Exponenciální funkce

$$f(x) = a^{x} = e^{x \cdot \ln a}$$

exponent

základ $\in \mathbb{R}^{+} \setminus \{1\}$

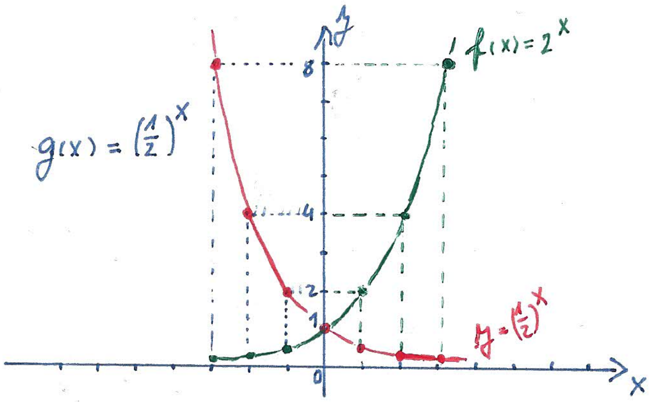
$$\boxed{f(x) = e^{x} = \exp(x) := 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!}}$$

$\boxed{D_f = \mathbb{R}}$

$D(f) = \mathbb{R}$; $H(f) = (0, \infty)$;

- je prostá
- je omezená zdola
- není sudá ani lichá
- není periodická

Př.: Načrtněte graf funkce $f(x) = 2^x$ a $g(x) = \left(\frac{1}{2}\right)^x$



Př.: Vyřešte exponenciální rovnici:

1.) $5^{3x+2} = 25^{x+1}$

$5^{3x+2} = (5^2)^{x+1}$

$3x+2 = 2x+2$

$\underline{\underline{x = 0}}$

2.) $8^x = 16^{2-x}$

$(2^3)^x = (2^4)^{2-x}$

$3x = 8-4x$

$\underline{\underline{x = \frac{8}{7}}}$

3.) $3^{x-2} = \left(\frac{1}{3}\right)^{-2x}$

$3^{x-2} = (3^{-1})^{-2x}$

$x-2 = 2x$

$\underline{\underline{x = -2}}$

4.) $2^{3x-4} = \left(\frac{1}{8}\right)^{x+1}$

$2^{3x-4} = (2^{-3})^{x+1}$

$3x-4 = -3x-3$

$\underline{\underline{x = \frac{1}{6}}}$

5.) $2^x \cdot 5^x = 0{,}1\,(10^{x-1})^5$

$10^x = 10^{-1} \cdot 10^{5x-5}$

$10^x = 10^{5x-6}$

$x = 5x-6$

$\underline{\underline{x = \frac{6}{4} = \frac{3}{2}}}$

6.) $4^x \cdot 5^{x+1} = 5 \cdot 20^{2-x}$

$20^x \cdot 5 = 5 \cdot 20^{2-x}$

$x = 2-x$

$\underline{\underline{x = 1}}$

Př. Vyriešte exponenciální rovnici :

1.)

$$3^{x+1} = 2^{2x+3}$$

$$3 \cdot 3^x = 8 \cdot 2^{2x}$$

$$3 \cdot 3^x = 8 \cdot 4^x$$

$$\frac{3^x}{4^x} \quad \frac{8}{3}$$

$$\left(\frac{3}{4}\right)^x = \frac{8}{3} \quad / \ln$$

$$\ln \left(\frac{3}{4}\right)^x = \ln \frac{8}{3}$$

$$x \cdot \ln \left(\frac{3}{4}\right) = \ln \frac{8}{3}$$

$$x = \frac{\ln \left(\frac{3}{4}\right)}{\ln \left(\frac{8}{3}\right)}$$

2.)

$$5^{2x-3} = 8^{3x+7} \quad / \ln$$

$$\ln 5^{2x-3} = \ln 8^{3x+7}$$

$$(2x-3) \ln 5 = (3x+7) \ln 8$$

$$(2 \ln 5) x - 3 \ln 5 = (3 \cdot \ln 8) x + 7 \cdot \ln 8$$

$$x (2 \ln 5 - 3 \ln 8) = 7 \cdot \ln 8 + 3 \ln 5$$

$$x (\ln 5^2 - \ln 8^3) = \ln 8^7 + \ln 5^3$$

$$x \ln \frac{5^2}{8^3} = \ln 8^7 \cdot 5^3$$

$$x = \frac{\ln 8^7 \cdot 5^3}{\frac{5^2}{8^3}}$$

Př. Vyriešte exponenciální nerovnici :

1.)

$$3^{2x} > 5^{x-1} \quad / \ln$$

$$\ln 3^{2x} > \ln 5^{x-1}$$

$$2x \cdot \ln 3 > (x-1) \ln 5$$

$$x \cdot \ln 3^2 > x \cdot \ln 5 - \ln 5$$

$$x (\ln 9 - \ln 5) > -\ln 5$$

$$x > \frac{-\ln 5}{\ln \frac{9}{5}}$$

$$x \in \left(-\frac{\ln 5}{\ln \frac{9}{5}}, \infty\right)$$

2.)

$$2^{8x-1} \leq \left(\frac{1}{3}\right)^{4x+6}$$

$$2^{8x-1} \leq (3^{-1})^{4x+6}$$

$$2^{8x-1} \leq 3^{-4x-6} \quad / \ln$$

$$\ln 2^{8x-1} \leq \ln 3^{-4x-6}$$

$$(8x-1) \ln 2 \leq (-4x-6) \ln 3$$

$$8x \ln 2 - \ln 2 \leq -4x \ln 3 - 6 \ln 3$$

$$x \ln 2^8 - \ln 2 \leq x \ln 3^{-4} - \ln 3^6$$

$$x (\ln 2^8 - \ln 3^{-4}) \leq \ln 2 - \ln 3^6$$

$$x \ln \frac{2^8}{3^{-4}} \leq \ln \frac{2}{3^6}$$

$$x \ln (2^8 \cdot 3^4) \leq \ln (2 \cdot 3^{-6})$$

$$x \leq \frac{\ln (2 \cdot 3^{-6})}{\ln (2^8 \cdot 3^4)}$$

$$x \in \left(-\infty, \frac{\ln (2 \cdot 3^{-6})}{\ln (2^8 \cdot 3^4)}\right>$$

Logaritmická funkce

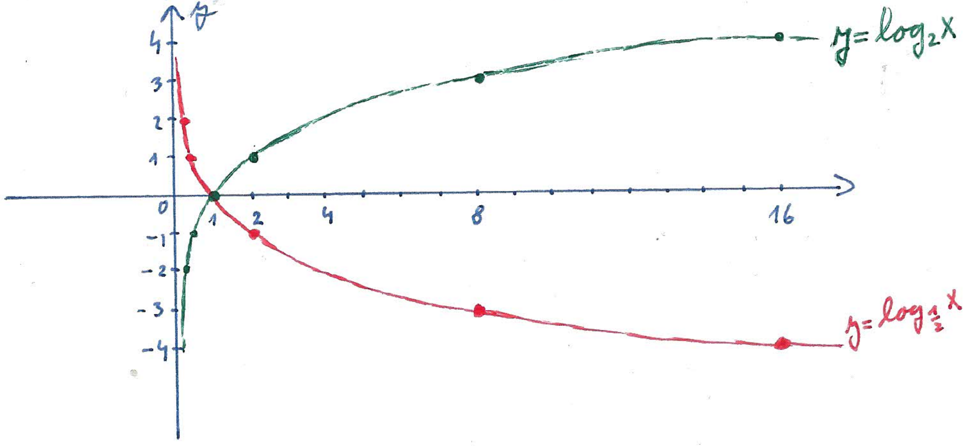
$\log_a x = y \iff a^y = x$

základ logaritmu $\in \mathbb{R}^+ \setminus \{1\}$

$D(f) = (0, \infty)$; $H(f) = \mathbb{R}$

- je prostá ; - není: sudá, lichá, periodická, omezená
- $\log_a x = \frac{\ln x}{\ln a}$

Př: Načrtněte graf funkce $f(x) = \log_2(x)$; $g(x) = \log_{\frac{1}{2}} x$



Př: Vyřešte logaritmickou rovnici:

1.) $x = \log_2 16$

$x = \log_2(2^4) = \underline{\underline{4}}$

2.) $x = \log_3 27$

$x = \log_3(3^3) = \underline{\underline{3}}$

3.) $x = \log_{\frac{1}{9}} 3$

$x = \log_{\frac{1}{9}}(9^{\frac{1}{2}}) = \log_{\frac{1}{9}}\left(\frac{1}{9}\right)^{-\frac{1}{2}} = \underline{\underline{-\frac{1}{2}}}$

4.) $x = \log_3 18 + \log_3 \frac{3}{2}$

$x = \log_3 18 \cdot \frac{3}{2} = \log_3 \overset{3^3}{\overset{\|}{27}} = \underline{\underline{3}}$

5.) $x = \log_{10} 500 - \log_{10} 5$

$x = \log_{10} \frac{500}{5} = \log_{10} 10^2 = \underline{\underline{2}}$

6.) $x = \log_{\frac{1}{5}} 5 + \log_5 \frac{1}{125}$

$x = -1 + (-3) = \underline{\underline{-4}}$

Př. Vyřešte logaritmickou rovnici.

1.) $\log_2 (\overbrace{4x+8}^{\Rightarrow x \in (-2,\infty)}) = 2$

$4x + 8 = 2^2$

$4x = -4$

$\underline{\underline{x = -1}}$

2.) $\overset{\Rightarrow x \in (0,\infty)}{\log_{10} x} - \log_{10} 5 = 2$

$\log_{10} \frac{x}{5} = 2$

$\frac{x}{5} = 10^2$

$\underline{\underline{x = 500}}$

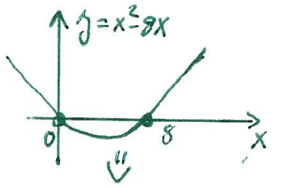
3.) $\log_3 (x^2 - 8x) = 2$

$x^2 - 8x = 3^2$

$x^2 - 8x - 9 = 0$

$(x-9)(x+1) = 0$

$x = \left\langle \begin{matrix} \underline{\underline{9}} \\ \underline{\underline{-1}} \end{matrix} \right.$



$\Downarrow$

$x \in (-\infty, 0) \cup (8, \infty)$

4.) Pozn.: U tohoto typu příkladu:

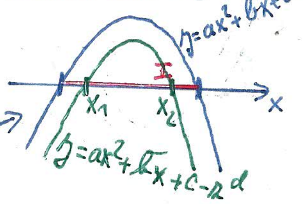
$\log_z (ax^2 + bx + c) = d$

musí I.) $x \in I = \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}, \frac{-b + \sqrt{b^2 - 4ac}}{2a} \right)$ pro $a < 0$

a $ax^2 + bx + c = z^d \Rightarrow ax^2 + bx + c - z^d = 0$

$\Rightarrow \underline{\underline{x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4a(c - z^d)}}{2a}}} \in I$

$\uparrow$ jistě



II.) pro $a > 0$ analogicky

Pozn. U typu př. jako v 1.) nalezené x také bude vždy vyhovovat.

Př. Vyřešte nerovnici:

1.) $\log_3 (-3x + 9) > 2$

I.) x musí splňovat: $-3x + 9 > 0$

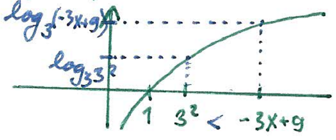
$-3x > -9 \quad /:(-3)$

$\underline{x < 3}$

II.) $\log_3 (-3x + 9) > 2$

$\log_3 (-3x + 9) > \log_3 (3^2)$

(y = $\log_3 x$ je rostoucí funkce:



)

$-3x + 9 > 3^2 \quad /-9$

$-3x > 0 \quad /:(-3)$

$\underline{x < 0}$

$\Rightarrow$ z I.) a II.): $\underline{\underline{x \in (-\infty, 0)}}$

2.) $\log_{\frac{1}{2}} (-5x + 10) > 3$

I.) x musí splňovat: $-5x + 10 > 0$

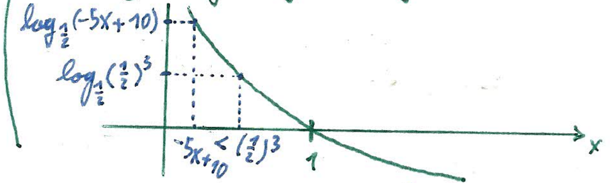
$-5x > -10 \quad /:(-5)$

$\underline{x < 2}$

II.) $\log_{\frac{1}{2}} (-5x + 10) > 3$

$\log_{\frac{1}{2}} (-5x + 10) > \log_{\frac{1}{2}} (\frac{1}{2})^3$

(y = log x je klesající funkce:



)

$-5x + 10 < (\frac{1}{2})^3$

$-5x + 10 < \frac{1}{8}$

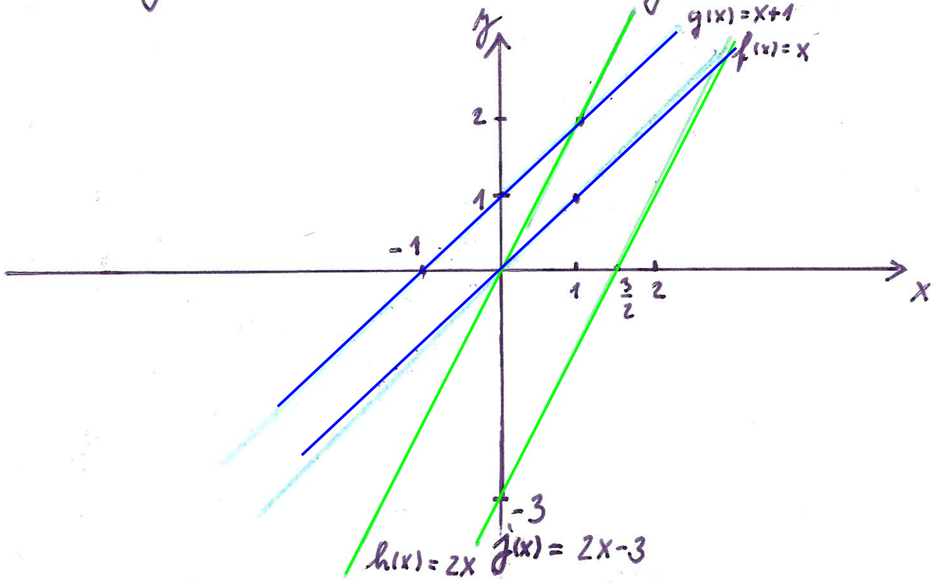
$x > \frac{\frac{1}{8} - 10}{-5} = \frac{-\frac{79}{8}}{-5} = \frac{79}{40}$

$\Rightarrow$ z I.) a II.): $\underline{\underline{x \in (\frac{79}{40}, 2)}}$

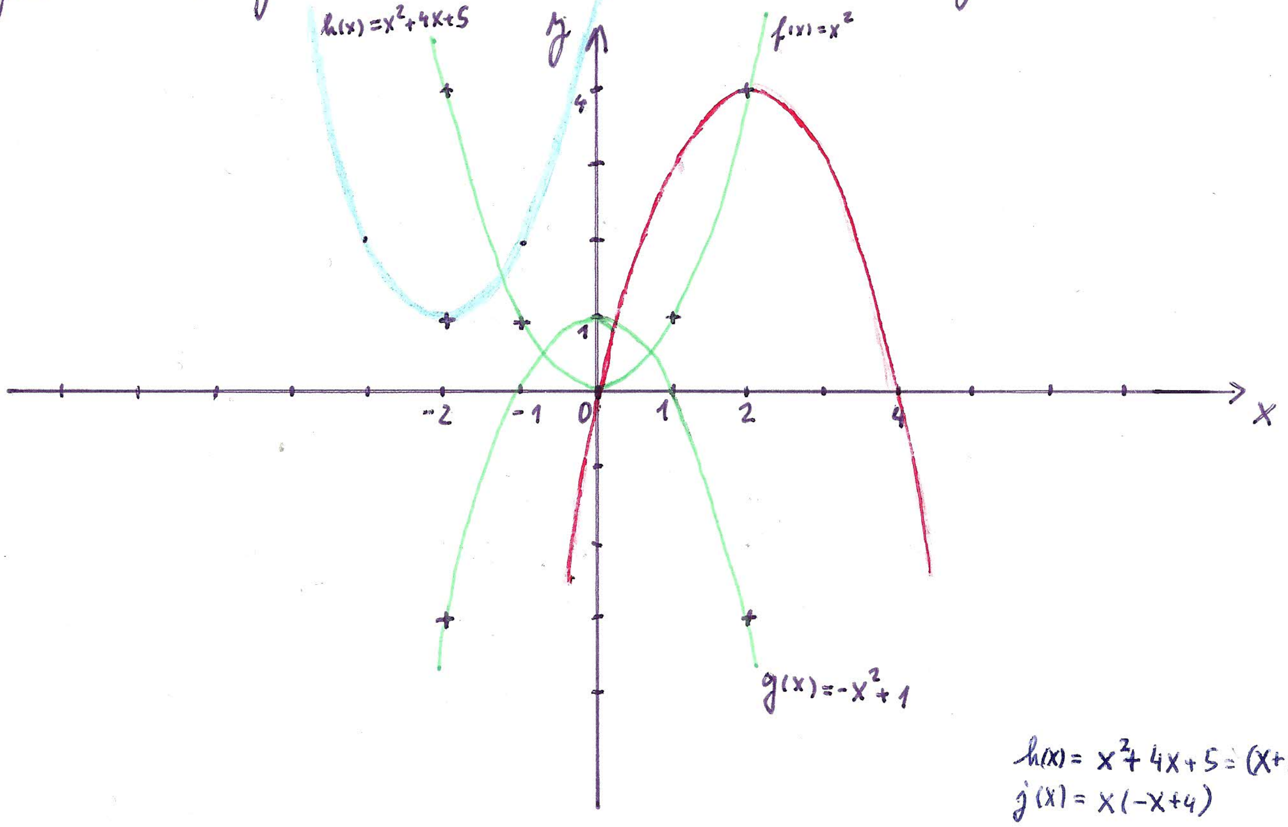
Elementární funkce a transformace grafu funkce

Př: Načrtněte grafy funkcí

1.) $f(x) = x$, $g(x) = x + 1$, $h(x) = 2x$, $j(x) = 2x - 3$



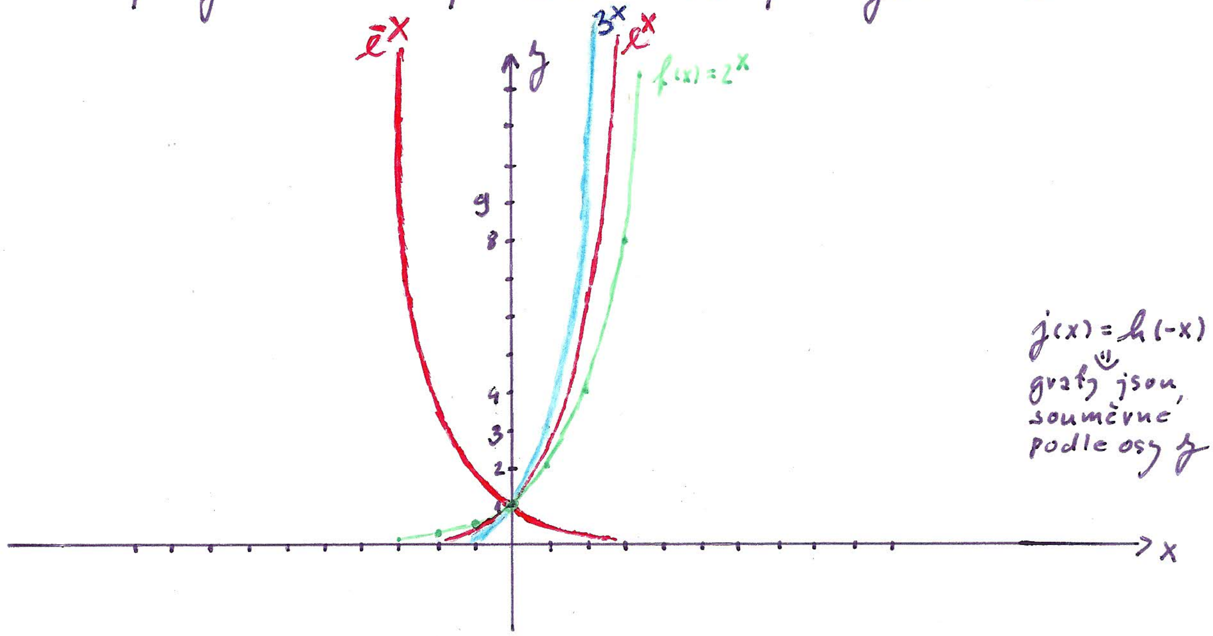
2.) $f(x) = x^2$, $g(x) = -x^2 + 1$, $h(x) = x^2 + 4x + 5$, $j(x) = -x^2 + 4x$



$h(x) = x^2 + 4x + 5 = (x+2)^2 + 1$

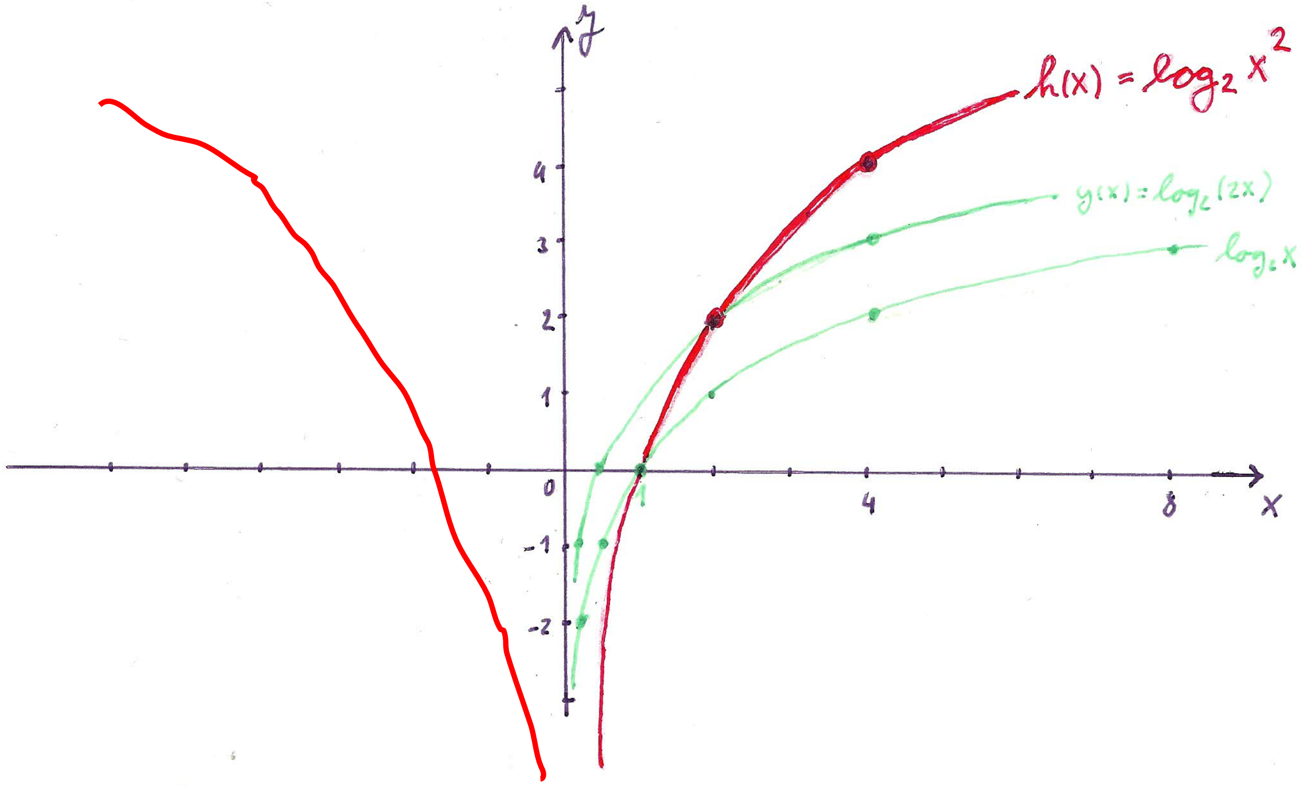
$j(x) = x(-x+4)$

3.) $f(x) = 2^x$, $g(x) = 3^x$, $h(x) = e^x$, $j(x) = e^{-x}$



$j(x) = h(-x)$
$\Downarrow$
grafy jsou souměrné podle osy y

4.) $f(x) = \log_2 x$, $g(x) = \log_2(2\cdot x)$, $h(x) = \log_2 x^2$, $j(x) = \ln(-x^2)$

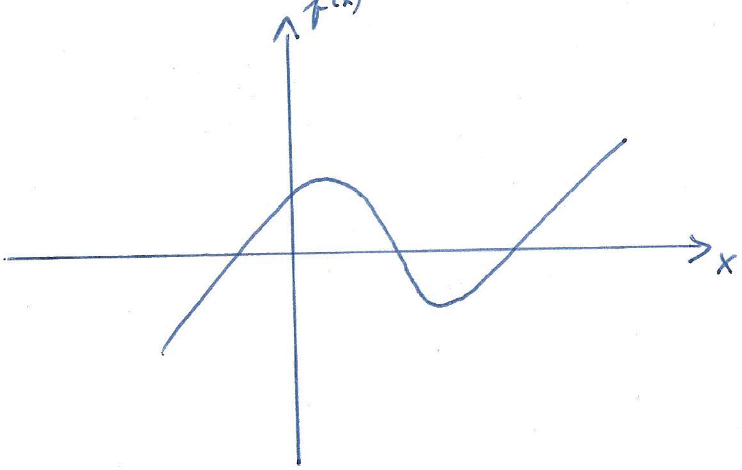


$g(x) = \log(2x) = \underbrace{\log_2 2}_{+1} + \log_2 x = \log_2 x + 1$

$h(x) = \log_2 x^2 = 2\cdot \log_2 x$

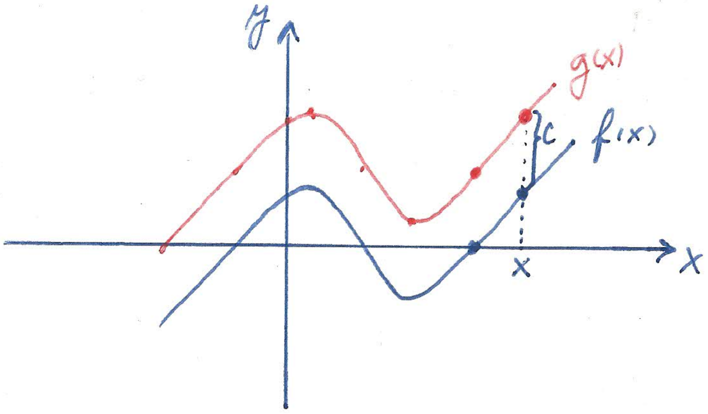
$j(x) = \ln(-x^2)$ ale $\forall x \in \mathbb{R}: -x^2 \leq 0 \Rightarrow D_j = \emptyset \Rightarrow$ nemá žádný graf

Transformace grafu funkce



1.) $g(x) = f(x) + c$

Graf fce g je stejný jako graf fce f, ale je posunuto c ve směru nahoru / dolů



2.) $g(x) = c \cdot f(x)$

(c=2)