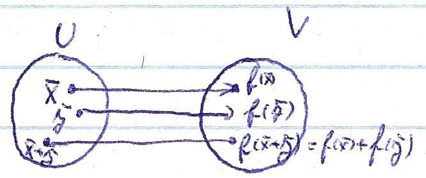
Lineární zobrazení

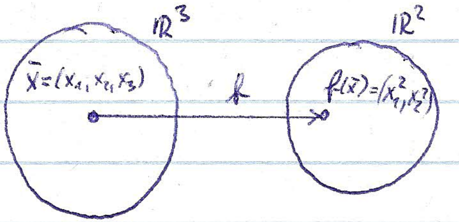
Def. (Lineární zobrazení): Zobrazení $f: U \to V$, kde $(U,+,\cdot)$ a $(V,+,\cdot)$ jsou vektorové prostory nazýváme lineárním zobrazením právě když platí:



① $\forall \bar{x},\bar{y} \in U: f(\bar{x}+\bar{y}) = f(\bar{x}) + f(\bar{y})$

② $\forall \alpha \in \mathbb{R}\ \forall \bar{x} \in U: f(\alpha\bar{x}) = \alpha \cdot f(\bar{x})$

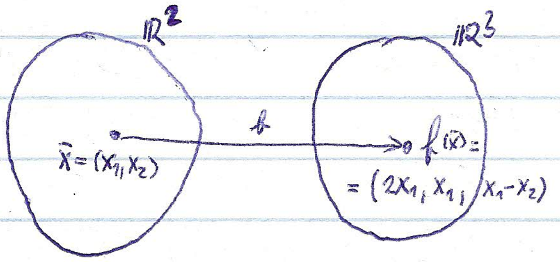
Př:



$$\Rightarrow \left.\begin{array}{l} f((2,2,2)) = (4,4) \\ f((3,1,0)) = (9,1) \end{array}\right\} (13,5)$$

$f((5,3,0)) = (25,9) \neq f((2,2,2)) + f((3,1,0)) \Rightarrow$ <u>není lineární</u>

Př:



1.) $\bar{x}=(x_1,x_2) \Rightarrow f(\bar{x}) = (2x_1, x_1, x_1-x_2)$

$\bar{y}=(y_1,y_2) \Rightarrow f(\bar{y}) = (2y_1, y_1, y_1-y_2)$

$\Downarrow$

$\bar{x}+\bar{y} = (x_1+y_1, x_2+y_2) \Rightarrow \underline{f(\bar{x}+\bar{y})} = (2(x_1+y_1), x_1+y_1, (x_1+y_1)-(x_2+y_2)) =$

$= (2x_1+2y_1, x_1+y_1, (x_1-x_2)+(y_1-y_2)) =$

$= \underline{f(\bar{x})+f(\bar{y})} \Rightarrow \boxed{f(\bar{x}+\bar{y}) = f(\bar{x})+f(\bar{y})}$

2.) $\alpha \in \mathbb{R}, \bar{x} \in \mathbb{R}^2\ (\bar{x}=(x_1,x_2)) \Rightarrow$

$f(\bar{x}) = (2x_1, x_1, x_1-x_2)$

$\alpha \cdot f(\bar{x}) = (2\alpha x_1, \alpha x_1, \alpha x_1 - \alpha x_2)$

$\alpha\bar{x} = (\alpha x_1, \alpha x_2) \Rightarrow f(\alpha\bar{x}) = f((\alpha x_1, \alpha x_2)) = (2\alpha x_1, \alpha x_1, \alpha x_1 - \alpha x_2) = \alpha f(\bar{x})$

$\Rightarrow \boxed{f(\alpha\bar{x}) = \alpha f(\bar{x})}$

$\Rightarrow$ <u>f je lineární zobrazení z $\mathbb{R}^2$ do $\mathbb{R}^3$ (značíme $f \in L(\mathbb{R}^2, \mathbb{R}^3)$.</u>

Pozn.: Lineární zobrazení $f: U \to V$ je jednoznačně určeno svými hodnotami na bázi vektorového prostoru U. Platí:

$$\bar{x} = k_1 \bar{e}_1 + k_2 \bar{e}_2 + \dots + k_n \bar{e}_n$$

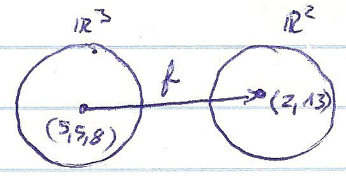
$$\Updownarrow$$

$$f(\bar{x}) = k_1 f(\bar{e}_1) + k_2 f(\bar{e}_2) + \dots + k_n f(\bar{e}_n)$$

Př.: Lineární zobrazení $f: R^3 \to R^2$ je dáno svými hodnotami:

$f((1,1,1)) = (1,2)$; $f((1,2,1)) = (1,-1)$; $f((1,1,2)) = (0,3)$

a) Určete $f((5,5,8)) = ?$



$$(5,5,8) = k_1(1,1,1) + k_2(1,2,1) + k_3(1,1,2) \Rightarrow$$

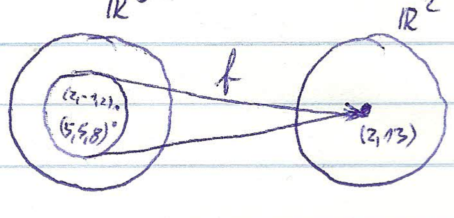
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 2 & 1 & 5 \\ 1 & 1 & 2 & 8 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right) \begin{array}{l} \to k_1 = 2 \\ \Rightarrow k_2 = 0 \\ \Rightarrow k_3 = 3 \end{array} \Rightarrow$$

$$f((5,5,8)) = 2 \cdot f((1,1,1)) + 0 \cdot f((1,2,1)) + 3 \cdot f((1,1,2)) = 2(1,2) + 0(1,-1) + 3(0,3) = \underline{\underline{(2,13)}}$$

b) Určete $\bar{x} \in \mathbb{R}^3$: $f(\bar{x}) = (2,13)$

$$f(\bar{x}) = (2,13) = k_1(1,2) + k_2(1,-1) + k_3(0,3) \Rightarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 2 & -1 & 3 & 13 \end{array}\right)_{-2r_1} \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -3 & 3 & 9 \end{array}\right)_{:3} \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & 3 \end{array}\right) \begin{array}{l} k_1 = 5-t \\ k_2 = -3+t \\ \Rightarrow k_3 = t \in \mathbb{R} \end{array}$$



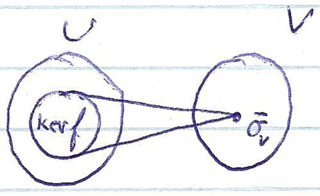
$\Rightarrow$ ~~$\bar{x} = ((5-t, -3+t, t))$, kde $t \in \mathbb{R}$~~

$$\bar{x} = (5-t)(1,1,1) + (-3+t)(1,2,1) + t(1,1,2)$$

$$\bar{x} = (5-t-3+t+t,\ 5-t-6+2t+t,\ 5-t-3+t+2t)$$

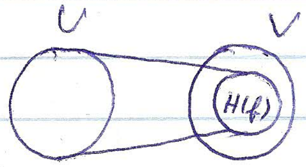
$$\underline{\underline{\bar{x} = (2+t, -1+2t; 2+2t), \text{ kde } t \in \mathbb{R}}}$$

Def: (Jádro lin. zobrazení) (Obor hodnot): Nechť $f: U \Rightarrow V$ je lineární zobrazení a $\bar{o}_v$ je nulový vektor vektorového prostoru $(V,+,\cdot)$. Jádrem lineárního zobrazení f nazýváme množinu



$$N(f) = \mathrm{Ker} f = \{ \bar{x} \in U \mid f(x) = \bar{o}_v \}$$

oborem hodnot zobrazení f nazýváme množinu



$$H(f) = \{ f(\bar{x}) \mid \bar{x} \in U \}$$

Př: Lineární zobrazení je dáno hodnotami $f((1,1,0)) = (1,2,-1)$, $f((0,1,1)) = (0,1,1)$; $f((1,0,1)) = (1,-1,-4)$. Určete jádro a obor hodnot f.

a) $\mathrm{Ker} f = \{ \bar{x} \in \mathbb{R}^3 \mid f(\bar{x}) = (0,0,0) \}$ $\Rightarrow$ ~~$f(\bar{x}) = (0,0,0) = k_1(0,1,1) + k_2(1,-1,$~~

$$f(\bar{x}) = (0,0,0) = k_1(1,2,-1) + k_2(0,1,1) + k_3(1,-1,-4)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 \\ -1 & 1 & -4 & 0 \end{array}\right)\begin{array}{l} \\ -2r_1 \\ +r_1 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right)\begin{array}{l} \\ \\ -r_2 \end{array} \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} k_1 = -t \\ k_2 = 3t \\ \Rightarrow k_3 = t \end{array}$$

$$\bar{x} = -t(1,1,0) + 3t(0,1,1) + t(1,0,1) = (0,2t,4t) = t(0,2,4) \Rightarrow$$

$$\underline{\mathrm{Ker} f = \{ t(0,2,4) \mid t \in \mathbb{R} \} = \langle (0,2,4) \rangle}$$

báze: $\underline{\{(0,2,4)\}}$

$\underline{\dim \mathrm{Ker} f = 1}$

b) $H(f) = \{ f(\bar{x}) \mid \bar{x} \in \mathbb{R}^3 \} = \{ k_1(1,2,-1) + k_2(0,1,1) + k_3(1,-1,-4) \mid k_1, k_2, k_3 \in \mathbb{R} \}$

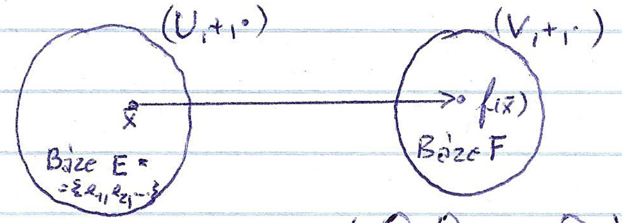
$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & -4 \end{pmatrix}\begin{array}{l} \\ \\ -r_1 \end{array} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -3 & -3 \end{pmatrix}\begin{array}{l} \\ \\ +3r_2 \end{array} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

báze: $\underline{\{(1,2,-1); (0,1,1)\}}$

$\underline{\dim H(f) = 2}$

$$H_f = \{ k_1(1,2,-1) + k_2(0,1,1) \mid k_1, k_2 \in \mathbb{R} \} = \underline{\langle (1,2,-1); (0,1,1) \rangle}$$

Pozn: Funkční hodnoty lineárního zobrazení $f: U \to V$ můžeme určit pomocí tzv. matice lineárního zobrazení :



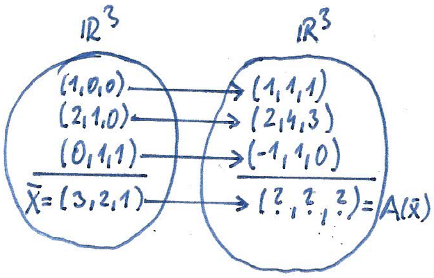
$$f^{T}(\bar{x})_{\langle F\rangle} = A_{EF} \cdot \bar{x}_{\langle E\rangle}$$

$$A_{EF} = \begin{pmatrix} f(\bar{e}_1)_{\langle F\rangle} & f(\bar{e}_2)_{\langle F\rangle} & \cdots & f(\bar{e}_n)_{\langle F\rangle} \end{pmatrix}$$

Př: Lineární zobrazení $A : \mathbb{R}^3 \to \mathbb{R}^3$ je dáno hodnotami :

$A((1,0,0)) = (1,1,1)$; $A((2,1,0)) = (2,4,3)$; $A((0,1,1)) = (-1,1,0)$

a) Určete $A((3,2,1))$.



$$\begin{array}{rcl} (3,2,1) & = & k_1(1,0,0)+k_2(2,1,0)+k_3(0,1,1) \quad (1) \\ \downarrow & & \quad\ \downarrow \qquad\quad\ \downarrow \qquad\quad\ \downarrow \\ A((3,2,1)) & = & k_1(1,1,1)+k_2(2,4,3)+k_3(-1,1,0) \quad (2) \end{array}$$

(1) ⇒ k_1, k_2, k_3 jsou řešením soustavy:

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right) \begin{array}{l} \Rightarrow k_1 = 1 \\ \Rightarrow k_2 = 1 \\ \Rightarrow k_3 = 1 \end{array} \Bigg\} \text{ dosadíme do } (2) \Rightarrow$$

$$\Rightarrow A((3,2,1)) = 1\cdot(1,1,1) + 1\cdot(2,4,3) + 1\cdot(-1,1,0) = \underline{\underline{(2,6,4)}}$$

b) Určete $\bar{x} \in \mathbb{R}^3$ takové, že $A(\bar{x}) = (2,6,4)$.

$$\begin{array}{rcl} ? = \bar{x} & = & k_1(1,0,0)+k_2(2,1,0)+k_3(0,1,1) \quad (1) \\ (2,6,4) = A(\bar{x}) & = & k_1(1,1,1)+k_2(2,4,3)+k_3(-1,1,0) \quad (2) \end{array}$$

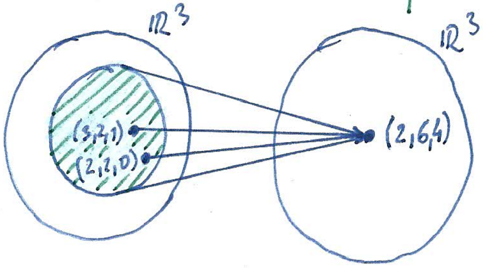
(2) ⇒ k_1, k_2, k_3 jsou řešením soustavy:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 1 & 4 & 1 & 6 \\ 1 & 3 & 0 & 4 \end{array}\right) \begin{array}{l} \\ -r_1 \\ -r_1 \end{array} \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & 1 & 2 \end{array}\right) \begin{array}{l} \\ :2 \\ \\ \end{array} \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{array}\right) \begin{array}{l} \\ \\ -r_2 \end{array} \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$$

$$\left.\begin{array}{lll} \Rightarrow k_2 + k_3 = 2 & \mid \text{zvolíme parametr} & \underline{k_3 = t \in \mathbb{R}} \\ \quad k_2 + t = 2 & \Rightarrow & \underline{k_2 = 2-t} \\ \Rightarrow k_1 + 2k_2 - k_3 = 2 & \mid \text{dosadíme } k_3 = t,\ k_2 = 2-t & \\ \quad k_1 + 4 - 2t - t = 2 & \Rightarrow & \underline{k_1 = -2+3t} \end{array}\right\} \text{dosadíme do } (1)$$

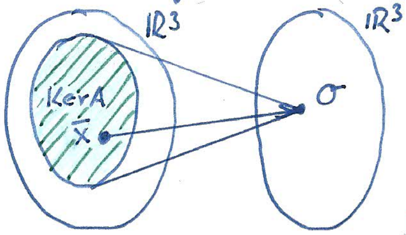
$$\bar{x} = (-2+3t)(1,0,0) + (2-t)(2,1,0) + t(0,1,1) = \underline{\underline{(2+t,\ 2,\ t)\ ;\ t \in \mathbb{R}}}$$

Všimněme si, že pro $t=1$ je $\bar{x}=(3,2,1)$, pro $t=0$ je $\bar{x}=(2,2,0)$, ...



Nekonečně mnoho vektorů se zobrazuje na (2,6,4), ale všechny mají tvar $(2+t, 2, t)$, $t \in \mathbb{R}$.

c) Určete jádro lineárního zobrazení A: $\mathrm{Ker}\,A = \{\bar{x} \in \mathbb{R}^3 \mid A(\bar{x}) = (0,0,0)\}$



Tzn. hledáme všechna $\bar{x}$, která se zobrazují na nulový vektor.

$$\boxed{\begin{aligned} ? = \bar{x} &= k_1(1,0,0) + k_2(2,1,0) + k_3(0,1,1) && (1) \\ (0,0,0) = A(\bar{x}) &= k_1(1,1,1) + k_2(2,4,3) + k_3(-1,1,0) && (2) \end{aligned}}$$

Ze (2) určíme k_1, k_2, k_3:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 1 & 4 & 1 & 0 \\ 1 & 3 & 0 & 0 \end{array}\right)\begin{matrix} \\ -r_1 \\ -r_1 \end{matrix} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \sim \dots \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{matrix} \Rightarrow k_1 - 2t - t = 0 \Rightarrow k_1 = 3t \\ \Rightarrow k_2 + k_3 = 0 \;/\; k_3 = t \Rightarrow k_2 = -t \\ \\ \end{matrix}$$

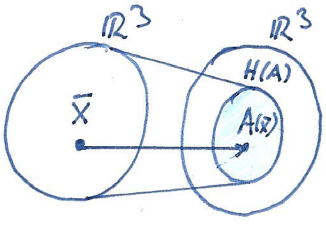
Dosadíme do (1):

$$\bar{x} = 3t(1,0,0) - t(2,1,0) + t(0,1,1) = (t, 0, t);\ t \in \mathbb{R} \quad \Rightarrow$$

$$\mathrm{Ker}\,A = \{t \cdot (1,0,1) \mid t \in \mathbb{R}\} = \langle (1,0,1) \rangle$$

Báze $\mathrm{Ker}\,A = \{(1,0,1)\}$ $\qquad$ $\dim \mathrm{Ker}\,A = d(A) = 1$ … defekt

d) Určete obor hodnot lin. zobr. A: $H(A) = \{A(\bar{x}) \mid \bar{x} \in \mathbb{R}^3\}$



Hledáme množinu všech funkčních hodnot zobrazení A. Z (2) plyne, že:

$$H(A) = \{k_1(1,1,1) + k_2(2,4,3) + k_3(-1,1,0) \mid k_1, k_2, k_3 \in \mathbb{R}\}$$

Vektory $(1,1,1); (2,4,3); (-1,1,0)$ by tvořily bázi H(A) kdyby byly nezávislé. Ověříme:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 3 \\ -1 & 1 & 0 \end{pmatrix}\begin{matrix} \\ -2r_1 \\ +r_1 \end{matrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix}\begin{matrix} \\ \\ -r_2 \end{matrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

jsou závislé! Bázi H(A) budou tvořit nenulové vektory, které zbyly po úpravě na schodový tvar.

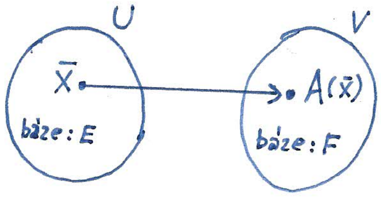
$\Rightarrow$ $H(A) = \langle (1,1,1); (0,2,1) \rangle$ $\qquad$ Báze $H(A) = \{(1,1,1); (0,2,1)\}$ $\qquad$ $\dim H(A) = 2$

Všimněme si: $A: U \to V \quad \Rightarrow \quad \dim U = d(A) + \dim H(A)$

d) Určete matici lin. zobrazení A vzhledem ke standardním bázím

$S_1 = S_2 = \{(1,0,0);(0,1,0);(0,0,1)\}$

Obecně: A_{EF}... matice lin. zobrazení A vzhledem k bázím E a F je matice splňující:



$$A(\bar{x})_F = A_{EF} \cdot \bar{x}_E$$

$$A_{EF} = \begin{pmatrix} A(\bar{e}_1)_F & A(\bar{e}_2)_F & \cdots & A(\bar{e}_m)_F \end{pmatrix}$$

$E = \{\bar{e}_1, \bar{e}_2, \dots, \bar{e}_m\} \Rightarrow$ Ve sloupcích A_{EF} jsou souřadnice vektorů $A(\bar{e}_i)$ vzhledem k bázi F.

$$\left(\begin{array}{ccc|ccc} \multicolumn{3}{c|}{\overbrace{\hphantom{1\ 0\ 0}}^{\bar{x}}} & \multicolumn{3}{c}{\overbrace{\hphantom{1\ 1\ 1}}^{A(\bar{x})}} \\ 1&0&0&1&1&1\\2&1&0&2&4&3\\0&1&1&-1&1&0\end{array}\right)\begin{matrix}\\ \\-2r_1\\ \\\end{matrix} \sim \left(\begin{array}{ccc|ccc}1&0&0&1&1&1\\0&1&0&0&2&1\\0&1&1&-1&1&0\end{array}\right)\begin{matrix}\\ \\-r_2\end{matrix} \sim$$

$$\begin{array}{c} \bar{e}_1 = \\ \bar{e}_2 = \\ \bar{e}_3 = \\ \\ \end{array} \left(\begin{array}{ccc|ccc}1&0&0&1&1&1\\0&1&0&0&2&1\\0&0&1&-1&-1&-1\\ &&& \multicolumn{3}{c}{\underbrace{\hphantom{-1\ -1\ -1}}_{A^T_{S_1S_2}}}\end{array}\right)\begin{array}{l}= A(\bar{e}_1)\\=A(\bar{e}_2)\\=A(\bar{e}_3)\\ \\ \end{array}$$

$$\Rightarrow \left.\begin{array}{l} A(\bar{e}_1)_{S_2} = (1,1,1)\\ A(\bar{e}_2)_{S_2}=(0,2,1)\\ A(\bar{e}_3)_{S_2}=(-1,-1,-1)\end{array}\right\} \Rightarrow A_{S_1S_2} = \begin{pmatrix}1&0&-1\\1&2&-1\\1&1&-1\end{pmatrix}$$

Pozn.: Např.

$$A(3,2,1)_{S_2} = \begin{pmatrix}1&0&-1\\1&2&-1\\1&1&-1\end{pmatrix}\overset{(3,2,1)_{S_1}}{\overset{\shortparallel}{\begin{pmatrix}3\\2\\1\end{pmatrix}}} = \begin{pmatrix}2\\6\\4\end{pmatrix} \Rightarrow$$

$A(3,2,1) = (2,6,4)$

e) Určete matici lin. zobrazení A vzhledem k bázím

$E = \{\underset{\bar{e}_1}{\underset{\shortparallel}{(1,0,0)}};\underset{\bar{e}_2}{\underset{\shortparallel}{(2,1,0)}};\underset{\bar{e}_3}{\underset{\shortparallel}{(0,1,1)}}\}$ a $F = S_2 = \{(1,0,0);(0,1,0);(0,0,1)\}$

$$\underbrace{A(\bar{e}_1)=(1,1,1)=A(\bar{e}_1)_{S_2}\ ;\ A(\bar{e}_2)=(2,4,3)=A(\bar{e}_2)_{S_2}\ ;\ A(\bar{e}_3)=(-1,1,0)=A(\bar{e}_3)_{S_2}}_{\text{víme ze zadání} \Rightarrow}$$

$$A_{EF} = \begin{pmatrix}1&2&-1\\1&4&1\\1&3&0\end{pmatrix}$$

Pozn.: Např.:

$$A((2,1,0))_{\langle S_2\rangle} = \begin{pmatrix}1&2&-1\\1&4&1\\1&3&0\end{pmatrix}\overset{(2,1,0)_E}{\overset{\shortparallel}{\begin{pmatrix}0\\1\\0\end{pmatrix}}} = \begin{pmatrix}2\\4\\3\end{pmatrix}$$

f) Určete A_{GH}, kde $G = \{(1,0,0);(3,1,0);(-1,0,1)\}$; $H = \{(1,1,1);(0,1,2);(0,0,1)\}$

$$\underbrace{\begin{pmatrix}1&0&-1\\1&2&-1\\1&1&-1\end{pmatrix}}_{A_{S_1S_2}}\underbrace{\begin{pmatrix}\underset{\bar{g}_1}{\begin{pmatrix}1\\0\\0\end{pmatrix}}&\underset{\bar{g}_2}{\begin{pmatrix}3\\1\\0\end{pmatrix}}&\underset{\bar{g}_3}{\begin{pmatrix}-1\\0\\1\end{pmatrix}}\end{pmatrix}}_{G} = \begin{pmatrix}\underset{A(\bar{g}_1)_{S_2}}{\begin{pmatrix}1\\1\\1\end{pmatrix}}&\underset{A(\bar{g}_2)_{S_2}}{\begin{pmatrix}3\\5\\4\end{pmatrix}}&\underset{A(\bar{g}_3)_{S_2}}{\begin{pmatrix}-2\\-2\\-2\end{pmatrix}}\end{pmatrix}$$

$$\Rightarrow A_{EF} = H^{-1}\cdot A_{S_1S_2}\cdot G = \begin{pmatrix}1&0&0\\1&1&0\\1&2&1\end{pmatrix}^{-1}\cdot\overbrace{\begin{pmatrix}1&0&-1\\1&2&-1\\1&1&-1\end{pmatrix}\cdot\begin{pmatrix}1&3&-1\\0&1&0\\0&0&1\end{pmatrix}}^{\shortparallel} =$$

$$= \begin{pmatrix}1&0&0\\1&1&0\\1&2&1\end{pmatrix}^{-1}\cdot\begin{pmatrix}1&3&-2\\1&5&-2\\1&4&-2\end{pmatrix} =$$

$$= \begin{pmatrix}1&0&0\\-1&1&0\\1&-2&1\end{pmatrix}\cdot\begin{pmatrix}1&3&-2\\1&5&-2\\1&4&-2\end{pmatrix} = \begin{pmatrix}1&3&-2\\0&2&0\\0&-3&0\end{pmatrix}$$

abychom převedli $A(\bar{g}_i)_{S_2}$ do souřadnic vzhledem k H řešíme soustavu se 3 pravými stranami:

$$\left(\begin{array}{ccc|ccc}1&0&0&1&3&-2\\1&1&0&1&5&-2\\1&2&1&1&4&-2\\ \multicolumn{3}{c|}{\underbrace{\hphantom{1\ 2\ 1}}_{H}} & \multicolumn{3}{c}{\underbrace{\hphantom{1\ 4\ -2}}_{A_{S_1S_2}\cdot G}}\end{array}\right) \quad \text{maticově} \quad H\cdot \overset{A_{EF}}{\overset{\shortparallel}{X}} = A_{S_1S_2}\cdot G$$

$$\Downarrow$$

$$A_{EF} = H^{-1}\cdot A_{S_1S_2}\cdot G$$

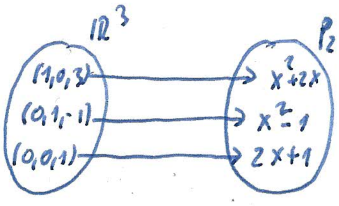
Potřebujeme určit H^{-1}:

$$\left(\begin{array}{ccc|ccc}1&0&0&1&0&0\\1&1&0&0&1&0\\1&2&1&0&0&1\\ \multicolumn{3}{c|}{H} & \multicolumn{3}{c}{E}\end{array}\right)\begin{matrix}\\-r_1\\-r_1\\ \\\end{matrix} \left(\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&-1&1&0\\0&2&1&-1&0&1\end{array}\right)\begin{matrix}\\ \\-2r_2\end{matrix} \sim$$

$$\sim \left(\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&-1&1&0\\0&0&1&1&-2&1\\ &&& \multicolumn{3}{c}{\underbrace{\hphantom{1\ -2\ 1}}_{H^{-1}}}\end{array}\right)$$

Př. Lineární zobrazení $A : \mathbb{R}^3 \to P_2$ je dáno hodnotami:

$A((1,0,3)) = x^2+2x$, $A((0,1,-1)) = x^2-1$, $A((0,0,1)) = 2x+1$



$P_2 = \{ ax^2+bx+c \mid a,b,c \in \mathbb{R} \}$

a) Určete $A(1,-1,6)$.

$k_1(1,0,3) + k_2(0,1,-1) + k_3(0,0,1) = (1,-1,6)$

$k_1(x^2+2x) + k_2(x^2-1) + k_3(2x+1) = A(1,-1,6)$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 3 & -1 & 1 & 6 \end{array}\right) -3r_1 \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 3 \end{array}\right) +r_2 \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right) \begin{array}{l} \Rightarrow k_1 = 1 \\ \Rightarrow k_2 = -1 \\ \Rightarrow k_3 = 2 \end{array} \Bigg\} \Rightarrow$$

$$A(1,-1,6) = 1(x^2+2x) - 1(x^2-1) + 2(2x+1) = \underline{\underline{6x + 3}}$$

b) Určete $\bar{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ tak, aby $A(\bar{x}) = 6x+3$

$k_1(1,0,3) + k_2(0,1,-1) + k_3(0,0,1) = \bar{x}$

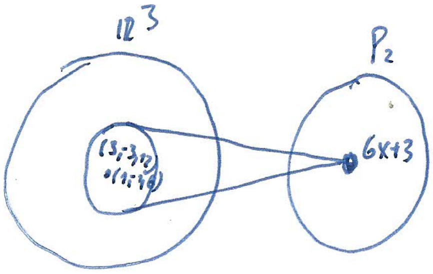
$k_1(x^2+2x) + k_2(x^2-1) + k_3(2x+1) = A(\bar{x}) = 6x+3$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & 0 & 2 & 6 \\ 0 & -1 & 1 & 3 \end{array}\right) -2r_1 \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -2 & 2 & 6 \\ 0 & -1 & 1 & 3 \end{array}\right) \begin{array}{l} \\ :2 \\ -\frac{1}{2}r_2 \end{array} \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \Rightarrow k_1 + t - 3 = 0 \quad \Rightarrow k_1 = 3-t \\ \Rightarrow -k_2 + k_3 = 3 \quad / k_3 = t / \Rightarrow k_2 = t-3 \end{array}$$

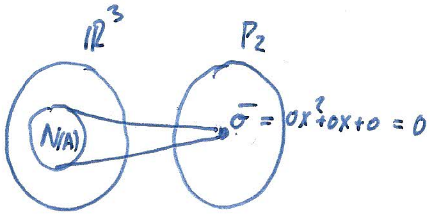
$\Rightarrow \bar{x} = (3-t)(1,0,3) + (t-3)(0,1,-1) + t(0,0,1) =$

$= (3-t,\ 0,\ 9-3t) + (0,\ t-3,\ 3-t) + (0,0,t) =$

$= \underline{\underline{(3-t,\ t-3,\ 12-3t),\ t \in \mathbb{R}}}$ (zvolíme-li $t=2 \Rightarrow \bar{x} = (1,-1,6)$)



c) Určete jádro lin. zobr. A (= nulový prostor lin. zobr. A = N(A)), jeho bázi a dimenzi N(A) (= dim N(A) = d(A) = defekt A).



$$\Rightarrow \quad N(A) = \text{Ker}\, A = \{\, \bar{x} \in \mathbb{R}^3 \mid A(\bar{x}) = 0 \,\}$$

$$k_1(1,0,3) + k_2(0,1,-1) + k_3(0,0,1) = \bar{x}$$

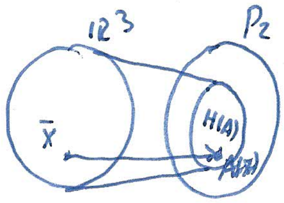
$$k_1(x^2+2x) + k_2(x^2-1) + k_3(2x+1) = A(\bar{x}) = 0x^2+0x+0 = 0$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \begin{array}{l} \\ -2r_1 \\ \\ \end{array} \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \begin{array}{l} \\ :(-2) \\ -\frac{1}{2}r_2 \end{array} \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \Rightarrow k_1 + t = 0 \Rightarrow k_1 = -t \\ \Rightarrow -k_2 + k_3 = 0 \quad |k_3 = t| \Rightarrow k_2 = t \end{array}$$

$$\Rightarrow \bar{x} = -t(1,0,3) + t(0,1,-1) + t(0,0,1) = (-t, t, -3t) = t(-1,1,-3)$$

$\Rightarrow \underline{\underline{N(A) = \{\, t(-1,1,-3) \mid t \in \mathbb{R} \,\}}}$ $\Rightarrow \underline{\underline{\text{Báze } N(A) = \{(-1,1,-3)\}}}$ $\Rightarrow \underline{\underline{d(A) = \dim N(A) = 1}}$

d) Určete obor hodnot zobrazení A (= H(A)), jeho bázi a jeho dimenzi (dim H(A) = h(A) = hodnost A)



$$H(A) = \{\, A(\bar{x}) \mid \bar{x} \in \mathbb{R}^3 \,\}$$

$$(\, k_1(1,0,3) + k_2(0,1,-1) + k_3(0,0,1) = \bar{x} \Rightarrow k_1(x^2+2x) + k_2(x^2-1) + k_3(2x+1) = A(\bar{x}))$$

$$\Rightarrow \quad H(A) = \{\, k_1(x^2+2x) + k_2(x^2-1) + k_3(2x+1) \mid k_1, k_2, k_3 \in \mathbb{R} \,\}$$

Tvoří x^2+2x, x^2-1 a $2x+1$ bázi H(A)? Musíme ověřit nezávislost! :

$$\underbrace{\begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \\ 0 & 2 & 1 \end{pmatrix}}_{\text{upravit na schodový tvar}} \begin{array}{l} \\ -r_1 \\ \\ \end{array} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{pmatrix} \begin{array}{l} \\ \\ +r_2 \end{array} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & -2 & -1 \\ \boxed{0 \;\; 0 \;\; 0} \end{pmatrix}$$

$\Rightarrow$ jsou lin. závislé! $\Rightarrow$ za bázi vezmeme polynomy reprezentované nenulovými řádky $\Rightarrow$

$\Rightarrow \underline{\underline{\text{Báze } H(A) = \{\, x^2+2x,\ -2x-1 \,\}}}$ $\Rightarrow$ $\underline{\underline{H(A) = \{\, k_1(x^2+2x) + k_2(-2x-1) \mid k_1, k_2 \in \mathbb{R} \,\}}}$

$\Downarrow$

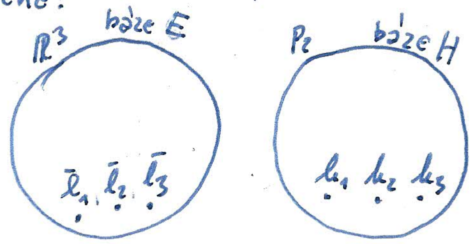
$\underline{\underline{\dim H(A) = 2 = h(A)}}$

$A: U \to V \Rightarrow$

Pozn.: Platí $d(A) + h(A) = \dim U$ a opravdu $\dim N(A) + \dim H(A) = 1 + 2 = 3 = \dim \mathbb{R}^3$

e) Určete matici lin. zobrazení A vzhledem ke standardním bázím $S_1 = \{\underset{\bar{e}_1}{(1,0,0)}; \underset{\bar{e}_2}{(0,1,0)}; \underset{\bar{e}_3}{(0,0,1)}\}$ a $S_2 = \{\underset{\bar{h}_1}{x^2}, \underset{\bar{h}_2}{x}, \underset{\bar{h}_3}{1}\}$

Obecně:



$$\Rightarrow A_{EH} = \begin{pmatrix} A(\bar{e}_1)_{\langle H\rangle} & A(\bar{e}_2)_{\langle H\rangle} & A(\bar{e}_3)_{\langle H\rangle} \end{pmatrix}$$

Vzhledem ke stand. bázím je to jednoduché:

$$\begin{array}{cc} \bar{x} & \longrightarrow \; A(\bar{x}) \end{array}$$

$$\underbrace{\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 2 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 & 1 \end{array}\right)}_{\text{upravíme na matici jednotkovou}} \begin{matrix} -3r_3 \\ +r_3 \\ \end{matrix} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -4 & -3 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 & 1 \end{array}\right)$$

stačí transponovat tuto matici a obdržíme matici lin. zobr. A vzhledem ke st. bázím.

$A((1,0,0)) = x^2-4x-3 \quad \Rightarrow A(\bar{e}_1)_{\langle S_2\rangle} = (1,-4,-3)$

$\Rightarrow A((0,1,0)) = x^2+2x \quad \Rightarrow A(\bar{e}_2)_{\langle S_2\rangle} = (0,1,0)$

$A((0,0,1)) = 2x+1 \quad \Rightarrow A(\bar{e}_3)_{\langle S_2\rangle} = (0,2,1)$

$$\Rightarrow \quad A_{S_1S_2} = \begin{pmatrix} 1 & 1 & 0 \\ -4 & 2 & 2 \\ -3 & 0 & 1 \end{pmatrix}$$

f) Určete matici A_{EH}, kde $E = \{(2,0,1); (0,1,1), (0,1,3)\}$, $H = \{x^2, x-1, x\ 1\}$

$$A_{EH} = \underset{\uparrow}{H^{-1}} \cdot A_{S_1S_2} \cdot \underset{\uparrow}{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 1 & 0 \\ -4 & 2 & 2 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

H má ve sloupcích souřadnice vektorů z H vzhledem k S_2

ve sloupcích souř. vektorů z G vzhledem k S_1

určíme H^{-1}:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \begin{matrix} \\ \\ +r_2 \end{matrix} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right) -r_3 \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right)_{H^{-1}} \Rightarrow$$

$$A_{EH} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ -4 & 2 & 2 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & -1 \\ -7 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 5 & -1 & -3 \\ -11 & 5 & 11 \end{pmatrix}$$

Platí:

$$\underset{\uparrow \text{sloupec}}{A(\bar{x})_{\langle H\rangle}} = A_{EH} \cdot \overset{\text{sloupec} \swarrow}{\bar{x}_{\langle E\rangle}}$$

g) Určete A((2,0,1))

α) Pomocí $A_{S_1S_2}$: $A((2,0,1))_{S_2} = A_{S_1S_2} \begin{pmatrix} (2,0,1)_{\langle S_1\rangle} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ -4 & 2 & 2 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \\ -5 \end{pmatrix} \Rightarrow A((2,0,1)) = 2x^2-6x-5$

nebo:

β) Pomocí A_{EH}: $A((2,0,1))_H = A_{EH} \begin{pmatrix} (2,0,1)_{\langle E\rangle} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 5 & -1 & -3 \\ -11 & 5 & 11 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -11 \end{pmatrix} \Rightarrow A((2,0,1)) = 2(x^2) + 5(x-1) - 11.(x) =$

$= 2x^2-6x-5$

$(2,0,1)_{\langle E\rangle} = (1,0,0)$

protože $(2,0,1) = 1(2,0,1) + 0(0,1,1) + 0(0,1,3)$

Př: Lineární zobrazení $A: \mathbb{R}^3 \to \mathbb{R}^2$ je dáno hodnotami $A((1,0,3)) = (1,2)$; $A((0,1,-1)) = (3,1)$; $A((1,1,1)) = (0,1)$

a) Určete $A((3,3,4))$

$$\boxed{\begin{array}{l} (3,3,4) = k_1(1,0,3) + k_2(0,1,-1) + k_3(1,1,1) \\ \quad\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \\ A((3,3,4)) = k_1(1,2) + k_2(3,1) + k_3(0,1) \end{array}}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 3 & -1 & 1 & 4 \end{array}\right)_{-3r_1} \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & -2 & -5 \end{array}\right)_{+r_2} \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -2 \end{array}\right) \begin{array}{l} \Rightarrow k_1 + 2 = 3 \Rightarrow \underline{k_1 = 1} \\ \Rightarrow k_2 + 2 = 3 \Rightarrow \underline{k_2 = 1} \\ \Rightarrow \underline{k_3 = 2} \end{array} \Rightarrow$$

$$A((3,3,4)) = 1\cdot(1,2) + 1\cdot(3,1) + 2(0,1) \quad = \underline{\underline{(4,5)}}$$

b) Určete $\bar{x} \in \mathbb{R}^3$ tak, aby $A(\bar{x}) = (3,2)$

(určitě to podle zadání bude např. $(0,1,-1) + (1,1,1) = (1,2,0)$)

$$\boxed{\begin{array}{l} \bar{x} = k_1(1,0,3) + k_2(0,1,-1) + k_3(1,1,1) \\ \downarrow \\ (3,2) = k_1(1,2) + k_2(3,1) + k_3(0,1) \end{array}}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 3 \\ 2 & 1 & 1 & 2 \end{array}\right)_{-2r_1} \sim \left(\begin{array}{ccc|c} 1 & 3 & 0 & 3 \\ 0 & -5 & 1 & -4 \end{array}\right) \begin{array}{l} k_1 + 3\lambda = 3 \qquad \Rightarrow \underline{k_1 = -3\lambda + 3} \\ \Rightarrow -5k_2 + k_3 = -4 \quad \Rightarrow \underline{k_2 = \lambda} \quad \Rightarrow \underline{k_3 = -4 + 5\lambda} \end{array}$$

$$\bar{x} = (-3\lambda+3)(1,0,3) + \lambda(0,1,-1) + (-4+5\lambda)(1,1,1) = (-3\lambda+3-4+5\lambda,\ \lambda-4+5\lambda,\ -9\lambda+9-\lambda-4+5\lambda) = \underline{\underline{(-1+2\lambda, -4+6\lambda, 5-5\lambda)}}, \lambda \in \mathbb{R}$$

c) Určete jádro lin. zobrazení A, jeho bázi a dimenzi

$$\boxed{\begin{array}{l} \bar{x} = k_1(1,0,3) + k_2(0,1,-1) + k_3(1,1,1) \\ \downarrow \\ (0,0) = k_1(1,2) + k_2(3,1) + k_3(0,1) \end{array}} \longrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right)_{-2r_1} \sim \left(\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & -5 & 1 & 0 \end{array}\right) \begin{array}{l} \Rightarrow k_1 = -3\lambda \\ \Rightarrow k_2 = \lambda \\ \Rightarrow k_3 = 5\lambda \end{array}$$

$$\Rightarrow \bar{x} = -3\lambda(1,0,3) + \lambda(0,1,-1) + 5\lambda(1,1,1) = (2\lambda, 6\lambda, -5\lambda),\ \lambda \in \mathbb{R}$$

$\Rightarrow$ $\underline{\underline{\mathrm{Ker}\,A = \{\lambda(2,6,-5) \mid \lambda \in \mathbb{R}\}}}$, $\underline{\underline{\text{báze } \mathrm{Ker}\,A = \{(2,6,-5)\}}}$, $\underline{\underline{\text{dimenze } \mathrm{Ker}\,A = 1}}$

d) Určete obor hodnot H_A, jeho bázi a dimenzi

$$\left.\begin{array}{l} x = k_1(1,0,3) + k_2(0,1,-1) + k_3(1,1,1) \\ \downarrow \\ A(x) = k_1(1,2) + k_2(3,1) + k_3(0,1) \end{array}\right\} \Rightarrow H_A = \{k_1(1,2) + k_2(3,1) + k_3(0,1) \mid k_1, k_2, k_3 \in \mathbb{R}\} \Rightarrow \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & -5 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & -5 \\ 0 & 0 \end{pmatrix}$$

$$\underline{\underline{H_A = \{k_1(1,2) + k_2(0,-5) \mid k_1, k_2 \in \mathbb{R}\}}}$$

$\underline{\underline{\text{báze } H_A = \{(1,2); (0,-5)\}}}$, $\underline{\underline{\dim H_A = 2}}$ $(\Rightarrow H_A = \mathbb{R}^2)$

Př.: Lineární zobrazení $f: P_3 \to P_2$ je dáno hodnotami:

$f(x^2) = 2x+1$, $f(x+2) = x+1$, $f(2x+5) = 3x+1$

a) Určete matici lin. zobr. f vzhledem ke st. bázím

P_3: $S_1 = \{x^2, x, 1\}$ P_2: $S_2 = \{x, 1\}$

$$\begin{array}{c} x \quad\quad f(x) \\ \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 2 & 5 & 3 & 1 \end{array}\right)_{-2r_2} \end{array} \sim \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & +1 & -1 \end{array}\right)_{-2r_3} \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right)$$

$$\underbrace{}_{E} \qquad \underbrace{}_{A^T_{S_1S_2}}$$

$\Rightarrow f(x^2) = 2x+1$

$\Rightarrow f(x) = -x+3$

$f(1) = x-1$

$$A_{S_1S_2} = \left(\underset{f(x^2)_{\langle S_2\rangle}}{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} \underset{f(x)_{\langle S_2\rangle}}{\begin{pmatrix} -1 \\ 3 \end{pmatrix}} \underset{f(1)_{\langle S_2\rangle}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} \right)$$

b) Určete matici lin. zobr. f vzhledem k bázím $G = \{x^2-1, x^2-x, x+1\}$ a $H = \{2x+2; 2x-2\}$

$$A_{GH} = \left(\begin{matrix} f(\bar g_1)_{\langle H\rangle} & f(\bar g_2)_{\langle H\rangle} & f(\bar g_3)_{\langle H\rangle} \end{matrix} \right) = \left(\begin{matrix} \bar h_{1\langle S_2\rangle} & \bar h_{2\langle S_2\rangle} & \bar h_{3\langle S_2\rangle} \end{matrix} \right)^{-1} \underbrace{\left(\qquad \right)}_{A_{S_1S_2}} \left(\begin{matrix} \bar g_{1\langle S_1\rangle} & \bar g_{2\langle S_1\rangle} & \bar g_{3\langle S_1\rangle} \end{matrix} \right) =$$

souř. vzhl. k H

$H \cdot \bar x = f(\bar g_1)^T_{\langle S_2\rangle}$

$\bar x = H^{-1} \cdot f(\bar g_1)_{\langle S_2\rangle}$

$$= \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 2 & -2 & 0 & 1 \end{array}\right)_{-r_1} \sim \left(\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 0 & -4 & -1 & 1 \end{array}\right)^{\cdot 2} \sim \left(\begin{array}{cc|cc} 4 & 4 & 2 & 0 \\ 0 & -4 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 4 & 0 & 1 & 1 \\ 0 & -4 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 4 & 0 & 1 & 1 \\ 0 & 4 & 1 & -1 \end{array}\right) \Rightarrow \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix}$$

$$A_{GH} = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 3 & 2 & 0 \\ -1 & -4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 3 & 1 & 2 \\ -3 & 3 & -2 \end{pmatrix}$$

Matice lineárního zobrazení

Nechť $E=\{\bar{e}_1,\bar{e}_2,\dots,\bar{e}_n\}$ je báze na U a $F=\{\bar{f}_1,\bar{f}_2,\dots,\bar{f}_m\}$ je báze na V. Jestliže $f:U\to V$ je lineární zobrazení, pak jsme schopni určit $f(x)$ pomocí hodnot $f(\bar{e}_1), f(\bar{e}_2),\dots, f(\bar{e}_n)$ následovně: Jestliže $\bar{u}=(\alpha_1,\alpha_2,\dots,\alpha_n)_{\langle E\rangle} \Rightarrow$

$$f(\bar{u}) = f(\alpha_1\bar{e}_1+\alpha_2\bar{e}_2+\dots+\alpha_n\bar{e}_n) = \alpha_1 f(\bar{e}_1)+\alpha_2 f(\bar{e}_2)+\dots+\alpha_n f(\bar{e}_n) \qquad (*)$$

Známe-li navíc souřadnice vektorů $f(\bar{e}_1), f(\bar{e}_2),\dots, f(\bar{e}_n)$ vzhledem k bázi F, tj.

$$\begin{aligned} f(\bar{e}_1) &= \beta_{11}\bar{f}_1+\beta_{12}\bar{f}_2+\dots+\beta_{1m}\bar{f}_m \\ f(\bar{e}_2) &= \beta_{21}\bar{f}_1+\beta_{22}\bar{f}_2+\dots+\beta_{2m}\bar{f}_m \\ &\vdots \\ f(\bar{e}_n) &= \beta_{m1}\bar{f}_1+\beta_{m2}\bar{f}_2+\dots+\beta_{mm}\bar{f}_m \end{aligned}$$

a dosadíme-li je do $(*)$, pak obdržíme souřadnice $f(\bar{u})$ vzhledem k F:

$$f(\bar{u}) = \underbrace{(\beta_{11}\alpha_1+\beta_{21}\alpha_2+\cancel{\beta_2}\dots+\beta_{m1}\alpha_n)}_{x_1^*}\bar{f}_1+\underbrace{(\beta_{12}\alpha_1+\beta_{22}\alpha_2+\dots+\beta_{m2}\alpha_n)}_{x_2^*}\bar{f}_2+\dots+\underbrace{(\beta_{1m}\alpha_1+\dots+\beta_{mm}\alpha_n)}_{x_m^*}\bar{f}_m$$

označme

$$\Rightarrow f(\bar{u})_{\langle F\rangle} = (x_1^*, x_2^*,\dots,x_m^*)_{\langle F\rangle}$$

Vidíme, že:

$$\begin{aligned} x_1^* &= \beta_{11}\alpha_1+\beta_{21}\alpha_2+\dots+\beta_{m1}\alpha_n \\ x_2^* &= \beta_{12}\alpha_1+\beta_{22}\alpha_2+\dots+\beta_{m2}\alpha_n \\ &\vdots \\ x_m^* &= \beta_{1m}\alpha_1+\beta_{2m}\alpha_2+\dots+\beta_{mm}\alpha_n \end{aligned}$$

Zapsáno maticově:

$$\underbrace{\begin{pmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_m^* \end{pmatrix}}_{\text{souřadnice } f(\bar{u}) \text{ vzhledem k bázi } F} = \underbrace{\begin{pmatrix} \overset{f(\bar{e}_1)_{\langle F\rangle}}{\beta_{11}} & \beta_{21} & \cdots & \beta_{m1} \\ \beta_{12} & \beta_{22} & \cdots & \beta_{m2} \\ \vdots & \vdots & & \\ \beta_{1m} & \beta_{2m} & \cdots & \beta_{mm} \end{pmatrix}}_{\text{tzv. matice lin. zobrazení } A_{EF} \text{ - jsou v ní ve sloupcích souřadnice } f(\bar{e}_1)_{\langle F\rangle}, f(\bar{e}_2)_{\langle F\rangle},\dots, f(\bar{e}_n)_{\langle F\rangle}} \underbrace{\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}}_{\text{souřadnice } \bar{u} \text{ vzhledem k bázi } E} \Rightarrow f(\bar{u})^T_{\langle F\rangle} = A_{EF}\cdot\bar{u}^T_{\langle E\rangle}$$