Lema 11: Platí:

$$\sum_{d\le x}\frac{\Lambda(d)}{d}=\ln x+O(1)$$

Důkaz: Nejprve ukážeme, že $\psi(x)=O(x)$. Podle definice $\psi(x)=\sum_{d\le x}\underbrace{\Lambda(d)}_{\ge 0}$. Proto $\psi(x)\ge 0$. $\Rightarrow$

Z Čebyševovy věty:

$$0\le\underbrace{\sum_{d\le x}\Lambda(d)}_{=\psi(x)}\le x\ln 4+O(\ln^2 x) \quad\Rightarrow \text{ Pro dost velká } x:$$

$$\left|\frac{\psi(x)}{x}\right|\le\left|\ln 4+\underbrace{O\left(\frac{\ln^2 x}{x}\right)}_{\to 0}\right|\le\ln 5 \qquad\Rightarrow\quad \psi(x)=O(x)$$

Z Lema 3,5 a Lema 5:

$$B(x)=\sum_{d\le x}\Lambda(d)\left[\frac{x}{d}\right]=x\ln x-x+O(\ln x)$$

$\left[\frac{x}{d}\right]$ nahradíme $\frac{x}{d}+\varepsilon_{\frac{x}{d}}$, kde $\varepsilon_{\frac{x}{d}}\in(-1,0\rangle$ $\Rightarrow$

$$\sum_{d\le x}\Lambda(d)\left(\frac{x}{d}+\varepsilon_{\frac{x}{d}}\right)=x\ln x-x+O(\ln x)$$

$$x\cdot\sum_{d\le x}\frac{\Lambda(d)}{d}+\sum_{d\le x}\left(\Lambda(d)\,\varepsilon_{\frac{x}{d}}\right)=x\ln x-x+O(\ln x)\quad /:x$$

$$\sum_{d\le x}\frac{\Lambda(d)}{d}+\frac{1}{x}\sum_{d\le x}\left(\Lambda(d)\,\varepsilon_{\frac{x}{d}}\right)=\ln x\underbrace{-1+O\left(\frac{\ln x}{x}\right)}_{O(1)}$$

$$\left(\sum_{d\le x}\frac{\Lambda(d)}{d}\right)-\ln x=O(1)-\frac{1}{x}\sum_{d\le x}\left(\Lambda(d)\,\varepsilon_{\frac{x}{d}}\right)$$

$$\left|\left(\sum_{d\le x}\frac{\Lambda(d)}{d}\right)-\ln x\right| \le |O(1)| + \left|\frac{1}{x}\sum_{d\le x}\left(\Lambda(d)\,\varepsilon_{\frac{x}{d}}\right)\right| \le$$

$$\le |O(1)| + \left|\frac{1}{x}\sum_{d\le x}\Lambda(d)\right| =$$

$$= |O(1)| + \left|\frac{1}{x}\,\psi(x)\right| =$$

$$= |O(1)| + \left|\frac{1}{x}\,O(x)\right| = |O(1)| + |O(1)| = O(1)$$

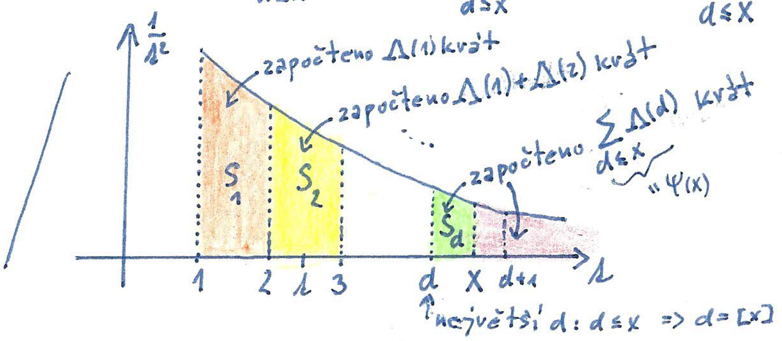
$$\Rightarrow \sum_{d\le x}\frac{\Lambda(d)}{d} = \ln x + O(1)$$

$\square$

Lema 12 Platí: $$\int_1^x \frac{\psi(t)}{t^2}\,dt = \ln x + O(1)$$

Důkaz: Lema 11: $\ln x + O(1) = \sum_{d\le x}\frac{\Lambda(d)}{d} \Rightarrow$

$$\ln x + O(1) = \sum_{d\le x}\frac{\Lambda(d)}{d} = \sum_{d\le x}\Lambda(d)\,\frac{1}{d} = \sum_{d\le x}\left(\Lambda(d)\int_d^\infty \frac{1}{t^2}\,dt\right) =$$



$t\in(1,2) \Rightarrow \psi(t) = \sum_{d\le t}\Lambda(d) = \Lambda(1) = \psi(1)$

$t\in\langle 2,3) \Rightarrow \psi(t) = \sum_{d\le t}\Lambda(d) = \sum_{d\le 2}\Lambda(d) = \Lambda(1)+\Lambda(2) = \psi(2)$

$t\in\langle 3,4) \Rightarrow \psi(t) = \Lambda(1)+\Lambda(2)+\Lambda(3) = \psi(3)$

$$= \Lambda(1)\int_1^2\frac{1}{t^2}\,dt + (\Lambda(1)+\Lambda(2))\int_2^3\frac{1}{t^2}\,dt + (\Lambda(1)+\Lambda(2)+\Lambda(3))\int_3^4\frac{1}{t^2}\,dt + \dots + \sum_{d\le x}\Lambda(d)\int_{[x]}^x\frac{1}{t^2}\,dt + \sum_{d\le x}\Lambda(d)\int_x^\infty\frac{1}{t^2}\,dt =$$

$$= \int_1^2\frac{\psi(t)}{t^2}\,dt + \int_2^3\frac{\psi(t)}{t^2}\,dt + \int_3^4\frac{\psi(t)}{t^2}\,dt + \dots + \int_{[x]}^x\frac{\psi(t)}{t^2}\,dt + \psi(x)\int_x^\infty\frac{1}{t^2}\,dt = \int_1^x\frac{\psi(t)}{t^2}\,dt + \psi(x)\left[\frac{-1}{t}\right]_x^\infty =$$

$$= \int_1^x\frac{\psi(t)}{t^2}\,dt + \underbrace{\frac{\psi(x)}{x}}_{\psi(x)=O(x)\ \dots\ \text{viz důkaz Lema 11}} = \int_1^x\frac{\psi(t)}{t^2}\,dt + O(1)$$

$$\Rightarrow \ln x + O(1) = \int_1^x\frac{\psi(t)}{t^2}\,dt + O(1) \Rightarrow$$

$$\ln x + O(1) = \int_1^x\frac{\psi(t)}{t^2}\,dt$$

$\square$

$\underset{\sim}{\text{Věta}}$ (Čebyševova 2): Jestliže $\lim\limits_{x\to\infty} \frac{\pi(x)}{\frac{x}{\ln x}}$ existuje, pak $\lim\limits_{x\to\infty} \frac{\pi(x)}{\frac{x}{\ln x}} = 1$, neboť:

$$\liminf_{x\to\infty} \frac{\pi(x)}{\frac{x}{\ln x}} \leq 1 \leq \limsup_{x\to\infty} \frac{\pi(x)}{\frac{x}{\ln x}}$$

<u>Důkaz:</u> Lema 10 říká:

$$\pi(x) = \frac{\psi(x)}{\ln x} + O\left(\frac{x}{\ln^2 x}\right) \quad / \cdot \frac{\ln x}{x}$$

$$\boxed{\frac{\pi(x)\ln x}{x} = \frac{\psi(x)}{x} + \underbrace{O\left(\frac{1}{\ln x}\right)}_{= o(1)}} \Rightarrow$$

Jestliže $\alpha = \limsup\limits_{x\to\infty} \frac{\pi(x)\ln x}{x}$, pak $\alpha = \limsup\limits_{x\to\infty} \frac{\psi(x)}{x}$ $\Rightarrow$

$$\Rightarrow \forall \varepsilon > 0 \quad \exists x_0 = x_0(\varepsilon) \quad : \quad \forall t > x_0 \ : \ \frac{\psi(t)}{t} \leq \alpha + \varepsilon$$

$$\Rightarrow \forall \varepsilon > 0 \quad \exists x_0 = x_0(\varepsilon) \quad : \quad \forall t > x_0 \ : \ \psi(t) \leq (\alpha+\varepsilon)\, t$$

$\Rightarrow$ Pro $x > x_0$ platí:

Lema 12: $$\ln x + O(1) = \int_1^x \frac{\psi(t)}{t^2}\,dt = \underbrace{\int_1^{x_0} \frac{\psi(t)}{t^2}\,dt}_{=\text{ konstanta pro dané } x_0 = O_\varepsilon(1)} + \int_{x_0}^x \frac{\psi(t)}{t^2}\,dt =$$

$$= O_\varepsilon(1) + \int_{x_0}^x \frac{\psi(t)}{t^2}\,dt \leq O_\varepsilon(1) + \int_{x_0}^x \frac{(\alpha+\varepsilon)\,t}{t^2}\,dt = O_\varepsilon(1) + (\alpha+\varepsilon)\int_{x_0}^x \frac{1}{t}\,dt \leq$$

$$\leq O_\varepsilon(1) + (\alpha+\varepsilon)\ln x \qquad \Rightarrow$$

Pro $x > x_0$ platí: $$\ln x + O(1) \leq O_\varepsilon(1) + (\alpha+\varepsilon)\ln x \qquad /:\ln x$$
$$1 + o(1) \leq o_\varepsilon(1) + (\alpha+\varepsilon)$$

Při $x \to \infty$ dostáváme: $$1 \leq \alpha + \varepsilon$$

Při $\varepsilon \to 0$ dostáváme: $$\boxed{1 \leq \alpha = \limsup_{x\to\infty} \frac{\pi(x)}{\frac{x}{\ln x}}}$$

Nyní označme $\beta = \liminf_{x\to\infty} \frac{\pi(x)\ln x}{x}$, pak $\beta = \liminf_{x\to\infty} \frac{\psi(x)}{x}$. $\Rightarrow$

$\Rightarrow \forall \varepsilon > 0 \; \exists x_0 = x_0(\varepsilon) : \forall s > x_0 : \frac{\psi(x)}{x} > \beta - \varepsilon$

$\Rightarrow \forall \varepsilon > 0 \; \exists x_0 = x_0(\varepsilon) : \forall s > x_0 : \psi(x) > (\beta - \varepsilon) x \quad \Rightarrow$

Pro $x > x_0$ platí:

Lema 12: $$\ln x + O(1) = \int_1^x \frac{\psi(s)}{s^2}\,ds \geq \int_{x_0}^x \frac{\psi(s)}{s^2}\,ds \geq \int_{x_0}^x \frac{(\beta-\varepsilon)s}{s^2}\,ds = (\beta-\varepsilon)\int_{x_0}^x \frac{1}{s}\,ds =$$
$$= (\beta-\varepsilon)(\ln x - \ln x_0) = (\beta-\varepsilon)\ln x \underbrace{-(\beta-\varepsilon)\ln x_0}_{O_\varepsilon(1)} \quad \Rightarrow$$

Pro $x > x_0$ platí: $$\ln x + O(1) \geq (\beta-\varepsilon)\ln x + O_\varepsilon(1) \qquad /:\ln x$$
$$1 + o(1) \geq (\beta-\varepsilon) + o_\varepsilon(1)$$

Při $x \to \infty$ dostáváme: $$1 \geq \beta - \varepsilon$$

Při $\varepsilon \to 0$ dostáváme: $$\boxed{1 \geq \beta = \liminf_{x\to\infty} \frac{\pi(x)}{\frac{x}{\ln x}}}$$

$\square$