Lema 8: Pro každé $x \in \mathbb{R}^+$, $x \geq 2$ platí:

$$\psi(x) \leq \pi(x) \cdot \ln x \leq 2\,\psi(x)$$

Důkaz:

$$\psi(x) = \sum_{d \leq x} \Lambda(d) = \sum_{d \leq x} \begin{cases} 0 \Leftrightarrow d \neq p^\alpha,\ \alpha \in \mathbb{N} \\ \ln p \Leftrightarrow d = p^\alpha,\ \alpha \in \mathbb{N} \end{cases} = \sum_{p^\alpha \leq x} \ln p$$

Poslední suma je sčítána přes všechny dvojice (p,α) takové, že $p^\alpha \leq x$. Kolik jich je pro pevně zvolené p?

$$p^1 < p^2 < \dots < p^\alpha \leq x < p^{\alpha+1}$$

$$\ln p^\alpha \leq \ln x < \ln p^{\alpha+1}$$

$$\alpha \ln p \leq \ln x < (\alpha+1) \ln p$$

$$\alpha \leq \frac{\ln x}{\ln p} < \alpha + 1$$

$$\alpha = \left[\frac{\ln x}{\ln p}\right] \Rightarrow \text{Je jich } \left[\frac{\ln x}{\ln p}\right] \Rightarrow$$

$$\boxed{\psi(x) = \sum_{p \leq x} \left[\frac{\ln x}{\ln p}\right] \ln p}$$

Pro $y \in \mathbb{R}$, $y \geq 1$: $[y] \leq y < [y]+1 \leq 2[y]$. Protože $p \leq x$ je $y = \frac{\ln x}{\ln p} \geq 1 \Rightarrow$

$$\psi(x) = \sum_{p \leq x} \underbrace{\left[\frac{\ln x}{\ln p}\right]}_{[y]} \ln p \leq \sum_{p \leq x} \frac{\ln x}{\ln p} \ln p = \sum_{p \leq x} \ln x = \pi(x) \cdot \ln x = \sum_{p \leq x} \ln x =$$

$$= \sum_{p \leq x} \underbrace{\frac{\ln x}{\ln p}}_{y} \ln p \leq \sum_{p \leq x} \underbrace{2\left[\frac{\ln x}{\ln p}\right]}_{2[y]} \ln p = 2 \sum_{p \leq x} \left[\frac{\ln x}{\ln p}\right] \ln p = 2\,\psi(x)$$

$\square$

Lema 9: Nechť $\pi(x) = |\{p \in \mathbb{P} \mid p \leq x\}|$ pro $x \in \mathbb{R}^+$. Potom

$$\lim_{x \to \infty} \frac{\pi(x)}{x} = 0.$$

Důkaz: Z Lema 8 plyne, že pro $x \in \mathbb{R}$, $x \geq 2$:

$$\pi(x) \ln x \leq 2\Psi(x)$$

Čebyševova věta říká, že $\Psi(x) \leq x \cdot \ln 4 + O(\ln^2 x)$. Proto

$$\pi(x) \ln x \leq 2x \ln 4 + O(\ln^2 x)$$

$$\pi(x) \leq \frac{x\, 2\ln 4}{\ln x} + O(\ln x)$$

$$\frac{\pi(x)}{x} \leq \frac{2\ln 4}{\ln x} + O\left(\frac{\ln x}{x}\right) \quad (*) \quad \Big/ \; f(x) = o(1) \Leftrightarrow \lim_{x\to\infty} f(x) = 0$$

$$\frac{\pi(x)}{x} \leq \frac{2\ln 4}{\ln x} + o(1)$$

$$0 \leq \lim_{x\to\infty} \frac{\pi(x)}{x} \leq \lim_{x\to\infty} \overset{\nearrow 0}{\frac{2\ln 4}{\ln x}} + \overset{\nearrow 0}{o(1)} = 0$$

$\square$

Lema 10: Platí:

$$\pi(x) = \frac{\psi(x)}{\ln x} + O\left(\frac{x}{\ln^2 x}\right)$$

Důkaz: Uvažujeme $x \geq 2$ $\quad (\exists p \in \mathbb{P} : p \leq x)$

$$0 \underset{\text{Lema 8}}{\leq} \pi(x)\ln x - \psi(x) = \sum_{p \leq x} \Big(\underbrace{\ln x - \underset{\leq \frac{\ln x}{\ln p}}{\left[\frac{\ln x}{\ln p}\right]} \ln p}_{\geq 0}\Big) =$$

$$= \sum_{p \leq \sqrt{x}} \Big(\ln x - \underbrace{\left[\frac{\ln x}{\ln p}\right]}_{\geq \left(\frac{\ln x}{\ln p} - 1\right)} \ln p\Big) + \sum_{\sqrt{x} < p \leq x} \Big(\ln x - \left[\frac{\ln x}{\ln p}\right] \ln p\Big) \leq$$

$\sqrt{x} < p \leq x$

$\frac{1}{2}\ln x < \ln p \leq \ln x$

$1 \leq \frac{\ln x}{\ln p} < 2$

$\left[\frac{\ln x}{\ln p}\right] = 1$

$$\leq \sum_{p \leq \sqrt{x}} \ln p + \sum_{\sqrt{x} < p \leq x} \ln x - \ln p \leq$$

$p \leq \sqrt{x} \Rightarrow \ln p \leq \ln\sqrt{x}$

$1 \leq \frac{\ln\sqrt{x}}{\ln p}$

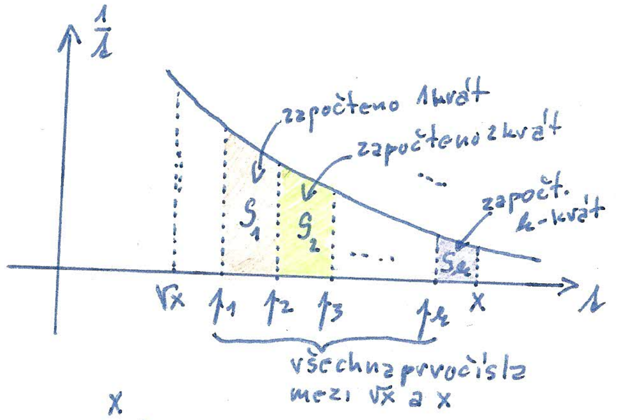
$\left[\frac{\ln\sqrt{x}}{\ln p}\right] \geq 1$

$\Rightarrow \ln p \leq \left[\frac{\ln\sqrt{x}}{\ln p}\right] \ln p$

$$\leq \underbrace{\sum_{p \leq \sqrt{x}} \left[\frac{\ln\sqrt{x}}{\ln p}\right] \ln p}_{\parallel} + \sum_{\sqrt{x} < p \leq x} \int_p^x \frac{1}{t}\, dt =$$

(Viz. důkaz Lema 8)

$$= \psi(\sqrt{x}) + \sum_{\sqrt{x} < p \leq x} \int_p^x \frac{1}{t}\, dt =$$



$$= \psi(\sqrt{x}) + 1 \cdot \int_{p_1}^{p_2} \frac{1}{t}\, dt + 2 \cdot \int_{p_2}^{p_3} \frac{1}{t}\, dt + \ldots + k \int_{p_k}^{x} \frac{1}{t}\, dt \leq$$

$$\leq \psi(\sqrt{x}) + \pi(p_1) \int_{p_1}^{p_2} \frac{1}{t}\, dt + \pi(p_2) \int_{p_2}^{p_3} \frac{1}{t}\, dt + \ldots + \pi(p_k) \int_{p_k}^{x} \frac{1}{t}\, dt =$$

$\forall t \in (p_1, p_2) :$ $\pi(t) = \pi(p_1)$

$$= \psi(\sqrt{x}) + \int_{p_1}^{p_2} \frac{\pi(t)}{t}\, dt + \int_{p_2}^{p_3} \frac{\pi(t)}{t}\, dt + \ldots + \int_{p_k}^{x} \frac{\pi(t)}{t}\, dt \leq$$

$$\leq \psi(\sqrt{x}) + \int_{\sqrt{x}}^{x} \frac{\pi(t)}{t}\, dt$$

- A tak pro $x \geq 2$:

$$\boxed{0 \leq \pi(x) \ln x - \psi(x) \leq \psi(\sqrt{x}) + \int_{\sqrt{x}}^{x} \frac{\pi(t)}{t}\, dt}$$

Dokažme, že $\int_{\sqrt{x}}^{x} \frac{\pi(t)}{t}\, dt = O\left(\frac{x}{\ln x}\right)$:

V důkazu Lema 9 nerovnost (*) říká, že

$$\frac{\pi(x)}{x} \leq \frac{2 \ln 4}{\ln x} + O\left(\frac{\ln x}{x}\right)$$

Tzn. pro dost velká x existuje $C \in \mathbb{R}$:

$$\frac{\pi(x)}{x} \leq \frac{2 \ln 4}{\ln x} + C \cdot \frac{\ln x}{x}$$

$$\Rightarrow \int_{\sqrt{x}}^{x} \frac{\pi(t)}{t}\, dt \leq \int_{\sqrt{x}}^{x} \left(\frac{2\ln 4}{\ln t} + C \frac{\ln t}{t}\right) dt = \int_{\sqrt{x}}^{x} \frac{2 \ln 4}{\ln t}\, dt + C \int_{\sqrt{x}}^{x} \frac{\ln t}{t}\, dt \leq$$

$$\leq \frac{2 \ln 4}{\ln \sqrt{x}} (x - \sqrt{x}) + C \left[\frac{\ln^2 t}{2}\right]_{\sqrt{x}}^{x} \leq \frac{x - \sqrt{x}}{\frac{1}{2} \ln x} 2 \ln 4 + C \frac{\ln^2 x}{2} \leq$$

$$\leq \frac{x}{\ln x} (4 \ln 4) + C \frac{\ln^2 x}{2} = \frac{x}{\ln x} \left(4 \ln 4 + C \frac{\ln^3 x}{2x}\right) \overset{\text{pro dost velká } x}{\leq'}$$

$$\lim_{x \to \infty} \frac{\ln^3 x}{x} \overset{l'H}{=} \lim_{x \to \infty} \frac{(3 \ln^2 x) \frac{1}{x}}{1} \overset{l'H}{=} \lim_{x \to \infty} \frac{(6 \ln x) \frac{1}{x}}{1} \overset{l'H}{=} \lim_{x \to \infty} \frac{6 \frac{1}{x}}{1} = 0$$

$\Rightarrow$ pro dost velká x : $\frac{\ln^3 x}{x} \leq \frac{2}{C}$

$$\leq' \frac{x}{\ln x} (4 \ln 4 + 1) \qquad \Rightarrow \int_{\sqrt{x}}^{x} \frac{\pi(t)}{t}\, dt = O\left(\frac{x}{\ln x}\right)$$

$$\Rightarrow \boxed{0 \leq \pi(x) \ln x - \psi(x) \leq \psi(\sqrt{x}) + O\left(\frac{x}{\ln x}\right)}$$

Čebyševova věta říká, že:

$$\sqrt{x}\ln 2 + O(\ln\sqrt{x}) \leq \psi(\sqrt{x}) \leq \sqrt{x}\ln 4 + O(\ln^2\sqrt{x})$$

$$\sqrt{x}\ln 2 + O(\tfrac{1}{2}\ln x) \leq \psi(\sqrt{x}) \leq \sqrt{x}\ln 4 + O(\tfrac{1}{4}\ln^2 x)$$

$$\sqrt{x}\ln 2 + O(\ln x) \leq \psi(\sqrt{x}) \leq \sqrt{x}\ln 4 + O(\ln^2 x) \qquad /: \frac{x}{\ln x}$$

$$\underbrace{\frac{\ln x}{\sqrt{x}}}_{\to 0}\ln 2 + \underbrace{O\left(\frac{\ln^2 x}{x}\right)}_{\to 0} \leq \frac{\psi(\sqrt{x})}{\frac{x}{\ln x}} \leq \underbrace{\frac{\ln x \ln 4}{\sqrt{x}}}_{\to 0} + \underbrace{O\left(\frac{\ln^3 x}{x}\right)}_{\to 0} \qquad \Rightarrow$$

Při $x \to \infty$: $\dfrac{\psi(\sqrt{x})}{\frac{x}{\ln x}} \to 0 \qquad \Rightarrow$

$\forall \varepsilon > 0\ \exists x_0 \in \mathbb{R} : \ x > x_0 \Rightarrow \left|\dfrac{\psi(\sqrt{x})}{\frac{x}{\ln x}}\right| < \varepsilon \quad \Rightarrow \quad \boxed{\psi(\sqrt{x}) = O\left(\frac{x}{\ln x}\right)} \quad \Rightarrow$

$$0 \leq \pi(x)\ln x - \psi(x) \leq O\left(\frac{x}{\ln x}\right) \qquad /: \ln x$$

$$0 \leq \pi(x) - \frac{\psi(x)}{\ln x} \leq O\left(\frac{x}{\ln^2 x}\right)$$

$$\pi(x) - \frac{\psi(x)}{\ln x} = O\left(\frac{x}{\ln^2 x}\right)$$

$\square$