Náhodná veličina

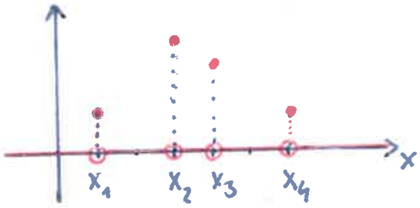
Funkci $X : \Omega \to \mathbb{R}$, kde Ω je množina elementárních jevů nazveme náhodnou veličinou (tj. náhodná veličina přiřazuje jevům čísla). Značíme je velkými písmeny.

Diskrétní NV

Nabývá jen konečně, nebo spočetně mnoho hodnot $x_1, x_2, \ldots$

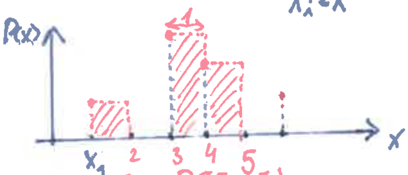
Pravděpodobnostní funkce

$$P(x) = P(X = x)$$



Distribuční funkce

$$F(x) = P(X < x) = \sum_{x_i < x} P(X = x_i)$$



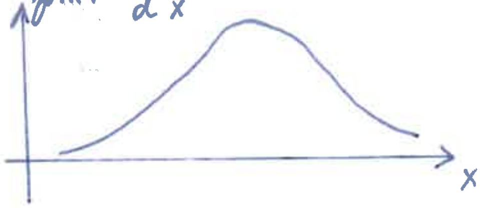
$S = P(X < 5)$

Spojitá NV

Nabývá hodnot z nějakého intervalu.

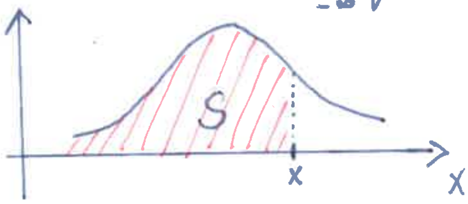
Hustota pravděpodobnosti

$$f(x) = \frac{d F_x(x)}{d x}$$



Distribuční funkce

$$F(x) = P(X < x) = \int_{-\infty}^{x} f(x)\,dx = S$$



$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Střední hodnota: $EX = \sum_{x_i} x_i \, P(X = x_i)$ **Střední hodnota**: $EX = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$

Rozptyl: $DX = \sigma_x^2 = \sum_{x_i} (x_i - EX)^2 P(X = x_i)$ **Rozptyl**: $DX = \sigma_x^2 = \int_{-\infty}^{\infty} (x - EX)^2 f(x)\,dx$

$$DX = EX^2 - (EX)^2$$

Směrodatná odchylka: $\sigma_x = \sqrt{DX}$ **Směrodatná odchylka**: $\sigma = \sqrt{DX}$

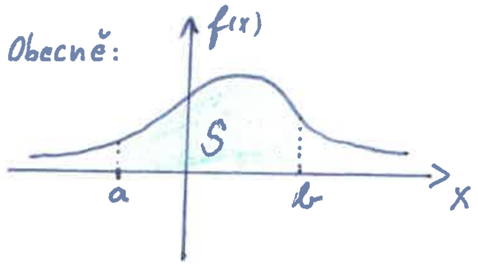
Spojitá náhodná veličina

Př. Náhodná veličina X je dána hustotou pravděpodobnosti:

$$f(x) = \begin{cases} c(1-x^2) & \text{pro} \quad x \in \langle -1, 1 \rangle \\ 0 & \text{pro} \quad x \in (-\infty, -1) \cup (1, \infty) \end{cases}$$

a) Určete konstantu c.

Obecně:



$$S = \int_a^b f(x)\,dx = P(a < X < b) \Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = P(-\infty < X < \infty) = 1$$

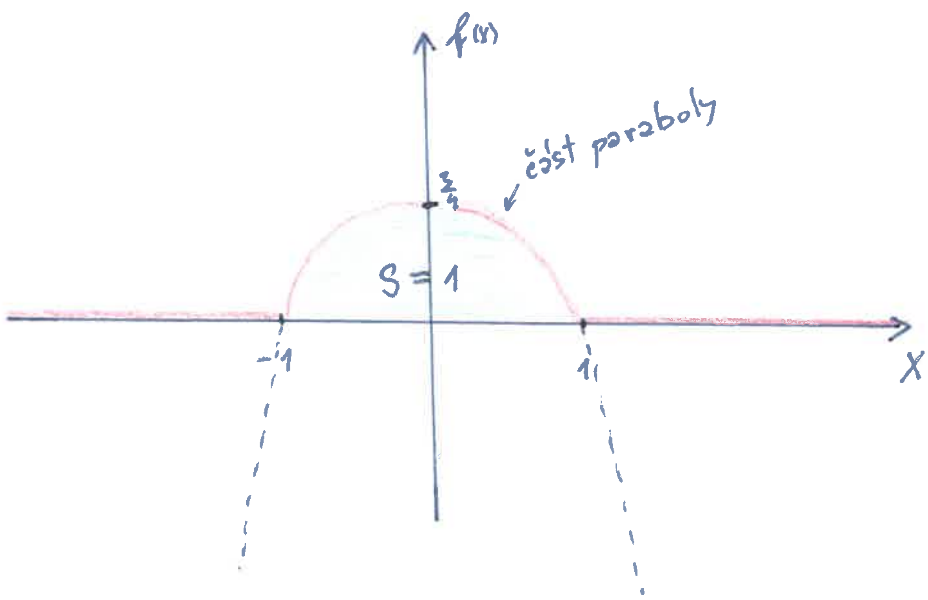
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{-1} f(x)\,dx + \int_{-1}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx = \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{1} c(1-x^2)\,dx + \int_{1}^{\infty} 0\,dx =$$

$$= c\int (1-x^2)\,dx = c\left[x - \frac{x^3}{3}\right] = c\Big[\underbrace{\left(1-\tfrac{1}{3}\right)}_{\frac{2}{3}} - \underbrace{\left(-1-\tfrac{-1}{3}\right)}_{-\frac{2}{3}}\Big] = \frac{4}{3}c = 1 \Rightarrow$$

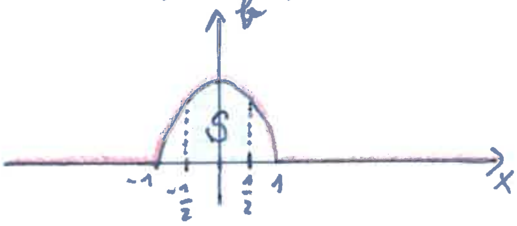
$$\Rightarrow \quad c = \frac{3}{4}$$

b) Načrtněte graf hustoty pravděpodobnosti f.

$$f(x) = \begin{cases} \frac{3}{4}(1-x^2) = \frac{3}{4}(1-x)(1+x) \ldots \text{kvadrat. fce} \; ; \; \text{průs. s osou } x: \; 1 \text{ a } -1 \; ; \; x=0 \Rightarrow y = \frac{3}{4} \\ 0 \end{cases}$$



c) Určete pravděpodobnost $P(-\frac{1}{2} < X \le \frac{1}{2})$



$$S = P(-\tfrac{1}{2} < X \le \tfrac{1}{2}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\,dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{3}{4}(1-x^2)\,dx = \frac{3}{4}\left[x - \frac{x^3}{3}\right]_{-\frac{1}{2}}^{\frac{1}{2}} =$$

$$= \frac{3}{4}\left[\left(\frac{1}{2} - \frac{\frac{1}{8}}{3}\right) - \left(-\frac{1}{2} - \frac{-\frac{1}{8}}{3}\right)\right] = \frac{3}{4}\left[\left(\frac{1}{2} - \frac{1}{24}\right) - \left(-\frac{1}{2} + \frac{1}{24}\right)\right] = \frac{3}{4}\left[1 - \frac{2}{24}\right] =$$

$$= \frac{3}{4} \cdot \frac{22}{24} = \frac{1}{4} \cdot \frac{22}{8} = \frac{11}{16} = \underline{\underline{0{,}6875}}$$

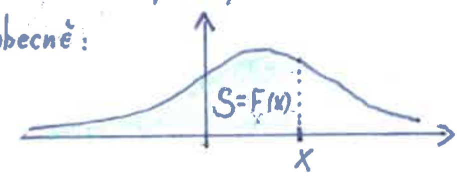
d) Určete pravděpodobnost $P(X > 0)$

$$P(X>0) = \int_0^\infty f(x)\,dx = \int_0^1 \frac{3}{4}(1-x^2)\,dx + \overbrace{\int_1^\infty 0\,dx}^{0} =$$

$$= \left[\frac{3}{4}\left(x - \frac{x^3}{3}\right)\right]_0^1 = \frac{3}{4}\left(1 - \frac{1}{3}\right) - \frac{3}{4}\left(0 - \frac{0}{3}\right) = \frac{3}{4} \cdot \frac{2}{3} = \underline{\underline{\frac{1}{2}}}$$

e) Určete předpis distribuční funkce $F(x)$.

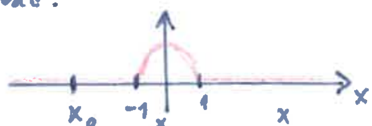
obecně:



$$F(x) = P(X < x) = \int_{-\infty}^{x} f(x)\,dx$$

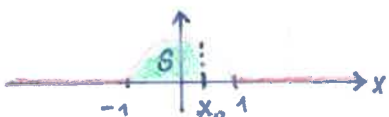
v tomto případě:

I. $x \in (-\infty; -1)$



$\Rightarrow$ Plocha $S = F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} 0\,dx = \underline{0}$

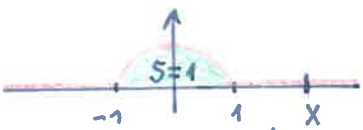
II. $x \in \langle -1, 1 \rangle$



$$\Rightarrow F(x_0) = \int_{-1}^{x_0} \frac{3}{4}(1-x^2)\,dx = \left[\frac{3}{4}\left(x - \frac{x^3}{3}\right)\right]_{-1}^{x_0} = \frac{3}{4}\left(x_0 - \frac{x_0^3}{3}\right) - \frac{3}{4}\left(-1 - \frac{-1}{3}\right) =$$

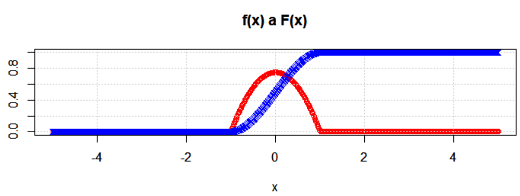
$$= \frac{3}{4}x_0 - \frac{1}{4}x_0^3 - \frac{3}{4}\left(-\frac{2}{3}\right) = \underline{-\frac{1}{4}x_0^3 + \frac{3}{4}x_0 + \frac{1}{2}}$$

III. $x \in (1, \infty)$



$$F(x_0) = \int f(x)\,dx = \int_{-\infty}^{-1} 0\,dx + \underbrace{\int_{-1}^{1} \frac{3}{4}(1-x^2)\,dx}_{=1} + \int_{1}^{x_0} 0\,dx = 1$$

$$\Rightarrow F(x) = \begin{cases} 0 & \text{pro } x \in (-\infty, -1) \\ -\frac{1}{4}x^3 + \frac{3}{4}x + \frac{1}{2} & \text{pro } x \in \langle -1, 1 \rangle \\ 1 & \text{pro } x \in (1, \infty) \end{cases}$$



f) Určete pravděpodobnost $P(0{,}1 < X < 3)$

$$P(0{,}1 < X < 3) = P(X < 3) - P(X \leq 0{,}1) = P(X < 3) - P(X < 0{,}1) =$$

$$= \underbrace{F(3)}_{1} - F(0{,}1) = 1 - \left(-\tfrac{1}{4}\cdot 0{,}1^3 + \tfrac{3}{4}\cdot 0{,}1 + \tfrac{1}{2}\right) = \underline{\underline{0{,}42525}}$$

g) Určete střední hodnotu náhodné veličiny X.

$$EX = \int_{-\infty}^{\infty} x\cdot f(x)\,dx = \underbrace{\int_{-\infty}^{-1} x\cdot 0\,dx}_{0} + \int_{-1}^{1} x\cdot\tfrac{3}{4}(1-x^2)\,dx + \underbrace{\int_{1}^{\infty} x\cdot 0\,dx}_{0} =$$

$$= \int_{-1}^{1} \tfrac{3}{4}(x - x^3)\,dx = \left[\tfrac{3}{4}\left(\tfrac{x^2}{2} - \tfrac{x^4}{4}\right)\right]_{-1}^{1} = \tfrac{3}{4}\left(\tfrac{1}{2} - \tfrac{1}{4}\right) - \tfrac{3}{4}\left(\tfrac{1}{2} - \tfrac{1}{4}\right) = \underline{\underline{0}}$$

h) Určete rozptyl náhodné veličiny X.

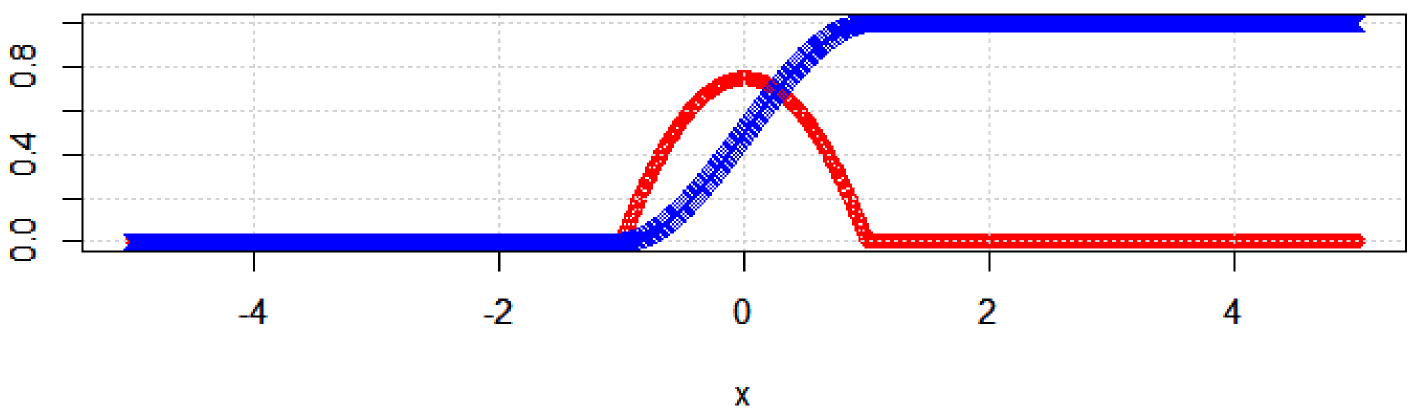
$$DX = \int_{-\infty}^{\infty} (x - EX)^2 f(x)\,dx = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{-1}^{1} x^2\cdot\tfrac{3}{4}(1-x^2)\,dx =$$

$$= \int \tfrac{3}{4}(x^2 - x^4)\,dx = \tfrac{3}{4}\left[\tfrac{x^3}{3} - \tfrac{x^5}{5}\right] = \tfrac{3}{4}\left[\left(\tfrac{1}{3} - \tfrac{1}{5}\right) - \left(\tfrac{-1}{3} - \tfrac{-1}{5}\right)\right] = \tfrac{3}{4}\left[\tfrac{2}{3} - \tfrac{2}{5}\right] = \tfrac{3}{4}\cdot\tfrac{10-6}{15} = \underline{\underline{\tfrac{1}{5}}}$$

i) Určete směrodatnou odchylku náhodné veličiny X.

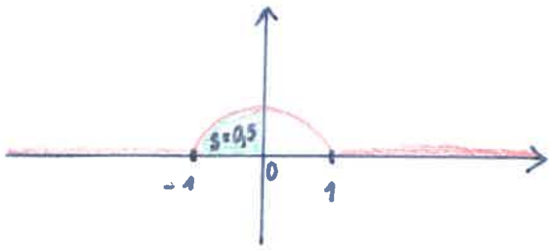
$$\sigma = \sqrt{DX} = \sqrt{\tfrac{1}{5}} \doteq \underline{\underline{0{,}4472}}$$

f(x) a F(x)



j) Určete medián náhodné veličiny X.

Je to hodnota $x_{0,5}$ pro kterou platí $F(x_{0,5}) = P(X < x_{0,5}) = 0{,}5$.



$$0{,}5 = \int_{-\infty}^{0} f(x)\,dx = F(0) \Rightarrow \underline{\underline{x_{0,5} = 0}}$$

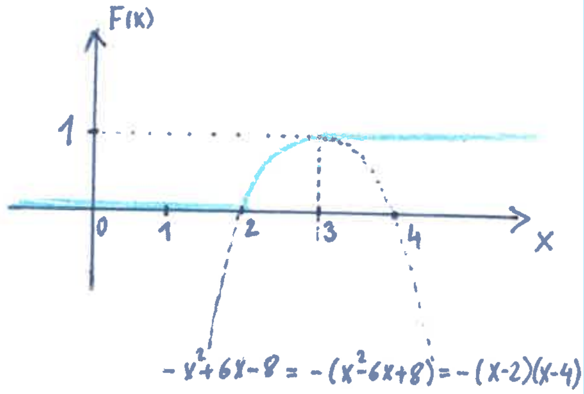
k) Určete modus náhodné veličiny X

U spojité náh. veličiny je modus hodnota $\hat{x}$ v níž $f(x)$ nabývá svého maxima. Z obrázku vidíme, že $f(x)$ je maximální, když $x = 0$ (vrchol paraboly) $\Rightarrow$

$$\underline{\underline{\hat{x} = 0}}$$

Př. Distribuční funkce náhodné veličiny X je dána předpisem:

$$F(x) = \begin{cases} 0 & \text{pro } x \in (-\infty, 2) \\ -x^2+6x-8 & \text{pro } x \in \langle 2, 3 \rangle \\ 1 & \text{pro } x \in (3, \infty) \end{cases}$$



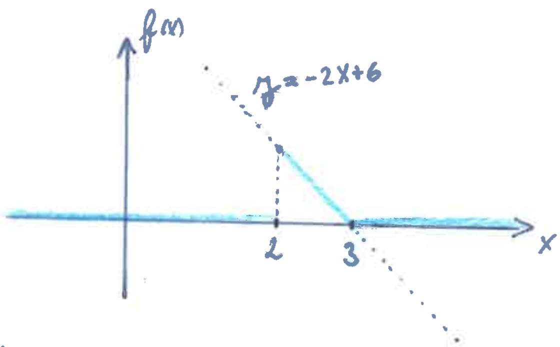
a) Určete pravděpodobnost $P(1 < X < \frac{5}{2})$

$$P(1 < X < \tfrac{5}{2}) = P(X < \tfrac{5}{2}) - P(X \leq 1) = F(\tfrac{5}{2}) - \underbrace{F(1)}_{0} = -(\tfrac{5}{2})^2 + 6 \cdot \tfrac{5}{2} - 8 - 0 = \underline{\underline{\tfrac{3}{4}}}$$

b) Určete předpis hustoty pravděpodobnosti NV X.

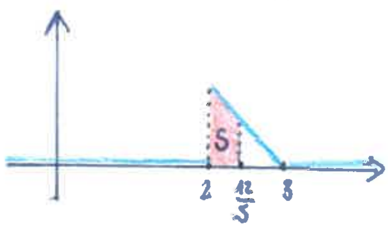
$f(x) = \dfrac{dF(x)}{dx}$ $\Rightarrow$ hustotu dostaneme tak, že zderivujeme $F(x)$:

$$f(x) = \begin{cases} (0)' = 0 & \text{pro } x \in (-\infty, 2) \\ (-x^2+6x-8)' = -2x+6 & \text{pro } x \in \langle 2, 3 \rangle \\ (1)' = 0 & \text{pro } x \in (3, \infty) \end{cases}$$



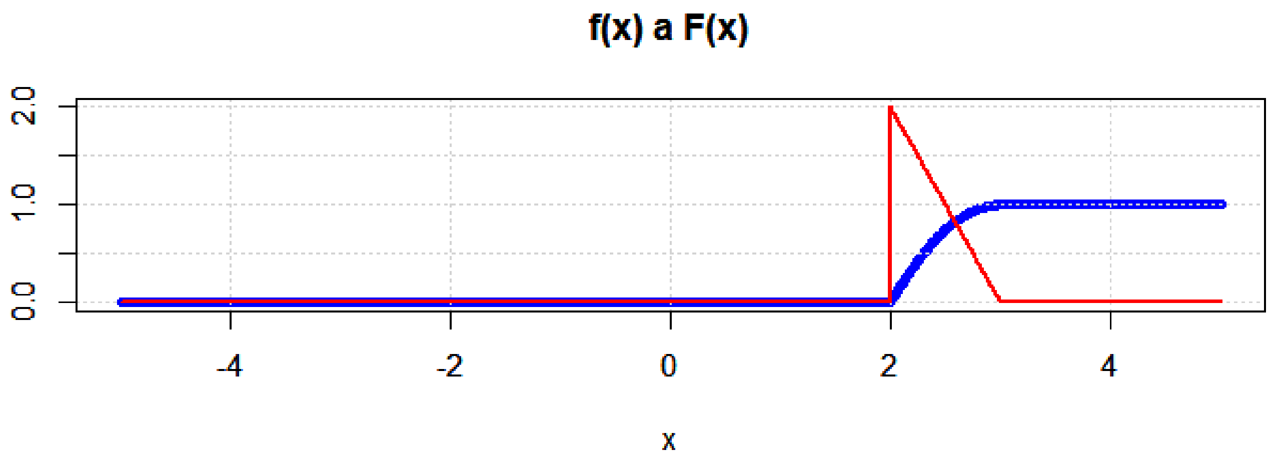
c) Určete pravděpodobnost $P(X < \frac{12}{5})$

$$S = P(X < \tfrac{12}{5}) = \int_{-\infty}^{\frac{12}{5}} f(x)\,dx = \int_{2}^{\frac{12}{5}} (-2x+6)\,dx = \ldots$$



NEBO:

$$P(X < \tfrac{12}{5}) = F(\tfrac{12}{5}) = -(\tfrac{12}{5})^2 + 6 \cdot \tfrac{12}{5} - 8 = \tfrac{-144}{25} + \tfrac{360}{25} - \tfrac{200}{25} = \underline{\underline{\tfrac{16}{25}}}$$



d) Určete střední hodnotu náhodné veličiny X.

$$EX = \int_{-\infty}^{\infty} x\cdot f(x)\,dx = \underbrace{\int_{-\infty}^{2} x\cdot 0\,dx}_{0} + \int_{2}^{3} x\cdot(-2x+6)\,dx + \underbrace{\int_{3}^{\infty} x\cdot 0\,dx}_{0} =$$

$$= \int (-2x^2+6x)\,dx = \left[-2\frac{x^3}{3}+6\frac{x^2}{2}\right] = \underbrace{\left(-2\frac{27}{3}+6\cdot\frac{9}{2}\right)}_{(-18+27)} - \underbrace{\left(-2\frac{8}{3}+6\cdot\frac{4}{2}\right)}_{\left(-\frac{16}{3}+12\right)} = \underline{\underline{\frac{7}{3}}}$$

e) Určete rozptyl náhodné veličiny X.

$$DX = \int_{-\infty}^{\infty} (x-EX)^2 f(x)\,dx = \int_{-\infty}^{\infty} (x^2-2EX+(EX)^2)\, f(x)\,dx =$$

$$= \underbrace{\int_{-\infty}^{\infty} x^2 f(x)\,dx} - \int_{-\infty}^{\infty} 2EX\cdot x\, f(x)\,dx + \int_{-\infty}^{\infty} (EX)^2 f(x)\,dx =$$

$$= EX^2 - 2EX\underbrace{\int_{-\infty}^{\infty} x\cdot f(x)\,dx}_{= EX} + EX^2 \underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{= 1} = EX^2 - 2(EX)^2 - (EX)^2 = EX^2 - (EX)^2$$

$$EX^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{2}^{3} x^2(-2x+6)\,dx = \int_{2}^{3}(-2x^3+6x^2)\,dx = \left[-2\frac{x^4}{4}+6\frac{x^3}{3}\right]_2^3 = \left[-\frac{x^4}{2}+2x^3\right]_2^3 =$$

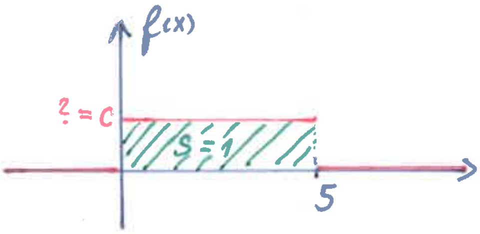
$$= \left(-\frac{81}{2}+54\right) - (-8+16) = -\frac{81}{2}+54-8 = -\frac{81}{2}+46 = \frac{11}{2} \quad \Rightarrow$$

$$DX = EX^2 - (EX)^2 = \frac{11}{2} - \frac{49}{9} = \frac{99-98}{18} = \underline{\underline{\frac{1}{18}}}$$

Př.: Předpokládejme, že doba čekání je NV X s hustotou pravděpodobnosti:

$$f(x) = \begin{cases} 0 & \text{pro } x \in (-\infty, 0) \\ C & \text{pro } x \in \langle 0, 5 \rangle \\ 0 & \text{pro } x \in (5, \infty) \end{cases}$$

a) Určete konstantu C:



$$\underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{= S = } = 1$$

$$C \cdot 5 = 1 \Rightarrow \underline{\underline{C = \frac{1}{5}}}$$

b) Určete distribuční funkci $F(x)$:

$$F(x_0) = \begin{cases} \int_{-\infty}^{x_0} 0\,dx = \underline{\underline{0}} & \text{pro } x_0 \in (-\infty, 0) \\ \int_{-\infty}^{0} 0\,dx + \int_{0}^{x_0} \frac{1}{5}\,dx = \left[\frac{1}{5}x\right]_0^{x_0} = \underline{\underline{\frac{1}{5}x_0}} & \text{pro } x_0 \in \langle 0, 5 \rangle \\ \int_{-\infty}^{0} 0\,dx + \int_{0}^{5} \frac{1}{5}\,dx + \int_{5}^{x_0} 0\,dx = \left[\frac{1}{5}x\right]_0^{5} = \underline{\underline{1}} & \text{pro } x_0 \in (5, \infty) \end{cases}$$

c) Určete EX a DX

$$EX = \int_0^5 x \cdot f(x)\,dx = \int_0^5 x \cdot \frac{1}{5}\,dx = \left[\frac{x^2}{2} \cdot \frac{1}{5}\right]_0^5 = \underline{\underline{\frac{5}{2}}}$$

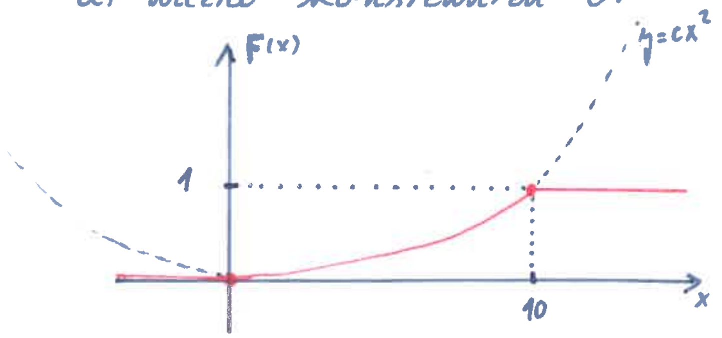
$$EX^2 = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \int_0^5 x^2 \cdot \frac{1}{5}\,dx = \left[\frac{x^3}{3} \cdot \frac{1}{5}\right]_0^5 = \frac{25}{3} \Rightarrow$$

$$DX = EX^2 - (EX)^2 = \frac{25}{3} - \left(\frac{5}{2}\right)^2 = \frac{4 \cdot 25 - 3 \cdot 25}{12} = \underline{\underline{\frac{25}{12}}}$$

Př. Předpokládejme, že hmotnost kapra (v kilogramech) v Brčálníku je spojitá náhodná veličina s distribuční funkcí:

$$F(x) = \begin{cases} 0 & \text{pro } x \in (-\infty, 0) \\ cx^2 & \text{pro } x \in \langle 0, 10 \rangle \\ 1 & \text{pro } x \in (10, \infty) \end{cases}$$

a) Určete konstantu C.



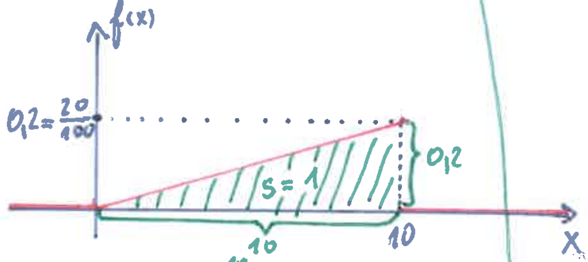
$\Rightarrow$ Chceme, aby $F(10) = 1$ $\Rightarrow$

$$C \cdot 10^2 = 1$$

$$C = \frac{1}{100}$$

b) Určete hustotu pravděpodobnosti NV X.

$$f(x) = \begin{cases} (0)' = 0 & \text{pro } x \in (-\infty, 0) \\ \left(\frac{x^2}{100}\right)' = \frac{2x}{100} & \text{pro } x \in \langle 0, 10 \rangle \\ (1)' = 0 & \text{pro } x \in (10, \infty) \end{cases}$$



Všimněme si, kdybychom určovali C z $f(x)$ $\Rightarrow$ $f(x) = 2Cx$ na $\langle 0, 10 \rangle$ $\Rightarrow \int_0^{10} 2c\,x\,dx = 1$

$$[cx^2]_0^{10} = 1$$

$C \cdot 100^2 = 1 \Rightarrow$ stejné jako

c) Určete pravděpodobnost, že kapr vylovený z Brčálníku má 5 až 10 kg.

$$P(5 \le X \le 10) = \int \frac{1}{50} x\,dx \overset{\text{NEBO}}{=} P(X < 10) - P(X < 5) = F(10) - F(5) =$$

$$= \frac{1}{100} \cdot 10^2 - \frac{1}{100} 5^2 = 1 - \frac{25}{100} = 0{,}75$$

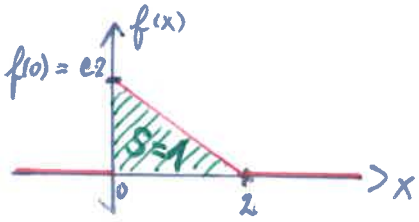
d) Určete EX.

$$EX = \int_0^{10} x \cdot \frac{x}{50}\,dx = \int \frac{1}{50} x^2\,dx = \left[\frac{1}{50} \frac{x^3}{3}\right] = \frac{1000}{50 \cdot 3} - 0 = \frac{20}{3} = 6{,}\overline{3} \text{ kg.}$$

Př. Předpokládejme, že délka života mrkve je náhodná veličina X, jejíž hustota pravděpodobnosti je:

$$f(x) = \begin{cases} C(2-x) & \text{pro} \quad x \in \langle 0,2 \rangle \\ 0 & \text{pro} \quad x \notin \langle 0,2 \rangle \end{cases}$$

a) Určete konstantu C.



I.) $S = 1$

$$\frac{C \cdot 2 \cdot 2}{2} = 1$$

$$\underline{\underline{C = \frac{1}{2}}}$$

II.) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^2 C(2-x)\,dx = 1$$

$$C \cdot \int_0^2 (2-x)\,dx = 1$$

$$C\left[2x - \frac{x^2}{2}\right]_0^2 = 1$$

$$C \cdot \left[4 - \frac{4}{2} - 0\right] = 1 \Rightarrow \underline{\underline{C = \frac{1}{2}}}$$

b) Určete distribuční funkci $F(x)$

$$F(x_0) = \int_{-\infty}^{x_0} f(x)\,dx \Rightarrow$$

$x \in (-\infty, 0) \Rightarrow F(x_0) = P(X < x_0) = \int_{-\infty}^{x_0} f(x)\,dx = \int_{-\infty}^{x_0} 0\,dx = 0$

$x_0 \in \langle 0,2 \rangle \Rightarrow F(x_0) = \int_{-\infty}^{x_0} f(x)\,dx = \underbrace{\int_{-\infty}^{0} 0\,dx}_{=0} + \int_0^{x_0} \frac{1}{2}(2-x)\,dx = \int_0^{x_0} \left(1 - \frac{1}{2}x\right) dx = \left[x - \frac{x^2}{4}\right]_0^{x_0} = x_0 - \frac{x_0^2}{4}$

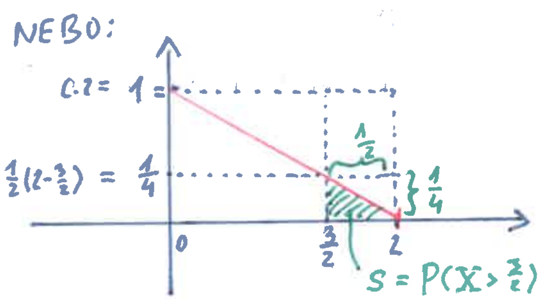
$x_0 \in (2, \infty) \Rightarrow F(x_0) = \int_{-\infty}^{x_0} f(x)\,dx = \underbrace{\int_{-\infty}^{0} 0\,dx}_{0} + \underbrace{\int_0^2 \frac{1}{2}(2-x)\,dx}_{1} + \underbrace{\int_2^{x_0} 0\,dx}_{0} = 1$

$$\Rightarrow F(x) = \begin{cases} 0 & \text{pro} \quad x \in (-\infty, 0) \\ x - \frac{x^2}{4} & \text{pro} \quad x \in \langle 0,2 \rangle \\ 1 & \text{pro} \quad x \in (2, \infty) \end{cases}$$

c) Určete pravděpodobnost, že se mrkev dožije více než 1,5 roku.

$$P(X > 1{,}5) = 1 - F\left(\frac{3}{2}\right) = 1 - \left(\frac{3}{2} - \frac{\left(\frac{3}{2}\right)^2}{4}\right) = 1 - \frac{3}{2} + \frac{9}{16} = \frac{16-24+9}{16} = \underline{\underline{\frac{1}{16}}}$$

NEBO:



$$\Rightarrow P\left(X > \frac{3}{2}\right) = \frac{\frac{1}{2} \cdot \frac{1}{4}}{2} = \underline{\underline{\frac{1}{16}}}$$

Př. Předpokládejme, že rychlost částic je náhodná veličina X (jednotkou je c = rychlost světla) s distribuční funkcí

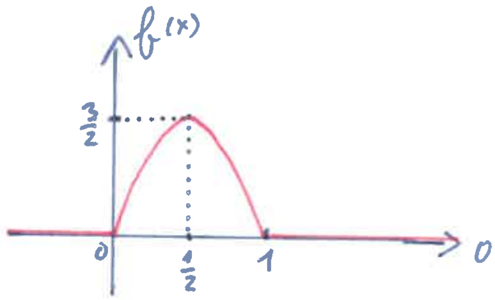
$$F(x) = \begin{cases} 0 & \text{pro } x \in (-\infty, 0) \\ c\left(\frac{x^3}{3} - \frac{x^2}{2}\right) & \text{pro } x \in \langle 0, 1 \rangle \\ 1 & \text{pro } x \in (1, \infty) \end{cases}$$

a) Určete konstantu C.

Chceme, aby: $F(1) = C \cdot \left(\frac{1^3}{3} - \frac{1^2}{2}\right) = 1 \Rightarrow C\left(\frac{1}{3} - \frac{1}{2}\right) = 1 \Rightarrow C\left(\frac{-1}{6}\right) = 1 \Rightarrow \underline{\underline{C = -6}}$

b) Určete hustotu pravděpodobnosti f:

$$f(x) = \begin{cases} (0)' = 0 & \text{pro } x \in (-\infty, 0) \\ \left(-6\left(\frac{x^3}{3} - \frac{x^2}{2}\right)\right)' = -6(x^2 - x) & \text{pro } x \in \langle 0, 1 \rangle \\ (0)' = 0 & \text{pro } x \in (1, \infty) \end{cases}$$



c) Odhadněte kolik procent částic překročí rychlost 0,9c.

$\Rightarrow$ Určíme $P(X > 0{,}9) = 1 - P(X \leq 0{,}9) = 1 - F(0{,}9) = 1 + 6\left(\frac{0{,}9^3}{3} - \frac{0{,}9^2}{2}\right) =$

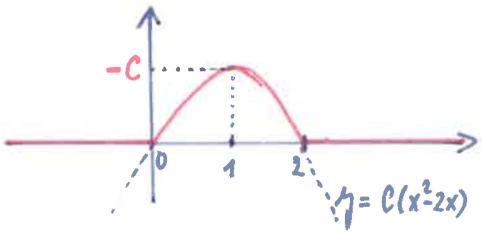
$$= 1 + \underbrace{2 \cdot 0{,}9^3}_{1{,}458} - \underbrace{3 \cdot 0{,}9^2}_{2{,}43} = 0{,}028$$

$\Rightarrow$ cca 2,8 % částic překročí rychlost 0,9 c.

Př. Předpokládejme, že životnost výrobku je náhodná veličina X s hustotou pravděpodobnosti:

$$f(x) = \begin{cases} 0 & \text{pro } x \in (-\infty, 0) \\ c\,x(x-2) & \text{pro } x \in \langle 0, 2 \rangle \\ 0 & \text{pro } x \in (2, \infty) \end{cases}$$

a) Určete konstantu c



$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_0^2 c\,x(x-2)\,dx = c \cdot \int_0^2 (x^2-2x)\,dx = c \cdot \left[\frac{x^3}{3} - x^2\right]_0^2$$

$$1 = c \cdot \left[\frac{8}{3} - 4 - 0\right] = c\,\frac{8-12}{3} = c\left(-\frac{4}{3}\right)$$

$$\Rightarrow \quad c = -\frac{3}{4}$$

b) Určete distribuční funkci $F(x)$:

$x_0 \in (-\infty, 0) \Rightarrow F(x_0) = \int_{-\infty}^{x_0} f(x)\,dx = \int_{-\infty}^{x_0} 0\,dx = \underline{\underline{0}}$

$x_0 \in \langle 0, 2 \rangle \Rightarrow F(x_0) = \int_{-\infty}^{0} 0\,dx + \int_0^{x_0} -\frac{3}{4}x(x-2)\,dx = \int_0^{x_0}\left(-\frac{3}{4}x^2 + \frac{3}{2}x\right)dx = \left[-\frac{x^3}{4} + \frac{3}{4}x^2\right]_0^{x_0} = \underline{\underline{-\frac{x_0^3}{4} + \frac{3}{4}x_0^2}}$

$x_0 \in (2, \infty) \Rightarrow F(x_0) = \underbrace{\int_{-\infty}^{0} 0\,dx}_{0} + \underbrace{\int_0^{2} -\frac{3}{4}x(x-2)\,dx}_{1\text{, viz a)}} + \underbrace{\int_2^{x_0} 0\,dx}_{0} = \underline{\underline{1}}$

c) Určete pravděpodobnost, že se výrobek pokazí dříve než za rok:

$$P(X < 1) = P(X \leq 1) = F(1) = -\frac{1^3}{4} + \frac{3}{4} \cdot 1^2 = \frac{2}{4} = \underline{\underline{\frac{1}{2}}}$$

d) Určete pravděpodobnost, že výrobek vydrží více než $\frac{1}{2}$ roku a méně než $\frac{3}{2}$ roku:

$$P\left(\frac{1}{2} < X < \frac{3}{2}\right) = F\left(\frac{3}{2}\right) - F\left(\frac{1}{2}\right) = -\frac{\left(\frac{3}{2}\right)^3}{4} + \frac{3 \cdot \left(\frac{3}{2}\right)^2}{4} - \left(-\frac{\left(\frac{1}{2}\right)^3}{4} + \frac{3\left(\frac{1}{2}\right)^2}{4}\right) =$$

$$= \frac{-\frac{27}{8} + \frac{27}{4}}{4} - \left(\frac{-\frac{1}{8} + \frac{3}{4}}{4}\right) = \frac{-\frac{26}{8} + \frac{24}{4}}{4} = \frac{\frac{-26+48}{8}}{4} = \frac{22}{8 \cdot 4} = \underline{\underline{\frac{11}{16}}}$$