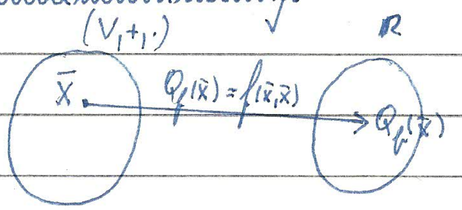
Kvadratické formy



Př: Je dána bilineární forma f. Určete k ní příslušející kvadratickou formu Q_f

a) $f: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, kde $f(\bar{x},\bar{y}) = 2x_1y_1 - 3x_1y_3 + 2x_2y_2 - x_3y_1 + x_3y_2$, $\bar{x}=(x_1,x_2,x_3)$, $\bar{y}=(y_1,y_2,y_3)$

$$\Rightarrow Q_f(\bar{x}) = f(\bar{x},\bar{x}) = 2x_1x_1 - 3x_1x_3 + 2x_2x_2 - x_3x_1 + x_3x_2 = \underline{\underline{2x_1^2 - 4x_1x_3 + 2x_2^2 + x_3x_2}}$$

(např.: $Q_f((1,2,-1)) = 2\cdot 1^2 - 4\cdot 1\cdot(-1) + 2\cdot 2^2 + (-1)\cdot 2 = 2+4+8-2 = 12$)

b) $f: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, kde $f(\bar{x},\bar{y}) = x_1y_1 + 4x_1y_2 - x_2y_1 + x_2y_2$; $\bar{x}=(x_1,x_2)$; $\bar{y}=(y_1,y_2)$

$$\Rightarrow Q_f(\bar{x}) = f(\bar{x},\bar{x}) = x_1x_1 + 4x_1x_2 - x_2x_1 + x_2x_2 = \underline{\underline{x_1^2 + 3x_1x_2 + x_2^2}}$$

(Např. $Q_f((3,2)) = 3^2 + 3\cdot 3\cdot 2 + 2^2 = 9 + 18 + 4 = 31$

c) $f: P_3 \times P_3 \to \mathbb{R}$, kde $f(a(x); b(x)) = 3a(1)b(2) - 2a(2)b(1)$

$$\Rightarrow Q_f(a(x)) = f(a(x), a(x)) = 3a(1)a(2) - 2a(2)a(1) = \underline{\underline{a(1)a(2)}}$$

(např. $Q_f(3x^2-2x+5) = \underbrace{(3\cdot 1^2 - 2\cdot 1 + 5)}_{=a(1)}\underbrace{(3\cdot 2^2 - 2\cdot 2 + 5)}_{=a(2)} = 6\cdot 13 = 78$)

Př: Je dána kv. forma $Q_f: \mathbb{R}^3 \to \mathbb{R}$, kde $Q_f(\bar{x}) = 2x_1^2 - x_2^2 + 4x_1x_3 + 2x_2x_3$; $\bar{x}=(x_1,x_2,x_3)$
Určete symetrickou bilineární formu f k níž Q_f přísluší a její matici vzhledem ke standardní bázi.

$\underline{\underline{f(\bar{x},\bar{y}) = 2x_1y_1 - x_2y_2 + 2x_1y_3 + 2x_3y_1 + x_2y_3 + x_3y_2}}$, kde $\bar{x}=(x_1,x_2,x_3)$; $\bar{y}=(y_1,y_2,y_3)$

$$\underline{\underline{B_{f\langle S\rangle} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & -1 & 1 \\ 2 & 1 & 0 \end{pmatrix}}} = (Q_f)_{\langle S\rangle}$$

Př: Nalezněte matici kv. formy Q_f vzhledem ke standardní bázi S.

a) $Q_f(\bar{x}) = x_1^2 - 2x_2^2 + 3x_3^2 - 4x_1x_2 + x_2x_3$, $\bar{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$

$$\Rightarrow (Q_f)_{\langle S \rangle} = \begin{pmatrix} 1 & -2 & 0 \\ -2 & -2 & \frac{1}{2} \\ 0 & \frac{1}{2} & 3 \end{pmatrix}$$

b) $Q_f(\bar{x}) = x_2^2 - x_3^2 + 2x_4^2 - 2x_1x_3 + 4x_2x_4$; $\bar{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$

$$(Q_f)_{\langle S \rangle} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ -1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix}$$

c) $Q_f(f(x)) = f(-2)f(-3)$; $f(x) \in P_3 = \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\}$

$$Q_f(f(x)) = Q_f(x_1x^2 + x_2x + x_3) = (x_1(-2)^2 + x_2(-2) + x_3)(x_1(-3)^2 + x_2(-3) + x_3) =$$
$$= (4x_1 - 2x_2 + x_3)(9x_1 - 3x_2 + x_3) =$$
$$= 36x_1^2 - 12x_1x_2 + 4x_1x_3$$
$$-18x_1x_2 + 6x_2x_2 - 2x_2x_3$$
$$9x_1x_3 - 3x_2x_3 + x_3^2 =$$
$$= 36x_1^2 - 30x_1x_2 + 13x_1x_3 + 6x_2^2 - 5x_2x_3 + x_3^2$$

$$\Rightarrow (Q_f)_{\langle S \rangle} = \begin{pmatrix} 36 & -15 & \frac{13}{2} \\ -15 & 6 & -\frac{5}{2} \\ \frac{13}{2} & -\frac{5}{2} & 1 \end{pmatrix}$$

d) $Q_f(b(x)) = b(0)b(5)$, $b(x) \in P_2 = \{ax + b \mid a, b \in \mathbb{R}\}$

$$Q_f(b(x)) = Q_f(x_1x + x_2) = (x_1 0 + x_2)(x_1 5 + x_2) = 5x_1x_2 + x_2^2$$

$$\Rightarrow (Q_f)_{\langle S \rangle} = \begin{pmatrix} 0 & \frac{5}{2} \\ \frac{5}{2} & 1 \end{pmatrix}$$

Př: Určete matici kvadratické formy Q_f vzhledem k bázi G.

a) $Q_f(\bar{x}) = x_1^2 - x_3^2 + 4x_1x_2$, $\bar{x} = (x_1, x_2, x_3)$; $G = \{(1,0,-1); (0,2,3); (0,0,-5)\}$

$$(Q_f)_{\langle S\rangle} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$(Q_f)_{\langle G\rangle} = \underbrace{\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & -5 \end{pmatrix}}_{\text{báze v řádcích}} \underbrace{\begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{(Q_f)_{\langle S\rangle}} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & 3 & -5 \end{pmatrix}}_{\text{Báze ve sloupcích}} = \begin{pmatrix} 1 & 2 & 1 \\ 4 & 0 & -3 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & 3 & -5 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 & 7 & -5 \\ 7 & -9 & 15 \\ -5 & 15 & -25 \end{pmatrix}}}$$

b)

Klasifikace kvadratických forem

Př: Klasifikujte kvadratickou formu

1.) $Q_f(\bar{x}) = 4x_1^2 + x_2^2 + 12x_3^2 - 2x_1x_2 - 12x_1x_3$, $\bar{x} = (x_1, x_2, x_3)$

$$\underset{r_1 \leftrightarrow r_2}{\begin{pmatrix} 4 & -1 & -6 \\ -1 & 1 & 0 \\ -6 & 0 & 12 \end{pmatrix}} \sim \left[\underset{s_1 \leftrightarrow s_2}{\begin{pmatrix} -1 & 1 & 0 \\ 4 & -1 & -6 \\ -6 & 0 & 12 \end{pmatrix}} \sim \underset{r_1 \xrightarrow{+} r_2}{\begin{pmatrix} 1 & -1 & 0 \\ -1 & 4 & -6 \\ 0 & -6 & 12 \end{pmatrix}} \right] \sim \left[\underset{s_1 \to s_2}{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & -6 \\ 0 & -6 & 12 \end{pmatrix}} \sim \underset{2r_2 \to r_3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -6 \\ 0 & -6 & 12 \end{pmatrix}} \right] \sim \left[\underset{2s_2 \to s_3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -6 \\ 0 & 0 & 0 \end{pmatrix}} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right]$$

$\Rightarrow$ Q_f je pozitivně semidefinitní

2.) $Q_f(\bar{x}) = 2x_1x_2 + 2x_1x_3 + 2x_2x_3$, $\bar{x} = (x_1, x_2, x_3)$

$$\underset{r_1 \leftrightarrow r_2}{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}} \sim \left[\underset{s_1 \leftrightarrow s_2}{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}} \sim \underset{r_2 \to r_1}{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}} \right] \sim \left[\underset{s_2 \to s_1}{\begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}} \sim \underset{2 \cdot r_2}{\begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}} \right] \sim$$

co teď?! jsme zas na začátku! potřebujeme dostat nějaká čísla na diagonálu

$$\sim \left[\underset{2 \cdot s_2}{\begin{pmatrix} 2 & 1 & 2 \\ 2 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix}} \sim \underset{\substack{-r_1 \to r_2 \\ -r_1 \to r_3}}{\begin{pmatrix} 2 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}} \right] \sim \left[\underset{\substack{-s_1 \to s_2 \\ -s_1 \to s_3}}{\begin{pmatrix} 2 & 2 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}} \sim \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \right] \Rightarrow$$

Q_f je indefininitní

Př: Určete typ kvadratické formy a bázi, vzhledem k níž má diagonální matici

1.) $Q_f(\bar{x}) = x_1^2 + 5x_2^2 + 5x_3^2 - 2x_1x_2 - 8x_2x_3$, kde $\bar{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$

$$\underset{r_1 \to r_2}{\begin{pmatrix} 1 & -1 & 0 \\ -1 & 5 & -4 \\ 0 & -4 & 5 \end{pmatrix}} \sim \left[\underset{s_1 \to s_2}{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 4 & -4 \\ 0 & -4 & 5 \end{pmatrix}} \sim \underset{r_2 \to r_3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & -4 \\ 0 & -4 & 5 \end{pmatrix}} \right] \sim \left[\underset{s_2 \to s_3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & -4 \\ 0 & 0 & 1 \end{pmatrix}} \sim \underset{= D}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \right] \Rightarrow Q_f \text{ je pozitivně definitní}$$

$$\underset{r_1 \to r_2}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \sim \underset{r_2 \to r_3}{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \sim \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \Rightarrow \text{báze } B = \{(1,0,0); (1,1,0); (1,1,1)\}$$

Zk: $$\underset{= B}{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}} \underset{= (Q_f)_{\langle S \rangle}}{\begin{pmatrix} 1 & -1 & 0 \\ -1 & 5 & -4 \\ 0 & -4 & 5 \end{pmatrix}} \underset{= B^T}{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 4 & -4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} = D \ \checkmark$$

2.) $Q_f(\bar{x}) = x_1^2 + x_2^2 + x_3^2 + 4x_1x_2 + 4x_1x_3$, kde $\bar{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$

$$\underset{\substack{-2r_1 \to r_2 \\ -2r_1 \to r_3}}{\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}} \sim \left[\underset{\substack{-2s_1 \to s_2 \\ -2s_1 \to s_3}}{\begin{pmatrix} 1 & 2 & 2 \\ 0 & -3 & -4 \\ 0 & -4 & -3 \end{pmatrix}} \sim \underset{-3 \cdot r_3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & -4 \\ 0 & -4 & -3 \end{pmatrix}} \right] \sim \left[\underset{-3s_3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & -4 \\ 0 & 12 & 9 \end{pmatrix}} \sim \underset{4r_2 \to r_3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 12 \\ 0 & 12 & -27 \end{pmatrix}} \right] \sim$$

$$\sim \left[\underset{4s_2 \to s_3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 12 \\ 0 & 0 & 21 \end{pmatrix}} \sim \underset{= D}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 21 \end{pmatrix}} \right] \Rightarrow Q_f \text{ je indefinitní}$$

$$\underset{\substack{-2r_1 \to r_2 \\ -2r_1 \to r_3}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \sim \underset{-3r_3}{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}} \sim \underset{4r_2 \to r_3}{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 6 & 0 & -3 \end{pmatrix}} \sim \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 4 & -3 \end{pmatrix} \Rightarrow \text{báze } B = \{(1,0,0); (-2,1,0); (-2,4,-3)\}$$

Zk: $$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 4 & -3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & -3 & -4 \\ 0 & 0 & -7 \end{pmatrix} \begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 21 \end{pmatrix} = D \ \checkmark$$