim \left(\begin{array}{ccc|ccc}1&0&2&1&0&0\\0&1&0&-1&1&0\\0&1&1&0&0&1\end{array}\right)\begin{array}{c}\\ \\-r_2\end{array} \sim \left(\underbrace{\begin{array}{ccc}1&0&2\\0&1&0\\0&0&1\end{array}}_{U}\middle|\underbrace{\begin{array}{ccc}1&0&0\\-1&1&0\\1&-1&1\end{array}}_{\tilde{L}}\right)$$

$$\underset{E}{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}} \sim \underset{P_1}{\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}} \sim \underset{P_1P_2}{\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}} = P$$

2.) $\bar{z} = \tilde{L}\cdot\bar{b}$ $= \begin{pmatrix}1&0&0\\-1&1&0\\1&-1&1\end{pmatrix}\begin{pmatrix}7\\8\\4\end{pmatrix} = \begin{pmatrix}7\\1\\3\end{pmatrix}$

3.) $U.\bar{y} = \bar{z}$: $\left(\begin{array}{ccc|c}1&0&2&7\\0&1&0&1\\0&0&1&3\end{array}\right)$ $\begin{array}{l} \Rightarrow y_1 + 2\cdot 3 = 7 \Rightarrow y_1 = 1 \\ \Rightarrow y_2 = 1 \\ \Rightarrow y_3 = 3 \end{array}$ $\Rightarrow \bar{y} = \begin{pmatrix}1\\1\\3\end{pmatrix}$

4.) $\bar{x} = P\bar{y}$

$$\bar{x} = \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}1\\1\\3\end{pmatrix} = \underline{\underline{\begin{pmatrix}1\\3\\1\end{pmatrix}}}$$

Zk:
$0 + 2\cdot 3 + 1 = 7$ ✓
$1 + 2\cdot 3 + 1 = 8$ ✓
$1 + 3 = 4$ ✓

Nalezení matice inverzní LU rozkladem

Nechť $AP = LU$. Potom matici A^{-1} určíme následujícím způsobem:

$$AP = LU \quad / \cdot P^{-1}$$

$$A = LUP^{-1} \quad \Rightarrow \quad A^{-1} = PU^{-1}L^{-1}$$

$$A^{-1} = P \cdot U^{-1} \cdot \tilde{L}$$

$\Rightarrow$ Postup: Určíme matice $P, U, \tilde{L}$, poté U^{-1} a dosadíme do $A^{-1} = PU^{-1}\tilde{L}$

Př: Nalezněte s využitím LU rozkladu matici inverzní k A.

1.) $A = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 2 & 7 \\ -7 & -9 & -35 \end{pmatrix}$

$$\alpha) \left(\begin{array}{ccc|ccc} 1 & 1 & 4 & 1 & 0 & 0 \\ 1 & 2 & 7 & 0 & 1 & 0 \\ -7 & -9 & -35 & 0 & 0 & 1 \end{array}\right)\begin{matrix} \\ -r_1 \\ +7r_1 \end{matrix} \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & -2 & -7 & 7 & 0 & 1 \end{array}\right)\begin{matrix} \\ \\ +2r_2 \end{matrix} \sim \left(\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 1 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{matrix}}_{U} & \underbrace{\begin{matrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 2 & 1 \end{matrix}}_{\tilde{L}} \end{array}\right), \quad P = E$$

neboť jsme nevyměnovali sloupce

$$\beta) \left(\begin{array}{ccc|ccc} 1 & 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right)\begin{matrix} -r_2 \\ \\ \cdot(-1) \end{matrix} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right)\begin{matrix} -r_3 \\ -3r_3 \\ \\ \end{matrix} \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 1 & -1 & 1 \\ 0 & 1 & +3 \\ 0 & 0 & -1 \end{matrix}}_{U^{-1}} \\ 0 & 0 & 1 & \end{array}\right)$$

$$\underline{\underline{A^{-1} = PU^{-1}\tilde{L} = E \cdot \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 1 & 1 \\ 14 & 7 & 3 \\ -5 & -2 & -1 \end{pmatrix}}}$$

2.) $A = \begin{pmatrix} 1 & 2 & -1 \\ 4 & 10 & -3 \\ 1 & 3 & -1 \end{pmatrix}$

$$\alpha) \left(\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 2 & -1 \\ 4 & 10 & -3 \\ 1 & 3 & -1 \end{matrix}}_{A} & \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{E} \end{array}\right)\begin{matrix} \\ -4r_1 \\ -r_1 \end{matrix} \sim \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & -4 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right)\begin{matrix} \\ \\ \cdot 2 \end{matrix} \sim \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & -4 & 1 & 0 \\ 0 & 2 & 0 & -2 & 0 & 2 \end{array}\right)\begin{matrix} \\ \\ -r_2 \end{matrix} \sim \left(\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 2 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{matrix}}_{U} & \underbrace{\begin{matrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & -1 & 2 \end{matrix}}_{\tilde{L}} \end{array}\right) \quad P = E$$

nesmíme vyměňovat řádky!

$$\beta) \left(\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 2 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{matrix}}_{U} & \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{E} \end{array}\right)\begin{matrix} -r_2 \\ \\ \cdot(-1) \end{matrix} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right)\begin{matrix} +2r_3 \\ -r_3 \\ \\ \end{matrix} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -2 \\ 0 & 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right) \Rightarrow U^{-1} = \begin{pmatrix} 1 & -1 & -2 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -1 \end{pmatrix}$$

$$\underline{\underline{A^{-1} = PU^{-1}\tilde{L} = E \begin{pmatrix} 1 & -1 & -2 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -4 \\ -1 & 0 & 1 \\ -2 & 1 & -2 \end{pmatrix}}}$$

Př: S využitím LU rozkladu nalezněte matici inverzní k A.

①

$$A = \begin{pmatrix} 0 & 1 & 5 \\ 1 & 3 & -1 \\ 2 & 1 & -28 \end{pmatrix}$$

1.)

$$\underset{A\quad E}{\left(\begin{array}{ccc|ccc} 0 & 1 & 5 & 1 & 0 & 0 \\ 1 & 3 & -1 & 0 & 1 & 0 \\ 2 & 1 & -28 & 0 & 0 & 1 \end{array}\right)} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 3 & 1 & -1 & 0 & 1 & 0 \\ 1 & 2 & -28 & 0 & 0 & 1 \end{array}\right)\begin{array}{l} \\ -3r_1 \\ -r_1 \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 1 & -16 & -3 & 1 & 0 \\ 0 & 2 & -33 & -1 & 0 & 1 \end{array}\right)\begin{array}{l} \\ \\ -2r_2 \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 1 & -16 & -3 & 1 & 0 \\ 0 & 0 & -1 & 5 & -2 & 1 \end{array}\right)$$

$$\underbrace{}_{U} \quad \underbrace{}_{\tilde{L}}$$

$$P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = P$$

2.)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 1 & -16 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right)\begin{array}{l} +5r_3 \\ -16r_3 \\ \\ \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 5 \\ 0 & 1 & 0 & 0 & 1 & -16 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right)$$

$$\underbrace{}_{U} \qquad \underbrace{}_{U^{-1}}$$

$$A = P U^{-1} L^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -16 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 5 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 26 & -10 & 5 \\ -83 & 33 & -16 \\ -5 & 2 & -1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -83 & 33 & -16 \\ 26 & -10 & 5 \\ -5 & 2 & -1 \end{pmatrix}}}$$

$$\text{ZK:} \begin{pmatrix} 0 & 1 & 5 \\ 1 & 3 & -1 \\ 2 & 1 & -28 \end{pmatrix}\begin{pmatrix} -83 & 33 & -16 \\ 26 & -10 & 5 \\ -5 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

②

$$A = \begin{pmatrix} 0 & 3 & 1 \\ 1 & 6 & 2 \\ 3 & 2 & 1 \end{pmatrix}$$

1.)

$$\underset{A\quad E}{\left(\begin{array}{ccc|ccc} 0 & 3 & 1 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right)} \overset{P_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 2 & 6 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right)\begin{array}{l} \\ -2r_1 \\ -r_1 \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & -1 & 3 & -1 & 0 & 1 \end{array}\right) \overset{P_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 3 & -1 & -1 & 0 & 1 \end{array}\right)\begin{array}{l} \\ \\ -3r_2 \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 5 & -3 & 1 \end{array}\right)$$

$$U \qquad \tilde{L}$$

$$\underset{E}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \sim \underset{P_1}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}} \sim \underset{P_1 P_2}{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}} = P$$

2.)

$$\underset{U\qquad E}{\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right)}\begin{array}{l} +3r_3 \\ \\ \cdot(-1) \end{array} \sim \underset{E\qquad U^{-1}}{\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right)}$$

$$A = P U^{-1} L^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 16 & -9 & 3 \\ -2 & 1 & 0 \\ -5 & 3 & -1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -2 & 1 & 0 \\ -5 & 3 & -1 \\ 16 & -9 & 3 \end{pmatrix}}}$$

LU rozklad, řešení soustav rovnic LU rozkladem

1. Je dána matice

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

   Rozložte tuto matici na součin dolní a horní trojúhelníkové matice (LU rozklad).

2. Je dána matice

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -3 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

   a) rozložte tuto matici na součin dolní a horní trojúhelníkové matice (LU rozklad)
   b) využijte LU rozkladu k řešení soustavy:

$$\begin{aligned} -x_1 + 2x_2 &= 5 \\ 2x_1 - 3x_2 + 2x_3 &= 0 \\ 2x_2 + 5x_3 &= 0 \end{aligned}$$

3. Je dána matice

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 2 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix}$$

   a) rozložte tuto matici na součin dolní a horní trojúhelníkové matice (LU rozklad)
   b) využijte LU rozkladu k řešení soustavy:

$$\begin{aligned} x_1 - x_2 - 2x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= 1 \\ -x_1 + 2x_3 &= -1 \end{aligned}$$

4. Je dána matice

$$A = \begin{bmatrix} -1 & 3 & 0 \\ 3 & -8 & 3 \\ 0 & 3 & 1 \end{bmatrix}$$

   a) rozložte tuto matici na součin dolní a horní trojúhelníkové matice (LU rozklad)
   b) využijte LU rozkladu k řešení soustavy:

$$\begin{aligned} -x_1 + 3x_2 &= 2 \\ 3x_1 - 8x_2 + 3x_3 &= 4 \\ 3x_2 + x_3 &= 14 \end{aligned}$$

c) využijte LU rozkladu k řešení soustavy:

$$\begin{aligned} -x_1 + 3x_2 =& 1 \\ 3x_1 - 8x_2 + 3x_3 =& 2 \\ 3x_2 + x_3 =& -1 \end{aligned}$$

5. Pomocí LU rozkladu vyřešte soustavu rovnic

$$\begin{aligned} x_1 + 2x_2 + 2x_3 =& 9 \\ 2x_1 + x_2 - 2x_3 =& 0 \\ 2x_1 - 2x_2 + x_3 =& 0 \end{aligned}$$

6. Pomocí LU rozkladu nalezněte inverzní matici k matici A

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & -1 \\ 6 & 4 & 1 \end{bmatrix}.$$

Výsledky

1. $LU = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 0 & 4 & 5 \\ 0 & 0 & -3 \end{bmatrix}$

2. a) $LU = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$,

   b) $[x_1, x_2, x_3] = [95, 50, -20]$

3. a) $LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 3 & 5 \\ 0 & 0 & 5 \end{bmatrix}$,

   b) $[x_1, x_2, x_3] = [\frac{3}{5}, 0, -\frac{1}{5}]$

4. a) $LU = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -8 \end{bmatrix}$,

   b) $[x_1, x_2, x_3] = [10, 4, 2]$,

   c) $[x_1, x_2, x_3] = [-4, -1, 2]$

5. $LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -6 \\ 0 & 0 & 9 \end{bmatrix}$, $[x_1, x_2, x_3] = [1, 2, 2]$,

6. $A^{-1} = \begin{bmatrix} 5 & 3 & -2 \\ -7 & -4 & 3 \\ -2 & -2 & 1 \end{bmatrix}.$