Spektrální rozklad

Věta: Nechť A je reálná symetrická matice a λ je vlastní vektor matice A, pak $\lambda \in \mathbb{R}$.

Věta: Nechť A je reál. symetrická matice, pak vl. vekt. přísl. různým vl. číslům jsou na sebe kolmé.

Věta: Nechť A je reálná symetrická matice potom existuje ortogonální matice Q (tzn $Q \cdot Q^T = E$) a diagonální matice D taková, že

$$\boxed{A = Q^T D Q}$$

(Řádky Q tvoří ortonormální vlastní vektory matice A, diag. prvky D jsou vlastní čísla A)

Př: Nalezněte spektrální rozklad matice $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

1.) Určíme λ:

$$\begin{vmatrix} (3-\lambda) & 1 \\ 1 & (3-\lambda) \end{vmatrix} = 0 \Rightarrow (3-\lambda)^2 - 1 = 0 \Rightarrow \lambda^2 - 6\lambda + 8 = 0 \Rightarrow (\lambda-2)(\lambda-4) = 0 \Rightarrow \lambda = \left\langle \begin{matrix} 2 \\ 4 \end{matrix} \right.$$

2.) Určíme vlastní vektory:

$$\alpha)\ \left(\begin{matrix} (3-2) & 1 \\ 1 & (3-2) \end{matrix} \middle| \begin{matrix} 0 \\ 0 \end{matrix}\right) \sim \left(\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \end{matrix}\right) \sim \left(\begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \end{matrix}\right) \Rightarrow \begin{matrix} v_1 = -t \\ v_2 = t \end{matrix} \Rightarrow \bar{v}_1 = t(-1,1),\ t \in \mathbb{R} - \{0\}$$

$$\beta)\ \left(\begin{matrix} (3-4) & 1 \\ 1 & (3-4) \end{matrix} \middle| \begin{matrix} 0 \\ 0 \end{matrix}\right) \sim \left(\begin{matrix} -1 & 1 \\ 1 & -1 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \end{matrix}\right) \sim \left(\begin{matrix} -1 & 1 \\ 0 & 0 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \end{matrix}\right) \Rightarrow \begin{matrix} v_1 = t \\ v_2 = t \end{matrix} \Rightarrow \bar{v}_2 = t(1,1),\ t \in \mathbb{R} - \{0\}$$

3.) Normalizujeme vlastní vektory:

$$\bar{q}_1 = \frac{1}{\sqrt{v_1 v_1}} \bar{v}_1 = \frac{1}{\sqrt{2}}(-1,1) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\bar{q}_2 = \frac{1}{\sqrt{v_2 v_2}} \bar{v}_2 = \frac{1}{\sqrt{2}}(1,1) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

4.) $\Rightarrow$

$$\underline{\underline{A = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}}$$

5.) Zkouška:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -2 & 4 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}}} \checkmark$$

Př: Nalezněte spektrální rozklad matice $A = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & 0 \\ 4 & 0 & 8 \end{pmatrix}$

1) Nalezneme vlastní čísla a vlastní vektory:

$$\begin{vmatrix} (2-\lambda) & 0 & 4 \\ 0 & (1-\lambda) & 0 \\ 4 & 0 & (8-\lambda) \end{vmatrix} = (1-\lambda) \begin{vmatrix} (2-\lambda) & 4 \\ 4 & (8-\lambda) \end{vmatrix} = (1-\lambda)\left[(2-\lambda)(8-\lambda) - 16\right] = (1-\lambda)\left[\lambda^2 - 10\lambda\right] = \lambda(1-\lambda)(\lambda-10) = 0$$

$$\Rightarrow \quad \lambda_{1,2,3} = \begin{cases} \underline{0} \\ \underline{1} \\ \underline{10} \end{cases}$$

$\boxed{\lambda_1 = 0}$ $\Rightarrow \left(\begin{array}{ccc|c} 2 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 0 & 8 & 0 \end{array}\right) \begin{array}{c} \\ \\ -2r_1 \end{array} \sim \left(\begin{array}{ccc|c} 2 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 = -2s \\ x_2 = 0 \\ x_3 = s \end{array} \Rightarrow \underline{\underline{\bar{X}_1 = s(-2,0,1), \; s \in \mathbb{C} - \{0\}}}$

$\boxed{\lambda_2 = 1}$ $\left(\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & 7 & 0 \end{array}\right) \begin{array}{c} \\ \\ -4r_1 \end{array} \sim \left(\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 0 & -9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = s \\ x_3 = 0 \end{array} \Rightarrow \underline{\underline{\bar{X}_2 = s(0,1,0), \; s \in \mathbb{C} - \{0\}}}$

$\boxed{\lambda_3 = 10}$ $\left(\begin{array}{ccc|c} -8 & 0 & 4 & 0 \\ 0 & -9 & 0 & 0 \\ 4 & 0 & -2 & 0 \end{array}\right) \begin{array}{c} \\ :-9 \\ \cdot 2 \end{array} \sim \left(\begin{array}{ccc|c} -8 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 8 & 0 & -4 & 0 \end{array}\right) \begin{array}{c} \\ \\ +r_1 \end{array} \sim \left(\begin{array}{ccc|c} -8 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 = s \\ x_2 = 0 \\ x_3 = 2s \end{array} \Rightarrow \underline{\underline{\bar{X} = s(1,0,2), \; s \in \mathbb{C} - \{0\}}}$

2.) Vlastní vektory podělíme jejich velikostí:

$$\bar{e}_1 = \frac{1}{|\bar{X}_1|} \cdot \bar{X}_1 = \frac{1}{\sqrt{(-2)^2 + 0^2 + 1^2}} (-2,0,1) = \frac{1}{\sqrt{5}} (-2,0,1)$$

$$\bar{e}_2 = \frac{1}{|\bar{X}_2|} \cdot \bar{X}_2 = \frac{1}{\sqrt{0^2 + 1^2 + 0^2}} (0,1,0) = (0,1,0)$$

$$\bar{e}_3 = \frac{1}{|\bar{X}_3|} \cdot \bar{X}_3 = \frac{1}{\sqrt{1^2 + 0^2 + 2^2}} = \frac{1}{\sqrt{5}} (1,0,2)$$

3) Rozklad:

$$A = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & 0 \\ 4 & 0 & 8 \end{pmatrix} = \underbrace{\begin{pmatrix} -\frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} -\frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \end{pmatrix}}_{Q^T}$$

Př: Nalezněte spektrální rozklad matice $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$$\begin{vmatrix} (1-\lambda) & 1 & 1 \\ 1 & (1-\lambda) & 1 \\ 1 & 1 & (1-\lambda) \end{vmatrix} = (1-\lambda)^3 + 1 + 1 - \left[(1-\lambda) + (1-\lambda) + (1-\lambda)\right] =$$

$$= (1-\lambda)^3 + 2 - 3(1-\lambda) = 1 - 3\lambda + 3\lambda^2 - \lambda^3 + 2 - 3 + 3\lambda =$$

$$= -\lambda^3 + 3\lambda^2 = -\lambda^2(\lambda - 3) = 0 \Rightarrow \lambda = \begin{cases} 0 \\ 3 \end{cases}$$

$\boxed{\lambda = 0}$ $\Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \begin{array}{c} \\ -r_1 \\ -r_1 \end{array} \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$ $x_1 + x_2 + x_3 = 0 \Rightarrow x_3 = t,\ x_2 = s.$
$x_1 + s + t = 0 \Rightarrow x_1 = -s - t$

$$\Rightarrow \bar{x} = \begin{pmatrix} -s-t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\boxed{\lambda = 3}$ $\left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $-2x_1 + t + t = 0 \Rightarrow x_1 = t$
$-3x_2 + 3x_3 = 0 \Rightarrow x_3 = t \Rightarrow x_2 = t$

$$\Rightarrow \bar{x} = \begin{pmatrix} t \\ t \\ t \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Problém: $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ není kolmé na $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ (neboť $(-1,1,0)\cdot(-1,0,1) = 1 \neq 0$)

$\Rightarrow$ musíme z nich vytvořit dva navzájem kolmé vektory:

$f_1 = (-1, 1, 0)$

$f_2 = (-1,0,1) + \alpha(-1,1,0)$ tak, aby $f_1 \cdot f_2 = (-1,1,0)\left[(-1,0,1) + \alpha(-1,1,0)\right] = 0$

$$1 + \alpha \cdot 2 = 0$$
$$\alpha = -\tfrac{1}{2}$$

$f_2 = (-1,0,1) - \frac{1}{2}(-1,1,0) = (-\frac{1}{2}, -\frac{1}{2}, 1) = -\frac{1}{2}(1,1,-2)$

$\Rightarrow$ Vektory $(-1,1,0)$; $(1,1,-2)$ a $(1,1,1)$ jsou navzájem kolmé a vlastní vektory matice A

$\Rightarrow$ normujeme je:

$(\lambda=0)$ $\bar{e}_1 = \frac{1}{\sqrt{(-1)^2+1^2+0^2}}(-1,1,0) = \frac{1}{\sqrt{2}}(-1,1,0)$

$(\lambda=0)$ $\bar{e}_2 = \frac{1}{\sqrt{1^2+1^2+(-2)^2}}\cdot(1,1,-2) = \frac{1}{\sqrt{6}}(1,1,-2)$

$(\lambda=3)$ $\bar{e}_3 = \frac{1}{\sqrt{1^2+1^2+1^2}}\cdot(1,1,1) = \frac{1}{\sqrt{3}}(1,1,1)$

$$\Rightarrow A = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

Zk: $\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{3}{\sqrt{3}} & \frac{3}{\sqrt{3}} & \frac{3}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = A$ ✓