Matice lineárního zobrazení

Nechť $E=\{\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n\}$ je báze na U a $F=\{\bar{f}_1, \bar{f}_2, \dots, \bar{f}_m\}$ je báze na V. Jestliže $f: U \to V$ je lineární zobrazení, pak jsme schopni určit $f(x)$ pomocí hodnot $f(\bar{e}_1), f(\bar{e}_2), \dots, f(\bar{e}_n)$ následovně: jestliže $\bar{u}=(\alpha_1, \alpha_2, \dots, \alpha_n)_{\langle E\rangle} \Rightarrow$

$$f(\bar{u}) = f(\alpha_1\bar{e}_1 + \alpha_2\bar{e}_2 + \dots + \alpha_n\bar{e}_n) = \alpha_1 f(\bar{e}_1) + \alpha_2 f(\bar{e}_2) + \dots + \alpha_n f(\bar{e}_n) \qquad (*)$$

Známe-li navíc souřadnice vektorů $f(\bar{e}_1), f(\bar{e}_2), \dots, f(\bar{e}_n)$ vzhledem k bázi F, tj.

$$f(\bar{e}_1) = \beta_{11}\bar{f}_1 + \beta_{12}\bar{f}_2 + \dots + \beta_{1m}\bar{f}_m$$
$$f(\bar{e}_2) = \beta_{21}\bar{f}_1 + \beta_{22}\bar{f}_2 + \dots + \beta_{2m}\bar{f}_m$$
$$\vdots$$
$$f(\bar{e}_n) = \beta_{n1}\bar{f}_1 + \beta_{n2}\bar{f}_2 + \dots + \beta_{nm}\bar{f}_m$$

a dosadíme-li je do $(*)$, pak obdržíme souřadnice $f(\bar{u})$ vzhledem k F:

$$f(\bar{u}) = (\underbrace{\beta_{11}\alpha_1 + \beta_{21}\alpha_2 + \beta_{2}\dots + \beta_{n1}\alpha_n}_{X_1^*})\bar{f}_1 + (\underbrace{\beta_{12}\alpha_1 + \beta_{22}\alpha_2 + \dots + \beta_{n2}\alpha_n}_{X_2^*})\bar{f}_2 + \dots + \underbrace{(\beta_{1m}\alpha_1 + \dots + \beta_{nm}\alpha_n)}_{X_m^*}\bar{f}_m$$

označme

$$\Rightarrow f(\bar{u})_{\langle F\rangle} = (X_1^*, X_2^*, \dots, X_m^*)_{\langle F\rangle}$$

Vidíme, že:

$$X_1^* = \beta_{11}\alpha_1 + \beta_{21}\alpha_2 + \dots + \beta_{n1}\alpha_n$$
$$X_2^* = \beta_{12}\alpha_1 + \beta_{22}\alpha_2 + \dots + \beta_{n2}\alpha_n$$
$$\vdots$$
$$X_m^* = \beta_{1m}\alpha_1 + \beta_{2m}\alpha_2 + \dots + \beta_{nm}\alpha_n$$

Zapsáno maticově:

$f(\bar{e}_1)_{\langle E\rangle}$

$$\left.\begin{array}{l}\underbrace{\begin{pmatrix} X_1^* \\ X_2^* \\ \vdots \\ X_m^* \end{pmatrix}}_{\text{souřadnice } f(\bar{u}) \text{ vzhledem k bázi } F} = \underbrace{\begin{pmatrix} \beta_{11} & \beta_{21} & \cdots & \beta_{n1} \\ \beta_{12} & \beta_{22} & \cdots & \beta_{n2} \\ \vdots & \vdots & & \\ \beta_{1m} & \beta_{2m} & \cdots & \beta_{nm} \end{pmatrix}}_{\text{tzv. matice lin. zobrazení}} \underbrace{\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}}_{\text{souřadnice } \bar{u} \text{ vzhledem k bázi } E}\end{array}\right\} \Rightarrow f(\bar{u})^T_{\langle F\rangle} = A_{EF} \cdot \bar{u}^T_{\langle E\rangle}$$

souřadnice $f(\bar{u})$ vzhledem k bázi F

tzv. matice lin. zobrazení A_{EF} - jsou v ní ve sloupcích souřadnice $f(\bar{e}_1)_{\langle F\rangle}, f(\bar{e}_2)_{\langle F\rangle}, \dots, f(\bar{e}_n)_{\langle F\rangle}$

souřadnice $\bar{u}$ vzhledem k bázi E

Příklad 1: Lineární zobrazení $f: U \to V$ je dáno svými hodnotami

$$f((2,1,-1)) = (0,3)$$
$$f((3,0,1)) = (4,3)$$
$$f((0,2,1)) = (-1,2)$$

Určete matici lineárního zobrazení A_{EF}, jestliže E a F jsou kanonické báze, tj $E = \{(1,0,0);(0,1,0);(0,0,1)\}$ a $F = \{(1,0),(0,1)\}$

$\Rightarrow$ Musíme určit hodnoty $f(\bar{e}_1)_{\langle F\rangle}, f(\bar{e}_2)_{\langle F\rangle}, f(\bar{e}_3)_{\langle F\rangle}$ $\Rightarrow$ známé hodnoty seřadíme do matice (a upravujeme tak, aby vlevo vznikla jednotková matice)

$$\begin{array}{ccc|cc} \multicolumn{3}{c}{\overbrace{\qquad\qquad}^{\bar{x}}} & \multicolumn{2}{c}{\overbrace{\qquad}^{f(\bar{x})}} \\ 2 & 1 & -1 & 0 & 3 \\ 3 & 0 & 1 & 4 & 3 \\ 0 & 2 & 1 & -1 & 2 \end{array} \sim \left(\begin{array}{ccc|cc} 2 & 1 & -1 & 0 & 3 \\ 6 & 0 & 2 & 8 & 6 \\ 0 & 2 & 1 & -1 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|cc} 2 & 1 & -1 & 0 & 3 \\ 0 & -3 & 5 & 8 & -3 \\ 0 & 2 & 1 & -1 & 2 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|cc} 2 & 1 & -1 & 0 & 3 \\ 0 & -3 & 5 & 8 & -3 \\ 0 & 6 & 3 & -3 & 6 \end{array}\right) \sim \left(\begin{array}{ccc|cc} 2 & 1 & -1 & 0 & 3 \\ 0 & -3 & 5 & 8 & -3 \\ 0 & 0 & 13 & 13 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|cc} 2 & 1 & -1 & 0 & 3 \\ 0 & -3 & 5 & 8 & -3 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|cc} 2 & 1 & 0 & 1 & 3 \\ 0 & -3 & 0 & 3 & -3 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|cc} 2 & 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|cc} 2 & 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right) \sim \begin{array}{ccc|cc} \multicolumn{3}{c}{\overbrace{\qquad}^{\bar{x}}} & \multicolumn{2}{c}{\overbrace{\quad}^{f(\bar{x})}} \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}$$

$$\Rightarrow \left.\begin{array}{l} f((1,0,0)) = (1,1) \\ f((0,1,0)) = (-1,1) \\ f((0,0,1)) = (1,0) \end{array}\right] \Rightarrow \begin{array}{l} f(\bar{e}_1) = (1,1) = 1\cdot(1,0) + 1\cdot(0,1) = 1\cdot f_1 + 1\cdot f_2 = (1,1)_{\langle F\rangle} \\ f(\bar{e}_2) = (-1,1) = -1(1,0) + 1.(0,1) = -1 f_1 + 1\cdot f_2 = (-1,1)_{\langle F\rangle} \\ f(\bar{e}_3) = (1,0) = 1\cdot(1,0) + 0.(0,1) = 1\cdot f_1 + 0\cdot f_2 = (1,0)_{\langle F\rangle} \end{array}$$

$$\Rightarrow \quad A_{EF} = \begin{array}{ccc} \left(\begin{matrix} 1 \\ 1 \end{matrix}\right. & \begin{matrix} -1 \\ 1 \end{matrix} & \left.\begin{matrix} 1 \\ 0 \end{matrix}\right) \\ \underbrace{\qquad}_{f(\bar{e}_1)_{\langle F\rangle}} & \underbrace{\qquad}_{f(\bar{e}_2)_{\langle F\rangle}} & \underbrace{\qquad}_{f(\bar{e}_3)_{\langle F\rangle}} \end{array}$$

Příklad 2.: Uvažujme stejné zobrazení f jako v příkladu 1. Určete $f((1,-1,3))$ a $f((2,0,4))$

Využijeme vztahu $\overset{T}{f}(\bar u)_{\langle F\rangle} = A_{EF}\, \overset{T}{u}_{\langle E\rangle}$

$$\Rightarrow f((1,-1,3))^T = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \Rightarrow f((1,-1,3))_{\langle F\rangle} = (5,0)$$

$$\Rightarrow f((2,0,4))^T = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \Rightarrow f((2,0,4))_{\langle F\rangle} = (6,2)$$

Příklad 3.: Uvažujme stejné zobrazení f jako v příkladu 1. Určete matici A_{GH}, kde báze $G = \{\underbrace{(1,2,0)}_{\bar g_1}, \underbrace{(-1,1,0)}_{\bar g_2}, \underbrace{(0,0,1)}_{\bar g_3}\}$ a báze $H = \{\underbrace{(1,1)}_{h_1} ; \underbrace{(1,-1)}_{h_2}\}$

$\Rightarrow$ musíme určit $f(\bar g_1)_{\langle H\rangle}, f(\bar g_2)_{\langle H\rangle}, f(\bar g_3)_{\langle H\rangle}$.

① nejprve určíme $f(\bar g_1)_{\langle F\rangle}, f(\bar g_2)_{\langle F\rangle}, f(\bar g_3)_{\langle F\rangle}$ (tj. souřadnice vzhledem ke kanonické bázi)

$$\Rightarrow f((1,2,0))^T_{\langle F\rangle} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \Rightarrow f((1,2,0))_{\langle F\rangle} = (-1,3)$$

$$f((-1,1,0))^T_{\langle F\rangle} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \Rightarrow f((-1,1,0))_{\langle F\rangle} = (-2,0)$$

$$f((0,0,1))^T_{\langle F\rangle} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow f((0,0,1))_{\langle F\rangle} = (1,0)$$

② Určíme souřadnice $f(\bar g_1), f(\bar g_2), f(\bar g_3)$ vzhledem k bázi $\langle H\rangle$. Tzn. hledáme čísla a_1, a_2, ~~a_3~~, b_1, b_2, ~~b_3~~ a c_1, c_2, ~~c_3~~ tak, aby platilo:

I.) $f(\bar g_1) = f((1,2,0)) = (-1,3) = a_1 \cdot \bar h_1 + a_2 \bar h_2 \Rightarrow f(\bar g_1)_{\langle H\rangle} = (a_1, a_2)$

II.) $f(\bar g_2) = f((-1,1,0)) = (-2,0) = b_1 \cdot \bar h_1 + b_2 \bar h_2 \Rightarrow f(\bar g_2)_{\langle H\rangle} = (b_1, b_2)$

III.) $f(\bar g_3) = f((0,0,1)) = (1,0) = c_1 \bar h_1 + c_2 \bar h_2 \Rightarrow f(\bar g_3)_{\langle H\rangle} = (c_1, c_2)$

ad I.) hledáme a_1, a_2 tak, aby $(-1,3) = a_1(1,1) + a_2(1,-1)$

tj: $(-1,3) = (a_1 + a_2 , a_1 - a_2)$

$$\Rightarrow \left.\begin{matrix} a_1 + a_2 = -1 \\ \underline{a_1 - a_2 = 3} \end{matrix}\right\} \left(\begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}\middle|\begin{matrix} -1 \\ 3 \end{matrix}\right) \sim \left(\begin{matrix} 1 & 1 \\ 0 & -2 \end{matrix}\middle|\begin{matrix} -1 \\ 4 \end{matrix}\right) \Rightarrow \left.\begin{matrix} a_1 = 1 \\ a_2 = -2 \end{matrix}\right\} \Rightarrow f(\bar g_1)_{\langle H\rangle} = (1,-2)$$

ad II.) hledáme b_1, b_2 tak, aby $(-2,0) = b_1(1,1) + b_2(1,-1)$

tj: $(-2,0) = (b_1 + b_2 , b_1 - b_2)$

$$\Rightarrow \left.\begin{matrix} b_1 + b_2 = -2 \\ b_1 - b_2 = 0 \end{matrix}\right\} \left(\begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}\middle|\begin{matrix} -2 \\ 0 \end{matrix}\right) \sim \left(\begin{matrix} 1 & 1 \\ 0 & -2 \end{matrix}\middle|\begin{matrix} -2 \\ 2 \end{matrix}\right) \Rightarrow \left.\begin{matrix} a_1 = -1 \\ a_2 = -1 \end{matrix}\right\} \Rightarrow f(\bar g_2)_{\langle H\rangle} = (-1,-1)$$

ad III.) hledáme c_1, c_2 tak, aby $(1,0) = c_1(1,1) + c_2(1,-1)$

$\Rightarrow$ obdobně jako v I.) a II) zjistíme, že $c_1 = \frac{1}{2}$, $c_2 = \frac{1}{2}$ $\Rightarrow$ $f(\bar{g}_3)_{\langle H \rangle} = (\frac{1}{2}, \frac{1}{2})$

$\Rightarrow$ Matici A_{GH} určíme tak, že do sloupců dáme ~~vek~~ souřadnice $f(\bar{g}_1)_{\langle H \rangle}$, $f(\bar{g}_2)_{\langle H \rangle}$ a $f(\bar{g}_3)_{\langle H \rangle}$ $\Rightarrow$

$$A_{GH} = \begin{pmatrix} 1 & -1 & \frac{1}{2} \\ -2 & -1 & \frac{1}{2} \end{pmatrix}$$

Potom platí: $f(\bar{u})_{\langle H \rangle} = A_{GH} \; \bar{u}_{\langle G \rangle}$

Př.: Lineární zobrazení $f: P_3 \to P_2$ je dáno hodnotami:

$f(x^2) = 2x+1$, $f(x+2) = x+1$, $f(2x+5) = 3x+1$

a) Určete matici lin. zobr. f vzhledem ke st. bázím

$\overset{P_3}{S_1 = \{x^2, x, 1\}}$ $\overset{P_2}{S_2 = \{x, 1\}}$

$$\left(\begin{array}{ccc|cc} & x & & f(x) & \\ 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 2 & 5 & 3 & 1 \end{array}\right)_{-2r_2} \sim \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & +1 & -1 \end{array}\right)_{-2r_3} \left(\begin{array}{ccc|cc} \underbrace{\begin{matrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{matrix}}_{E} & & & \underbrace{\begin{matrix}2 & 1 \\ -1 & 3 \\ 1 & -1\end{matrix}}_{A^T_{S_1S_2}} & \end{array}\right) \Rightarrow \begin{array}{l} f(x^2) = 2x+1 \\ f(x) = -x+3 \\ f(1) = x-1 \end{array}$$

$$A_{S_1S_2} = \left(\underbrace{\begin{pmatrix}1\\2\end{pmatrix}}_{= f(x^2)_{<S_2>}} \underbrace{\begin{pmatrix}-1\\3\end{pmatrix}}_{= f(x)_{<S_2>}} \underbrace{\begin{pmatrix}1\\-1\end{pmatrix}}_{= f(1)_{<S_2>}} \right)$$

b) Určete matici lin. zobr. f vzhledem k bázím $G = \{x^2-1,\ x^2-x,\ x+1\}$ a $H = \{2x+1;\ 2x-2\}$

$$A_{GH} = \left(f(\bar{g}_1)_{<H>}\ f(\bar{g}_2)_{<H>}\ f(\bar{g}_3)_{<H>} \right) = \left(h_1\,_{<S_2>}\ h_2\,_{<S_2>}\ h_3\,_{<S_2>} \right)^{-1} \underbrace{\left(\qquad \right)}_{A_{S_1S_2}} \left(\bar{g}_1\,_{<S_1>}\ \bar{g}_2\,_{<S_1>}\ \bar{g}_3\,_{<S_1>} \right) =$$

souř. vzhl. k H

$H \cdot \bar{x} = f(\bar{g}_1)^T_{<S_2>}$

$\bar{x} = H^{-1} \cdot f(\bar{g}_1)_{<S_2>}$

$$= \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 2 & -2 & 0 & 1 \end{array}\right)_{-r_1} \sim \left(\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 0 & -4 & -1 & 1 \end{array}\right)^{\cdot 2} \sim \left(\begin{array}{cc|cc} 4 & 4 & 2 & 0 \\ 0 & -4 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 4 & 0 & 1 & 1 \\ 0 & -4 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 4 & 0 & 1 & 1 \\ 0 & 4 & 1 & -1 \end{array}\right) \Rightarrow \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix}$$

$$A_{GH} = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 3 & 2 & 0 \\ -1 & -4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 3 & 1 & 2 \\ -3 & 3 & -2 \end{pmatrix}$$