Čebyševovy nerovnosti

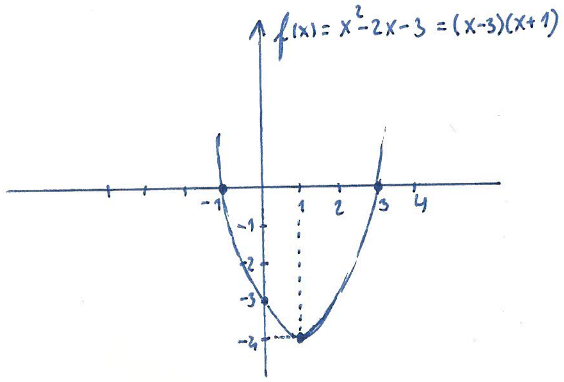
Def. ($O(g(n))$): Nechť $f(n)$ a $g(n)$ jsou posloupnosti reálných čísel. Potom

$$f(n) = O(g(n)) \iff \exists n_0 \in \mathbb{N}\ \exists C \in \mathbb{R}\ \forall n \in \mathbb{N}: n \geq n_0 \Rightarrow |f(n)| \leq C|g(n)|$$

Def. ($O(g(x))$): Nechť $f(x)$ a $g(x)$ jsou reálné funkce reálné proměnné. Potom

$$f(x) = O(g(x)) \iff \exists a \in \mathbb{R}\ \exists c \in \mathbb{R}\ \forall x \in \mathbb{R}: x \geq a \Rightarrow |f(x)| \leq C|g(x)|$$

Př.: Nechť $f(n) = n^2 - 2n - 3$



$$\Rightarrow \forall n \in \mathbb{N}: \quad -4 \leq n^2 - 2n - 3 \leq n^2 - 2n - 3 + 2n + 3$$

$$-4 \leq f(n) \leq n^2 \quad /: n^2 > 0$$

$$\underset{\downarrow 0^-}{\frac{-4}{n^2}} \leq \frac{f(n)}{n^2} \leq 1$$

$\Rightarrow$ Pro dost velká n platí:

$$-\frac{1}{2} \leq \frac{f(n)}{n^2} \leq 1 \quad \Rightarrow \quad \frac{|f(n)|}{n^2} \leq 1 \quad \Rightarrow \underline{f(n) = O(n^2)}$$

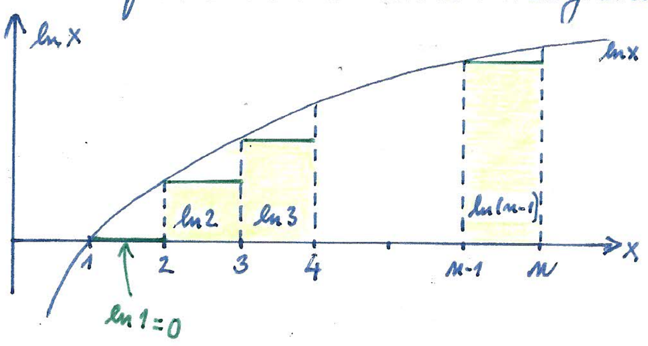
Ale!
$$\frac{|f(n)|}{n} = \frac{|n^2 - 2n - 3|}{|n|} = \left|\frac{n^2 - 2n - 3}{n}\right| = \left|n - 2 - \frac{3}{n}\right| \to +\infty$$

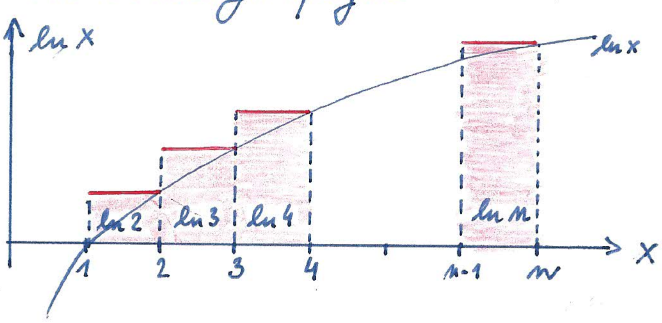
$$\Rightarrow \underline{f(n) \neq O(n)}$$

Lema 1: Platí: $\ln n! = \sum_{k=1}^{n} \ln k = n \cdot \ln n - n + O(\ln n)$

Důkaz: První rovnost plyne z toho, že logaritmus součinu je součet logaritmů.

Z definice Riemannova integrálu (viz. obrázky) plyne:





$$\overbrace{-\ln n + \ln n}^{=0} + \sum_{k=1}^{n-1} \ln k \leq \int_1^n \ln x \, dx \leq \sum_{k=2}^{n} \ln k + \overbrace{\ln 1}^{=0}$$

$$-\ln n + \sum_{k=1}^{n} \ln k \leq \int_1^n \ln x \, dx \leq \sum_{k=1}^{n} \ln k$$

$$\Big/ \int_1^n 1 \cdot \ln x \, dx = \begin{vmatrix} u' = 1 & v = \ln x \\ u = x & v' = \frac{1}{x} \end{vmatrix} = [x \cdot \ln x]_1^n - \int_1^n 1 \, dx = [x \cdot \ln x - x]_1^n = \underbrace{(n \cdot \ln n - n) - (1 \cdot \ln 1 - 1)}_{n \cdot \ln n - n + 1} \Big/$$

$$-\ln n + \sum_{k=1}^{n} \ln k \leq n \cdot \ln n - n + 1 \leq \sum_{k=1}^{n} \ln k \qquad \Big/ -1 - \sum_{k=1}^{n} \ln k$$

$$-1 - \ln n \leq n \cdot \ln n - n - \sum_{k=1}^{n} \ln k \leq -1 \qquad \Big/ \cdot (-1)$$

$$1 + \ln n \geq \underbrace{\sum_{k=1}^{n} \ln k - (n \cdot \ln n - n)}_{\text{označme } f(n)} \geq 1 \qquad \Big/ \cdot \frac{1}{\ln n}$$

$$\underbrace{\frac{1}{\ln n}}_{\to 0} + 1 \geq \frac{f(n)}{\ln n} \geq \underbrace{\frac{1}{\ln n}}_{\to 0} \quad \Rightarrow \quad f(n) = O(\ln n)$$

$$\Rightarrow \quad f(n) = \sum_{k=1}^{n} \ln k - (n \cdot \ln n - n) = O(\ln n)$$

$$\Rightarrow \quad \sum_{k=1}^{n} \ln k = n \cdot \ln n - n + O(\ln n)$$

$\square$

Def. (von Mangoldtova fce): Von Mangoldtova funkce $\Lambda : \mathbb{N} \to \mathbb{R}$ je dána předpisem:

$$\Lambda(d) = \begin{cases} \ln p & \Leftrightarrow \exists \alpha \in \mathbb{N}: d = p^{\alpha} \quad (p \text{ je prvočíslo}) \\ 0 & \Leftrightarrow \text{jindy} \quad (\text{tzn. } \forall \alpha \in \mathbb{N}: d \neq p^{\alpha}) \end{cases}$$

Př.:

		„2^1"	„3^1"	„2^2"	„5^1"		„7^1"	„2^3"	„3^2"		„11^1"	
d	1	2	3	4	5	6	7	8	9	10	11	...
$\Lambda(d)$	0	$\ln 2$	$\ln 3$	$\ln 2$	$\ln 5$	0	$\ln 7$	$\ln 2$	$\ln 3$	0	$\ln 11$	...

Př.: Určete $\sum_{p^{\alpha}/m} \ln p$ kde p je prvočíslo a $\alpha \in \mathbb{N}$.

1.) $m = 1 \Rightarrow \sum_{p^{\alpha}/1} \ln p = 0 = \ln 1$

2.) $m = 2 \Rightarrow \sum_{p^{\alpha}/2} \ln p = \ln 2$

3.) $m = 3 \Rightarrow \sum_{p^{\alpha}/3} \ln p = \ln 3$

4.) $m = 4 \Rightarrow \sum_{p^{\alpha}/4} \ln p = \underbrace{\ln 2}_{\text{za } p^{\alpha}=2} + \underbrace{\ln 2}_{\text{za } p^{\alpha}=4} = 2 \cdot \ln 2 = \ln 2^2 = \ln 4$

⋮

10) $m = 10 \Rightarrow \sum_{p^{\alpha}/10} \ln p = \underbrace{\ln 2}_{\text{za } p^{\alpha}=2} + \underbrace{\ln 5}_{\text{za } p^{\alpha}=5} = \ln 10$

Obecně: $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \Rightarrow \ln m = \sum_{i=1}^{n} \ln p_i^{\alpha_i} = \sum_{i=1}^{n} \alpha_i \ln p_i = \sum_{p_i/m} \left(\sum_{\alpha=1}^{\alpha_i} 1 \right) \ln p_i$

$$= \sum_{p^{\alpha}/m} \ln p$$

Př.

$$\ln 1 = \sum_{p^\alpha | 1} \ln p = 0$$

$$\ln 2 = \sum_{\substack{p^\alpha | 2 \\ p^\alpha = 2}} \ln p = \ln 2$$

$$\ln 3 = \sum_{\substack{p^\alpha | 3 \\ p^\alpha = 3}} \ln p = \ln 3$$

$$\ln 4 = \sum_{\substack{p^\alpha | 4 \\ p^\alpha = 2,\, 2^2}} \ln p = 2 \cdot \ln 2$$

$$\ln 5 = \sum_{\substack{p^\alpha | 5 \\ p^\alpha = 5}} \ln p = \ln 5$$

$$\ln 6 = \sum_{\substack{p^\alpha | 6 \\ p^\alpha = 2, 3}} \ln p = \ln 2 + \ln 3$$

$$\ln 7 = \sum_{\substack{p^\alpha | 7 \\ p^\alpha = 7}} \ln p = \ln 7$$

$$\ln 8 = \sum_{\substack{p^\alpha | 8 \\ p^\alpha = 2,\, 2^2,\, 2^3}} \ln p = 3 . \ln 2$$

$$\ln 9 = \sum_{\substack{p^\alpha | 9 \\ p^\alpha = 3,\, 3^2}} \ln p = 2 \cdot \ln 3$$

$$\sum_{m=1}^{9} \ln m = ?$$

$$\sum_{p^\alpha \le n} \ln p \left[\frac{n}{p^\alpha}\right] \Leftarrow$$

p^α	počet $\ln p$ za toto p^α v součtu
2	$4 = \left[\frac{9}{2}\right]$ $m = 2 = 1 \cdot 2$ $m = 4 = 2 \cdot 2$ $m = 6 = 3 \cdot 2$ $m = 4 \cdot 2$
2^2	$2 = \left[\frac{9}{2^2}\right]$ $m = 4$ $m = 2 \cdot 4 = 8$
2^3	$1 = \left[\frac{9}{2^3}\right]$ $m = 8$
3	$3 = \left[\frac{9}{3}\right]$ $m = 3$ $m = 2 \cdot 3 = 6$ $m = 3 \cdot 3 = 9$
3^2	$1 = \left[\frac{9}{3^2}\right]$ $m = 9$
5	$1 = \left[\frac{9}{5}\right]$ $m = 5$
7	$1 = \left[\frac{9}{7}\right]$ $m = 7$
p^α	$\left[\frac{n}{p^\alpha}\right]$ $m = p^\alpha < 2p^\alpha < \ldots < k \cdot p^\alpha \le n$

$k = \left[\frac{n}{p^\alpha}\right]$

má smysl uvažovat jen $p^\alpha \le n$

Lema 2: Platí:

$$\sum_{d \le n} \Lambda(d) \left[\frac{n}{d}\right] = n \cdot \ln n - n + O(\ln n)$$

Důkaz: Podle Lema 1 je $\ln n! = n \cdot \ln n - n + O(\ln n)$.
Stačí tedy dokázat rovnost:

$$\ln n! = \sum_{d \le n} \Lambda(d) \left[\frac{n}{d}\right].$$

Uvažme, že $(\ln 12 = \ln(2^2 \cdot 3) = \ln 2^2 + \ln 3 = 2 \cdot \ln 2 + \ln 3 = \ln 2 + \ln 2 + \ln 3)$

$$\ln m = \sum_{p^\alpha \mid m} \ln p$$

- na levé straně rovnosti je suma přes všechny dvojice (α, p) takové, že $p^\alpha \mid m$. Potom:

$$\ln n! = \sum_{1 \le m \le n} \ln m = \sum_{1 \le m \le n} \sum_{p^\alpha \mid m} \ln p = \sum_{p^\alpha \le n} \ln p \underbrace{\sum_{\substack{1 \le m \le n \\ p^\alpha \mid m}} 1}_{\substack{\text{jsou to čísla} \\ 1 \le p^\alpha, 2p^\alpha, \dots, k \cdot p^\alpha \le n \\ \text{těch je } \left[\frac{n}{p^\alpha}\right]}} =$$

$$= \sum_{p^\alpha \le n} \ln p \left[\frac{n}{p^\alpha}\right]$$

Připomeňme: $\Lambda(d) = \begin{cases} \ln p \iff d = p^\alpha \\ 0 \iff d \ne p^\alpha \end{cases} \Rightarrow$

$$\ln n! = \sum_{d \le n} \Lambda(d) \left[\frac{n}{d}\right]$$

$\square$

Defn (Funkce B, Ψ a χ): Definujme funkce B, Ψ a χ předpisem:

1.) $\forall n \in \mathbb{N}_0$: $B(n) = \sum_{d \le n} \Lambda(d)\left[\frac{n}{d}\right]$ a $\forall x \in \mathbb{R}^+$: $B(x) = B([x])$

2.) $\forall x \in \mathbb{R}^+$: $\Psi(x) = \sum_{d \le x} \Lambda(d)$

3.) $\forall x \in \mathbb{R}$: $\chi(x) = [x] - 2\left[\frac{x}{2}\right]$

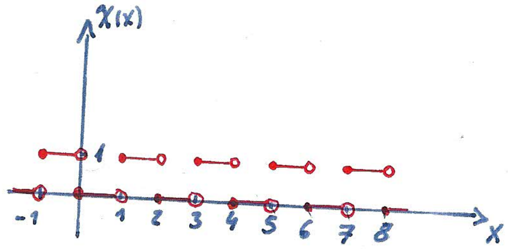
Lema 3 (Periodicita χ): Funkce χ je periodická s periodou 2.

Důkaz: Uvažme, že $\forall x \in \mathbb{R}$:

$$\chi(x+2) = [x+2] - 2\left[\frac{x+2}{2}\right] = [x] + 2 - 2\left[\frac{x}{2}+1\right] = [x] - 2\left[\frac{x}{2}\right] = \chi(x).$$

Navíc:

$$\left.\begin{array}{l} \forall x \in \langle 0,1) : \chi(x) = [x] - 2\left[\frac{x}{2}\right] = 0 \\ \forall x \in \langle 1,2) : \chi(x) = [x] - 2\Big[\underbrace{\tfrac{x}{2}}_{\in \langle \frac{1}{2},1)}\Big] = 1 \end{array}\right\} \Rightarrow$$



($\Rightarrow$ perioda nemůže být < 2) $\square$

Lema 3,5: Pro každé $x \in \mathbb{R}^+$: $B(x) = \sum_{d \le x} \Lambda(d)\left[\frac{x}{d}\right]$.

Důkaz: $\forall x \in \mathbb{R}^+$: $B(x) = B([x]) = \sum_{d \le [x]} \Lambda(d)\left[\frac{[x]}{d}\right] = \sum_{d \le x} \Lambda(d)\left[\frac{[x]}{d}\right] = \sum_{d \le x} \Lambda(d)\left[\frac{x}{d}\right]$,

neboť: $d \in \mathbb{N}, x \in \mathbb{R}^+ \Rightarrow$ $[x] = k \cdot d + r$, kde $r \in \{0,1,\dots,d-1\}$ $\Rightarrow x = k \cdot d + r + \underbrace{\varepsilon_x}_{\in\langle 0,1)} = k \cdot d + \underbrace{c}_{\in \langle 0,d)}$

$$\Rightarrow \left[\frac{[x]}{d}\right] = \left[\frac{kd+r}{d}\right] = \left[k + \frac{r}{d}\right] = k + \Big[\overbrace{\underbrace{\tfrac{r}{d}}_{<1}}^{0 \le}\Big] = k$$

$$\left[\frac{x}{d}\right] = \left[\frac{kd+c}{d}\right] = \left[k + \frac{c}{d}\right] = k + \Big[\underbrace{\tfrac{c}{d}}_{\in\langle 0,1)}\Big] = k$$

$\square$

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Def. $(O(g(x)))$: Nechť $f(x)$ a $g(x)$ jsou reálné funkce reálné proměnné. Potom

$$f(x) = O(g(x)) \iff \exists a \in \mathbb{R}\ \exists c \in \mathbb{R}\ \forall x \in \mathbb{R}: x \geq a \Rightarrow |f(x)| \leq C|g(x)|$$

Lemma: Nechť $f(n) = O(\ln n)$ a $\exists x_0 \in \mathbb{R}\ \forall x \in \mathbb{R}: x \geq x_0 \Rightarrow f^*(x) = f([x])$.

Potom $f^*(x) = O(\ln x)$

Důkaz: $f(n) = O(\ln n) \Rightarrow$

$$\exists n_0 \in \mathbb{N}\ \exists c \in \mathbb{R}\ \forall n \in \mathbb{N}: n \geq n_0 \Rightarrow \left|\frac{f(n)}{\ln n}\right| \leq C.$$

a dále: $\forall x \in \mathbb{R}: x \geq \max\{x_0, n_0\} \Rightarrow \left|\frac{f^*(x)}{\ln x}\right| = \left|\frac{f([x])}{\ln x}\right| \leq \left|\frac{f([x])}{\ln [x]}\right| \leq C.$

$\Rightarrow$

$$\exists a = \max\{x_0, n_0\} \in \mathbb{R}\ \exists c \in \mathbb{R}: \forall x \in \mathbb{R}: x \geq a \Rightarrow \left|\frac{f^*(x)}{\ln x}\right| \leq C,$$

$$\Rightarrow f^*(x) = O(\ln x)$$

$\square$