Lema 5: Platí:

$$B(x) = x \cdot \ln x - x + O(\ln x)$$

Důkaz: Víme (Lema 2), že

$$B(n) = \sum_{d \le n} \Lambda(d)\left[\frac{n}{d}\right] = n \cdot \ln n - n + O(\ln n)$$

$$\Rightarrow \quad f(n) = B(n) - (n \cdot \ln n - n) = O(\ln n)$$

Z Lema 4 pak plyne:

$$f^*(x) = f([x]) = B([x]) - ([x] \cdot \ln [x] - [x]) = O(\ln x)$$

a tak:

$$B(x) = B([x]) = [x] \cdot \ln [x] - [x] + O(\ln x) =$$

$$= (x - \overset{\in [0,1)}{\varepsilon_x}) \ln [x] - x + \overset{= O(\ln x)}{\varepsilon_x} + O(\ln x) =$$

$$= x \cdot \ln [x] \underbrace{- \varepsilon_x \ln [x] + \varepsilon_x}_{O(\ln x)} + O(\ln x) \Rightarrow$$

Stačí dokázat, že: $x \cdot \ln [x] = x \cdot \ln x + O(\ln x)$:

$$0 \le x \cdot \ln x - x \cdot \ln [x] \le x \cdot \ln x - x \cdot \ln (x-1)$$ / pro dost velká x

$$0 \le x \cdot \ln x - x \cdot \ln [x] \le x \cdot (\ln x - \ln (x-1))$$

$$0 \le x \cdot \ln x - x \cdot \ln [x] \le \ln \left(\frac{x}{x-1}\right)^x$$

$$= \ln \left(1 + \frac{1}{x-1}\right)^{x-1+1} \to \ln e = 1$$

$$\Rightarrow \limsup_{x \to \infty} \frac{|x \cdot \ln x - x \cdot \ln [x]|}{\ln x} \le 1$$

$$\Rightarrow \quad x \cdot \ln x - x \cdot \ln [x] = O(\ln x)$$

$\square$

Lema 6: Označme $B_2(x) = B(x) - 2B(\frac{x}{2})$. Potom $\forall x \in \mathbb{R}^+$:

$$\Psi(x) - \Psi(\tfrac{x}{2}) \leq B_2(x) \leq \Psi(x)$$

Důkaz: Pro každé $x \in \mathbb{R}^+$:

$$B_2(x) = B(x) - 2B(\tfrac{x}{2}) = \qquad \left(\text{Lema 3.5}: B(x) = \sum_{d\leq x} \Lambda(d)[\tfrac{x}{d}]\right)$$

$$= \sum_{d\leq x} \Lambda(d)[\tfrac{x}{d}] - 2\sum_{d\leq \frac{x}{2}} \Lambda(d)[\tfrac{x}{2d}] = \qquad \left(\text{pro } d > \tfrac{x}{2} \text{ je } [\tfrac{x}{2d}] = 0\right)$$

$$= \sum_{d\leq x} \Lambda(d)[\tfrac{x}{d}] - 2\sum_{d\leq x} \Lambda(d)[\tfrac{x}{2d}] =$$

$$= \sum_{d\leq x} \Lambda(d)\big(\underbrace{[\tfrac{x}{d}] - 2[\tfrac{x}{2d}]}_{\chi(\frac{x}{d})}\big) \Rightarrow \boxed{B_2(x) = \sum_{d\leq x} \Lambda(d)\chi(\tfrac{x}{d})}$$

1.) $$B_2(x) = \sum_{d\leq x} \Lambda(d)\underbrace{\chi(\tfrac{x}{d})}_{\in\{0,1\}} \leq \sum_{d\leq x} \Lambda(d) = \Psi(x)$$

2.) $$B_2(x) = \sum_{d\leq x} \Lambda(d)\chi(\tfrac{x}{d}) =$$

$$= \underbrace{\sum_{d\leq \frac{x}{2}} \Lambda(d)\chi(\tfrac{x}{d})}_{\geq 0} + \sum_{\frac{x}{2}<d\leq x} \Lambda(d)\chi(\tfrac{x}{d}) \geq \qquad \left(\tfrac{x}{2} < d \leq x \Leftrightarrow 1 \leq \tfrac{x}{d} < 2 \Rightarrow \Rightarrow \chi(\tfrac{x}{d}) = 1\right)$$

$$\geq \sum_{\frac{x}{2}<d\leq x} \Lambda(d) =$$

$$= \sum_{d\leq x} \Lambda(d) - \sum_{d\leq \frac{x}{2}} \Lambda(d) = \Psi(x) - \Psi(\tfrac{x}{2})$$

$\square$

Lema 7: Nechť $B_2(x) = B(x) - 2B(\frac{x}{2})$. Potom
$$B_2(x) = x\ln 2 + O(\ln x)$$

Důkaz: Lema 5 říká, že $B(x) = x\ln x - x + O(\ln x)$ $\Rightarrow$

$$B_2(x) = x\ln x - x + O(\ln x) - 2\left(\tfrac{x}{2}\ln\tfrac{x}{2} - \tfrac{x}{2} + O(\ln \tfrac{x}{2})\right) =$$

$$= x\ln x - x - x\ln\tfrac{x}{2} + x - 2O(\ln\tfrac{x}{2}) + O(\ln x) =$$

$$= x\ln x + x\ln\tfrac{2}{x} - 2O(\ln\tfrac{x}{2}) + O(\ln x) =$$

$$= x(\ln x + \ln\tfrac{2}{x}) - 2O(\ln\tfrac{x}{2}) + O(\ln x) =$$

$$= x\ln 2 + O(\ln x - \ln 2) + O(\ln x) =$$

$$= x\ln 2 + O(\ln x) + O(\ln x) =$$

$$= x\ln 2 + O(\ln x).$$

$\square$

Pozn: $f(x) = O(\ln x - \ln 2) \Rightarrow f(x) = O(\ln x)$ neboť:

$f(x) = O(\ln x - \ln 2) \Leftrightarrow \exists a \in \mathbb{R}\ \exists c \in \mathbb{R}\ \forall x \in \mathbb{R}:\ x \geq a \Rightarrow |f(x)| \leq c|\ln x - \ln 2|$

pro $x \geq 2$ je $|\ln x - \ln 2| < |\ln x|$ $\Rightarrow$

$\exists a^* = \max\{a, 2\}\ \exists c \in \mathbb{R}\ \forall x \in \mathbb{R}:\ x \geq a \Rightarrow |f(x)| \leq c|\ln x - \ln 2| < c|\ln x|$

$\Rightarrow f(x) = O(\ln x)$

Věta (Čebyševova): Pro funkci $\Psi(x)$ platí:

$$x\ln 2 + O(\ln x) \leq \Psi(x) \leq x\cdot\ln 4 + O(\ln^2 x)$$

Důkaz: Z Lema 6 plyne:

1.) $\Psi(x) \geq B_2(x) =$ (Lema 6: $B(x)_2 \leq \Psi(x)$)

$= x\ln 2 + O(\ln x)$ (Lema 7: $B_2(x) = x\ln 2 + O(\ln x)$)

2.) Zvolme $k_x = \left[\frac{\ln x}{\ln 2}\right] \Rightarrow$

$$k_x \leq \frac{\ln x}{\ln 2} < k_x + 1$$
$$k_x(\ln 2) \leq \ln x < (k_x+1)\ln 2$$
$$2^{k_x} \leq x < 2^{k_x+1}$$
$$\frac{1}{2} \leq \frac{x}{2^{k_x+1}} < 1$$

(Budeme uvažovat $x \geq 2$)

$$\Rightarrow \Psi\left(\frac{x}{2^{k_x+1}}\right) = \sum_{d \leq \frac{x}{2^{k_x+1}}} \Lambda(d) = 0$$

Z Lema 6 plyne:

$$\left.\begin{array}{l} \Psi(x) \leq B_2(x) + \Psi\left(\frac{x}{2}\right) \\ \Downarrow \\ \Psi\left(\frac{x}{2}\right) \leq B_2\left(\frac{x}{2}\right) + \Psi\left(\frac{x}{4}\right) \\ \Downarrow \\ \vdots \\ \Psi\left(\frac{x}{2^{k_x}}\right) \leq B\left(\frac{x}{2^{k_x}}\right) + \underbrace{\Psi\left(\frac{x}{2^{k_x+1}}\right)}_{=0} \end{array}\right\} \Rightarrow$$

$$\Psi(x) \leq B_2(x) + \Psi\left(\frac{x}{2}\right) \leq$$
$$\leq B_2(x) + B_2\left(\frac{x}{2}\right) + \Psi\left(\frac{x}{4}\right) \leq \dots$$
$$\leq \sum_{0 \leq j \leq k_x} B_2\left(\frac{x}{2^j}\right)$$

- Pro $x \geq 2$ proto platí:

$$\boxed{\psi(x) \leq \sum_{0 \leq j \leq k_x} B_2\left(\frac{x}{2^j}\right)}$$

Lema 7 říká: $B_2\left(\frac{x}{2^j}\right) = \frac{x}{2^j}\ln 2 + O\left(\ln \frac{x}{2^j}\right) \Rightarrow \forall j \in \{0, 1, \dots, k_x\}$:

$$B_2\left(\frac{x}{2^j}\right) = \frac{x}{2^j}\ln 2 + O(\ln x - \ln 2^j)$$

$$B_2\left(\frac{x}{2^j}\right) = \frac{x}{2^j}\cdot\ln 2 + O(\ln x)$$

$\Rightarrow$

$$\psi(x) \leq \sum_{0 \leq j \leq k_x}\left(\frac{x}{2^j}\ln 2 + O(\ln x)\right) =$$

$$= x\cdot\ln 2\cdot\sum_{0 \leq j \leq k_x}\frac{1}{2^j} + \sum_{0 \leq j \leq k_x} O(\ln x) =$$

$$= x\cdot\ln 2\cdot\frac{1-\left(\frac{1}{2}\right)^{k_x+1}}{1-\frac{1}{2}} + (k_x+1)\,O(\ln x) \leq$$

$$\leq x\cdot 2\cdot\ln 2 + \left(\left[\frac{\ln x}{\ln 2}\right]+1\right)O(\ln x) \leq$$

$$\leq x\cdot\ln 4 + \left(\frac{\ln x}{\ln 2} + \underset{\leq \frac{\ln x}{\ln 2}}{1}\right)O(\ln x) \leq \qquad \Big/\ \text{uvažujeme } x \geq 2 \Rightarrow \frac{\ln x}{\ln 2} \geq 1$$

$$\leq x\cdot\ln 4 + \underbrace{\left(\frac{2}{\ln 2}\ln x\right)\overbrace{O(\ln x)}^{g(x)}}_{f(x)}$$

$g(x) = O(\ln x) \Rightarrow \exists a \in \mathbb{R}\ \exists c \in \mathbb{R}\ \forall x \in \mathbb{R}: x > a \Rightarrow \left|\overbrace{\left(\frac{2}{\ln 2}\ln x\right)\cdot g(x)}^{f(x)}\right| \leq$

$$\leq \left|\frac{2}{\ln 2}\ln x\right|\cdot c\,|\ln x| =$$

$$= \underbrace{\left(\frac{2c}{\ln 2}\right)}_{=c^*}\cdot|\ln^2 x| \qquad \Rightarrow f(x) = O(\ln^2 x)$$

$\square$