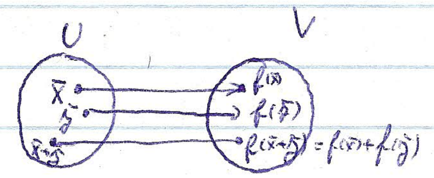
Lineární zobrazení

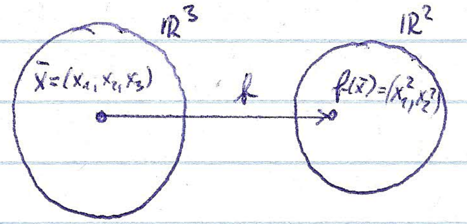
Def. (Lineární zobrazení): Zobrazení $f: U \to V$, kde $(U,+,\cdot)$ a $(V,+,\cdot)$ jsou vektorové prostory nazýváme lineárním zobrazením právě když platí:



① $\forall \bar{x}, \bar{y} \in U: f(\bar{x}+\bar{y}) = f(\bar{x}) + f(\bar{y})$

② $\forall \alpha \in \mathbb{R}\ \forall \bar{x} \in U: f(\alpha \bar{x}) = \alpha \cdot f(\bar{x})$

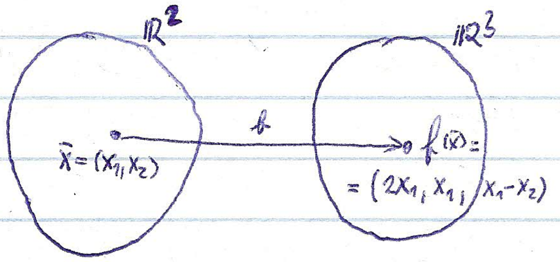
Př:



$\Rightarrow f((2,2,2)) = (4,4)$
$f((3,1,0)) = (9,1)$ $\}$ $(13,5)$

$f((5,3,0)) = (25,9) \neq f((2,2,2)) + f((3,1,0)) \Rightarrow$ není lineární

Př:



1.) $\bar{x} = (x_1, x_2) \Rightarrow f(\bar{x}) = (2x_1, x_1, x_1 - x_2)$
$\bar{y} = (y_1, y_2) \Rightarrow f(\bar{y}) = (2y_1, y_1, y_1 - y_2)$

$\bar{x}+\bar{y} = (x_1+y_1, x_2+y_2) \Rightarrow f(\bar{x}+\bar{y}) = (2(x_1+y_1), x_1+y_1, (x_1+y_1)-(x_2+y_2)) =$
$= (2x_1+2y_1, x_1+y_1, (x_1-x_2)+(y_1-y_2)) =$
$= f(\bar{x}) + f(\bar{y}) \Rightarrow \boxed{f(\bar{x}+\bar{y}) = f(\bar{x}) + f(\bar{y})}$

2.) $\alpha \in \mathbb{R}, \bar{x} \in \mathbb{R}^2$ $(\bar{x} = (x_1, x_2)) \Rightarrow$

$f(\bar{x}) = (2x_1, x_1, x_1 - x_2)$
$\alpha \cdot f(\bar{x}) = (2\alpha x_1, \alpha x_1, \alpha x_1 - \alpha x_2)$

$\alpha \bar{x} = (\alpha x_1, \alpha x_2) \Rightarrow f(\alpha \bar{x}) = f((\alpha x_1, \alpha x_2)) = (2\alpha x_1, \alpha x_1, \alpha x_1 - \alpha x_2) = \alpha f(\bar{x})$

$\Rightarrow \boxed{f(\alpha \bar{x}) = \alpha f(\bar{x})}$

$\Rightarrow$ f je lineární zobrazení z $\mathbb{R}^2$ do $\mathbb{R}^3$ (značíme $f \in L(\mathbb{R}^2, \mathbb{R}^3)$).

Pozn.: Lineární zobrazení $f: U \to V$ je jednoznačně určeno svými hodnotami na bázi vektorového prostoru U. Platí:

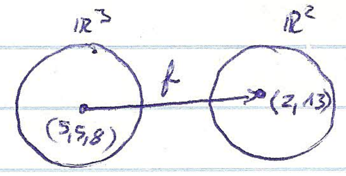
$$\bar{x} = k_1\bar{e}_1 + k_2\bar{e}_2 + \dots + k_n\bar{e}_n$$

$$\Updownarrow$$

$$f(\bar{x}) = k_1 f(\bar{e}_1) + k_2 f(\bar{e}_2) + \dots + k_n f(\bar{e}_n)$$

Př: Lineární zobrazení $f: \mathbb{R}^3 \to \mathbb{R}^2$ je dáno svými hodnotami:
$f((1,1,1)) = (1,2)$; $f((1,2,1)) = (1,-1)$; $f((1,1,2)) = (0,3)$

a) Určete $f((5,5,8)) = ?$



$$(5,5,8) = k_1(1,1,1) + k_2(1,2,1) + k_3(1,1,2) \Rightarrow$$

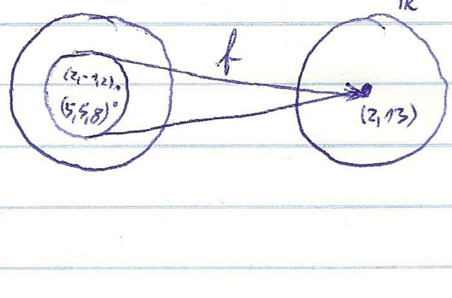
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 2 & 1 & 5 \\ 1 & 1 & 2 & 8 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right) \begin{array}{l} \Rightarrow k_1 = 2 \\ \Rightarrow k_2 = 0 \\ \Rightarrow k_3 = 3 \end{array} \Rightarrow$$

$$f((5,5,8)) = 2 \cdot f((1,1,1)) + 0 \cdot f((1,2,1)) + 3 \cdot f((1,1,2)) = 2(1,2) + 0(1,-1) + 3(0,3) = \underline{\underline{(2,13)}}$$

b) Určete $\bar{x} \in \mathbb{R}^3$: $f(\bar{x}) = (2,13)$

$$f(\bar{x}) = (2,13) = k_1(1,2) + k_2(1,-1) + k_3(0,3) \Rightarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 2 & -1 & 3 & 13 \end{array}\right)_{-2r_1} \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -3 & 3 & 9 \end{array}\right)_{:3} \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & 3 \end{array}\right) \begin{array}{l} k_1 = 5 - t \\ k_2 = -3 + t \\ \Rightarrow k_3 = t \in \mathbb{R} \end{array}$$



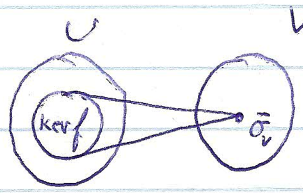
$\Rightarrow$ ~~$\bar{x} = ((5-t, -3+t, t)$ kde $t \in \mathbb{R}$~~

$$\bar{x} = (5-t)(1,1,1) + (-3+t)(1,2,1) + t(1,1,2)$$

$$\bar{x} = (5-t-3+t+t,\ 5-t-6+2t+t,\ 5-t-3+t+2t)$$

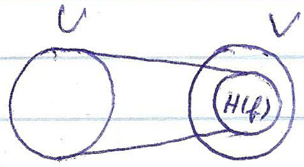
$$\underline{\underline{\bar{x} = (2+t,\ -1+2t;\ 2+2t),\ \text{kde } t \in \mathbb{R}}}$$

Def: (Jádro lin. zobrazení) (Obor hodnot): Nechť $f: U \to V$ je lineární zobrazení a $\bar{o}_v$ je nulový vektor vektorového prostoru $(V,+,\cdot)$. Jádrem lineárního zobrazení f nazýváme množinu



$$N(f) = \operatorname{Ker} f = \{ \bar{x} \in U \mid f(x) = \bar{o}_v \}$$

oborem hodnot zobrazení f nazýváme množinu



$$H(f) = \{ f(\bar{x}) \mid \bar{x} \in U \}$$

Př: Lineární zobrazení je dáno hodnotami $f((1,1,0)) = (1,2,-1)$, $f((0,1,1)) = (0,1,1)$; $f((1,0,1)) = (1,-1,-4)$. Určete jádro a obor hodnot f.

a) $\operatorname{Ker} f = \{ \bar{x} \in \mathbb{R}^3 \mid f(\bar{x}) = (0,0,0) \} \Rightarrow \cancel{f(\bar{x}) = (0,0,0) = k_1(0,1,1) + k_2(1,-1,}$

$$f(\bar{x}) = (0,0,0) = k_1(1,2,-1) + k_2(0,1,1) + k_3(1,-1,-4)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 \\ -1 & 1 & -4 & 0 \end{array}\right)\begin{array}{l} \\ -2r_1 \\ +r_1 \end{array} \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right)\begin{array}{l} \\ \\ -r_2 \end{array} \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} k_1 = -t \\ k_2 = 3t \\ \Rightarrow k_3 = t \end{array}$$

$$\bar{x} = -t(1,1,0) + 3t(0,1,1) + t(1,0,1) = (0, 2t, 4t) = t(0,2,4) \Rightarrow$$

$$\underline{\underline{\operatorname{Ker} f = \{ t(0,2,4) \mid t \in \mathbb{R} \} = \langle (0,2,4) \rangle}}$$

báze: $\underline{\underline{\{(0,2,4)\}}}$

$\underline{\underline{\dim \operatorname{Ker} f = 1}}$

b) $H(f) = \{ f(\bar{x}) \mid \bar{x} \in \mathbb{R}^3 \} = \{ k_1(1,2,-1) + k_2(0,1,1) + k_3(1,-1,-4) \mid k_1, k_2, k_3 \in \mathbb{R} \}$

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & -4 \end{pmatrix}\begin{array}{l} \\ \\ -r_1 \end{array} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -3 & -3 \end{pmatrix}\begin{array}{l} \\ \\ +3r_2 \end{array} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

báze: $\underline{\underline{\{(1,2,-1);(0,1,1)\}}}$

$\underline{\underline{\dim H(f) = 2}}$

$$H_f = \{ k_1(1,2,-1) + k_2(0,1,1) \mid k_1, k_2 \in \mathbb{R} \} = \underline{\underline{\langle (1,2,-1);(0,1,1) \rangle}}$$