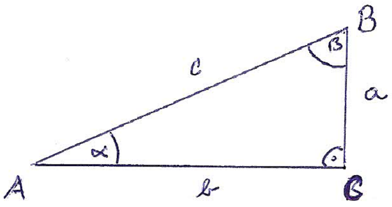
Goniometrické funkce

Jsou to funkce: $\sin x$, $\cos x$, $\operatorname{tg} x$, $\operatorname{cotg} x$.

Pravoúhlý trojúhelník:



$\Rightarrow$ Pro $\alpha \in (0°, 90°) = (0, \frac{\pi}{2})$ platí:

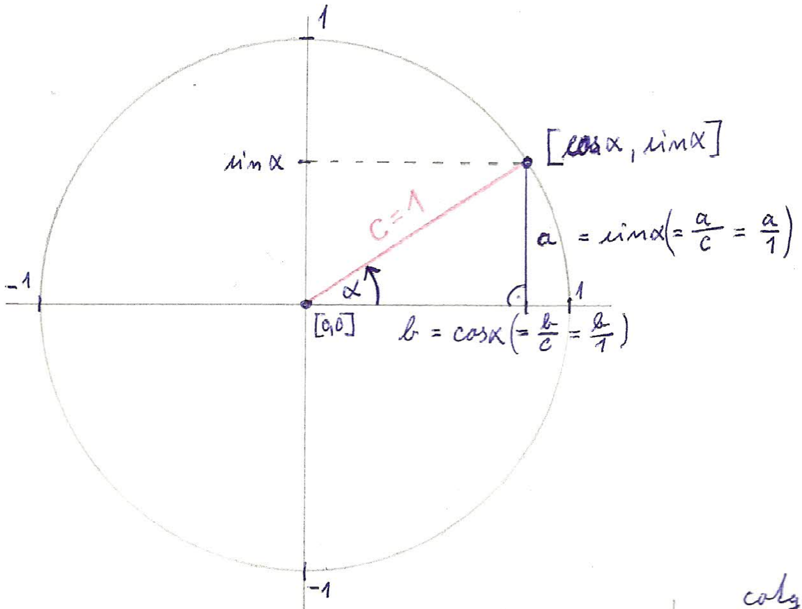
$$\sin \alpha = \frac{a}{c}$$

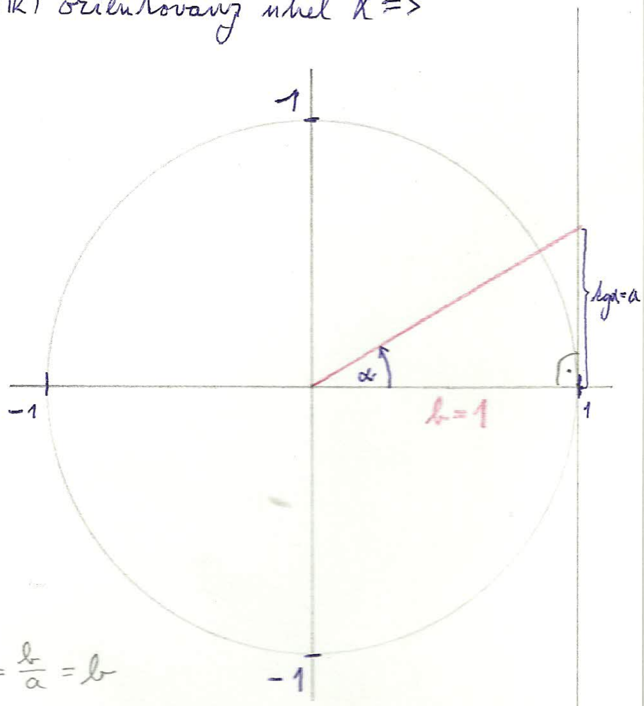
$$\cos \alpha = \frac{b}{c}$$

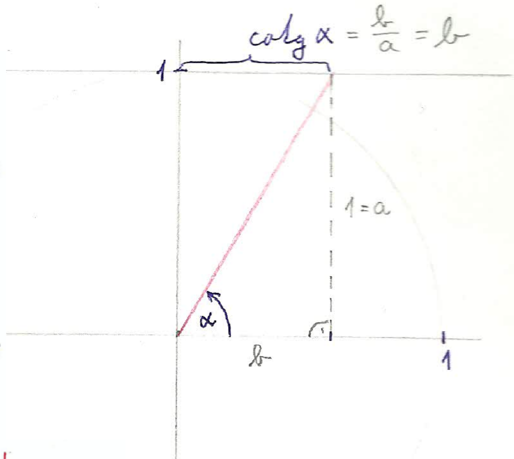
$$\operatorname{tg} \alpha = \frac{a}{b} = \frac{\sin \alpha}{\cos \alpha}$$

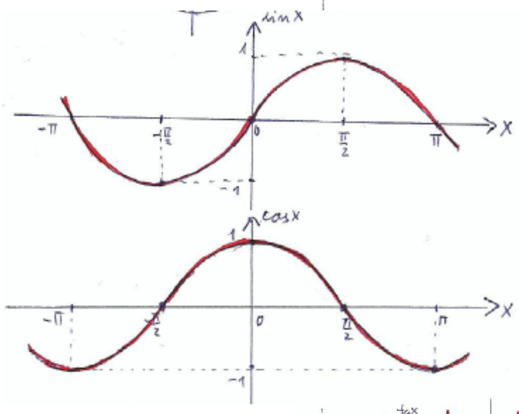
$$\operatorname{cotg} \alpha = \frac{b}{a} = \frac{\cos \alpha}{\sin \alpha}$$

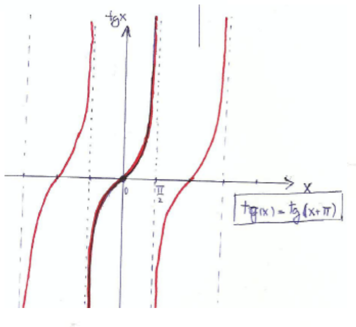
Zobecnění na jednotkové kružnici: Uvažujme libovolný ($\in \mathbb{R}$) orientovaný úhel α $\Rightarrow$

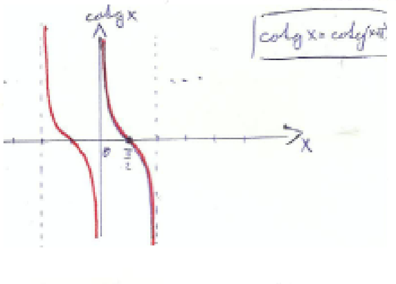








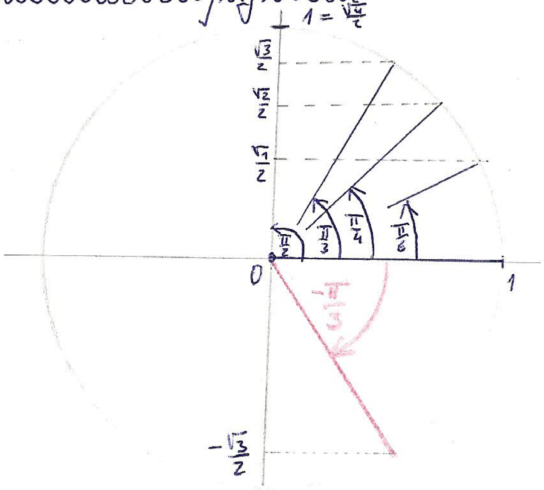




Definiční obory:

$f(x)$	D_f
$\sin x$	$\mathbb{R}$
$\cos x$	$\mathbb{R}$
$\operatorname{tg} x$	$\mathbb{R} - \{\frac{\pi}{2} + k\pi \mid k \in \mathbb{Z}\}$
$\operatorname{cotg} x$	$\mathbb{R} - \{0 + k\pi \mid k \in \mathbb{Z}\}$

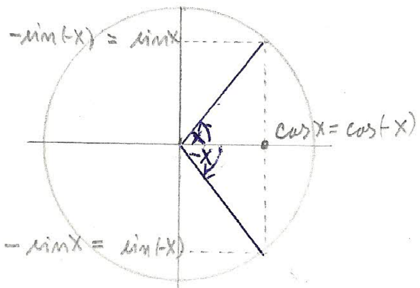
Vybrané hodnoty gon. fcí:



α	$\sin\alpha$	$\cos\alpha$	$\operatorname{tg}\alpha$	$\operatorname{cotg}\alpha$
$\frac{\pi}{2}$	$1=\frac{\sqrt{4}}{2}$	0	není def.	0
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1
$\frac{\pi}{6}$	$\frac{1}{2}=\frac{\sqrt{1}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$
0	$0=\frac{\sqrt{0}}{2}$	1	0	není def.

Př: $\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} = -\sin\left(\frac{\pi}{3}\right)$

Vlastnosti gon. fcí:

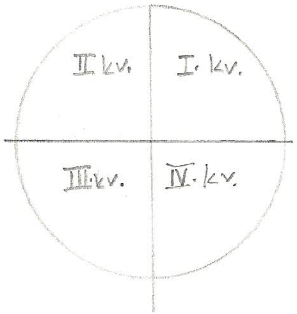


1.) $\forall x \in \mathbb{R}: \quad \sin x = -\sin(-x)$ (~~sudá~~ lichá funkce)

2.) $\forall x \in \mathbb{R}: \quad \cos x = \cos(-x)$ (sudá funkce)

3.) $\forall x \in \mathbb{R}: \quad \sin^2 x + \cos^2 x = 1$ (Pro $x \in (0, 90°)$ je to Pythagorova věta)

„Kvadrant"



Úhel x patří do:

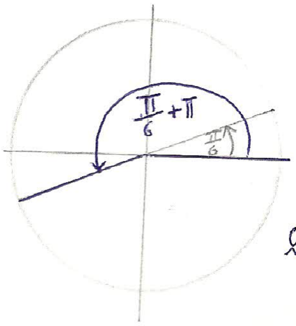
I. kvadrantu $\Leftrightarrow \quad x \in \left(0 + k2\pi,\ \frac{\pi}{2} + k\cdot 2\pi\right)$, kde $k \in \mathbb{Z}$

II. kvadrantu $\Leftrightarrow \quad x \in \left(\frac{\pi}{2} + k\cdot 2\pi,\ \pi + k\cdot 2\pi\right)$, kde $k \in \mathbb{Z}$

III. kvadrantu $\Leftrightarrow \quad x \in \left(\pi + k2\pi,\ \frac{3}{2}\pi + k\cdot 2\pi\right)$, kde $k \in \mathbb{Z}$

IV. kvadrantu $\Leftrightarrow \quad x \in \left(\frac{3}{2}\pi + k2\pi,\ 2\pi + k\cdot 2\pi\right)$, kde $k \in \mathbb{Z}$

Př: $x = \frac{31}{6}\pi \quad \Rightarrow \quad x = \frac{1}{6}\pi + \frac{30}{6}\pi = \frac{1}{6}\pi + 5\pi = \frac{1}{6}\pi + \pi + 2\cdot 2\pi \Rightarrow$



$\Rightarrow \quad \underline{\underline{x \in \text{III. kvadrantu}}}$

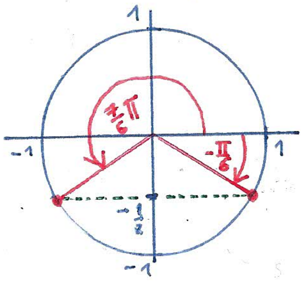
$\frac{1}{6}\pi + \pi = \underline{\underline{\frac{5}{6}\pi}}$ ... tzv. základní velikost úhlu x

Obecně: základní velikost úhlu $x \in \mathbb{R}$ je úhel $\alpha \in \langle 0, 2\pi)$:

$x = \alpha + k2\pi$, kde $k \in \mathbb{Z}$

Př.: Nalezněte všechna $x \in \mathbb{R}$ splňující:

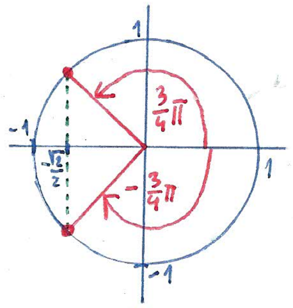
1.) $\sin 3X = -\frac{1}{2}$



$\Rightarrow \quad 3X = -\frac{\pi}{6} + k2\pi \;/{:}3, \quad$ nebo $\quad 3X = \frac{7}{6}\pi + k\cdot 2\pi \;/{:}3$

$\underline{\underline{X = -\frac{\pi}{18} + k\frac{2\pi}{3};\ k\in\mathbb{Z}}}$, nebo $\underline{\underline{X = \frac{7}{18}\pi + k\frac{2\pi}{3};\ k\in\mathbb{Z}}}$

2.) $\cos(5X+1) = -\frac{\sqrt{2}}{2}$



$\Rightarrow$

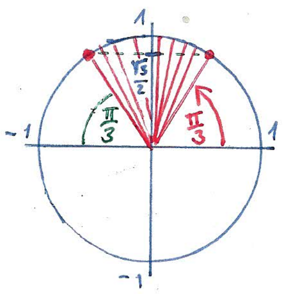
$5X+1 = \frac{3}{4}\pi + k2\pi \quad$ nebo $\quad 5X+1 = -\frac{3}{4}\pi + k2\pi$

$5X = \frac{3}{4}\pi - 1 + k2\pi \quad$ nebo $\quad 5X = -\frac{3}{4}\pi - 1 + k2\pi$

$\underline{\underline{X = \frac{3}{20}\pi - \frac{1}{5} + k\frac{2\pi}{5}, k\in\mathbb{Z}}}$ nebo $\underline{\underline{X = -\frac{3}{20}\pi - \frac{1}{5} + k\frac{2\pi}{5}, k\in\mathbb{Z}}}$

3.) $\sin(2x) \geq \frac{\sqrt{3}}{2} \quad \Rightarrow \quad 2x \in \left\langle \frac{\pi}{3} + k2\pi, \frac{2\pi}{3} + k2\pi \right\rangle \quad /{:}2$

$\underline{\underline{x \in \left\langle \frac{\pi}{6} + k\pi, \frac{2\pi}{6} + k\pi \right\rangle, k\in\mathbb{Z}}}$

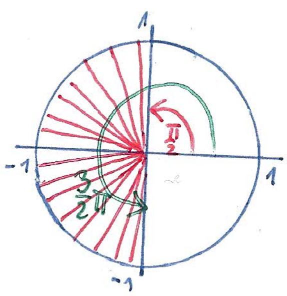


4.) $\cos(1-x) < 0 \quad \Rightarrow$

$\frac{\pi}{2} + k2\pi < 1-x < \frac{3}{2}\pi + k2\pi$

$\frac{\pi}{2} - 1 + k2\pi < -x < \frac{3}{2}\pi - 1 + k2\pi \quad /\cdot(-1)$

$1 - \frac{\pi}{2} - k2\pi > x > 1 - \frac{3}{2}\pi - k2\pi$



$\underline{\underline{x \in \left(1 - \frac{3}{2}\pi - k2\pi;\ 1 - \frac{\pi}{2} - k2\pi\right);\ k\in\mathbb{Z}}}$

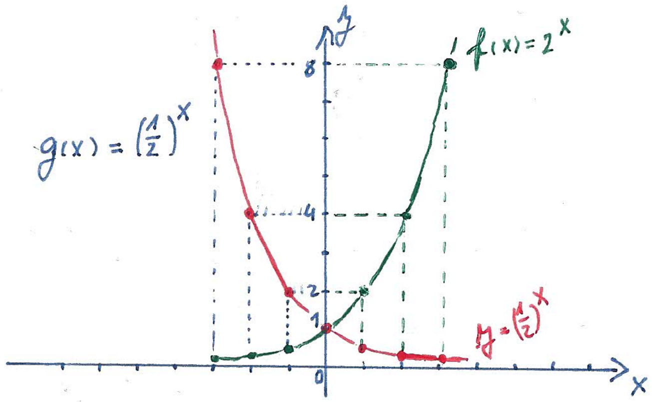
Exponenciální funkce

$f(x) = a^{x}$ ← exponent, ← základ $\in \mathbb{R}^+ \setminus \{1\}$ $= e^{x \cdot \ln a}$

$D(f) = \mathbb{R}$; $H(f) = (0, \infty)$;

- je prostá
- je omezená zdola
- není sudá ani lichá
- není periodická

Př.: Načrtněte graf funkce $f(x) = 2^x$ a $g(x) = \left(\frac{1}{2}\right)^x$



Př.: Vyřešte exponenciální rovnici:

1.) $5^{3x+2} = 25^{x+1}$

$5^{3x+2} = (5^2)^{x+1}$

$3x+2 = 2x+2$

$\underline{\underline{x = 0}}$

2.) $8^x = 16^{2-x}$

$(2^3)^x = (2^4)^{2-x}$

$3x = 8-4x$

$\underline{\underline{x = \frac{8}{7}}}$

3.) $3^{x-2} = \left(\frac{1}{3}\right)^{-2x}$

$3^{x-2} = (3^{-1})^{-2x}$

$x-2 = 2x$

$\underline{\underline{x = -2}}$

4.) $2^{3x-4} = \left(\frac{1}{8}\right)^{x+1}$

$2^{3x-4} = (2^{-3})^{x+1}$

$3x-4 = -3x-3$

$\underline{\underline{x = \frac{1}{6}}}$

5.) $2^x \cdot 5^x = 0{,}1\,(10^{x-1})^5$

$10^x = 10^{-1} \cdot 10^{5x-5}$

$10^x = 10^{5x-6}$

$x = 5x-6$

$\underline{\underline{x = \frac{6}{4} = \frac{3}{2}}}$

6.) $4^x \cdot 5^{x+1} = 5 \cdot 20^{2-x}$

$20^x \cdot 5 = 5 \cdot 20^{2-x}$

$x = 2-x$

$\underline{\underline{x = 1}}$

Př. Vyriešte exponenciální rovnici:

1.)

$$3^{x+1} = 2^{2x+3}$$

$$3 \cdot 3^{x} = 8 \cdot 2^{2x}$$

$$3 \cdot 3^{x} = 8 \cdot 4^{x}$$

$$\frac{3^{x}}{4^{x}} \quad \frac{8}{3}$$

$$\left(\frac{3}{4}\right)^{x} = \frac{8}{3} \quad / \ln$$

$$\ln \left(\frac{3}{4}\right)^{x} = \ln \frac{8}{3}$$

$$x \cdot \ln \left(\frac{3}{4}\right) = \ln \frac{8}{3}$$

$$\underline{\underline{x = \frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{8}{3}\right)}}}$$

2.)

$$5^{2x-3} = 8^{3x+7} \quad / \ln$$

$$\ln 5^{2x-3} = \ln 8^{3x+7}$$

$$(2x-3) \ln 5 = (3x+7) \ln 8$$

$$(2 \ln 5)x - 3 \ln 5 = (3 \cdot \ln 8)x + 7 \cdot \ln 8$$

$$x(2 \ln 5 - 3 \ln 8) = 7 \cdot \ln 8 + 3 \ln 5$$

$$x(\ln 5^2 - \ln 8^3) = \ln 8^7 + \ln 5^3$$

$$x \ln \frac{5^2}{8^3} = \ln 8^7 \cdot 5^3$$

$$\underline{\underline{x = \frac{\ln 8^7 \cdot 5^3}{\frac{5^2}{8^3}}}}$$

Př. Vyriešte exponenciální nerovnici:

1.)

$$3^{2x} > 5^{x-1} \quad / \ln$$

$$\ln 3^{2x} > \ln 5^{x-1}$$

$$2x \cdot \ln 3 > (x-1) \ln 5$$

$$x \cdot \ln 3^2 > x \cdot \ln 5 - \ln 5$$

$$x(\ln 9 - \ln 5) > -\ln 5$$

$$x > \frac{-\ln 5}{\ln \frac{9}{5}}$$

$$\underline{\underline{x \in \left(-\frac{\ln 5}{\ln \frac{9}{5}}, \infty\right)}}$$

2.)

$$2^{8x-1} \leq \left(\frac{1}{3}\right)^{4x+6}$$

$$2^{8x-1} \leq (3^{-1})^{4x+6}$$

$$2^{8x-1} \leq 3^{-4x-6} \quad / \ln$$

$$\ln 2^{8x-1} \leq \ln 3^{-4x-6}$$

$$(8x-1) \ln 2 \leq (-4x-6) \ln 3$$

$$8x \ln 2 - \ln 2 \leq -4x \ln 3 - 6 \ln 3$$

$$x \ln 2^8 - \ln 2 \leq x \ln 3^{-4} - \ln 3^6$$

$$x(\ln 2^8 - \ln 3^{-4}) \leq \ln 2 - \ln 3^6$$

$$x \ln \frac{2^8}{3^{-4}} \leq \ln \frac{2}{3^6}$$

$$x \ln(2^8 \cdot 3^4) \leq \ln(2 \cdot 3^{-6})$$

$$x \leq \frac{\ln(2 \cdot 3^{-6})}{\ln(2^8 \cdot 3^4)}$$

$$\underline{\underline{x \in \left(-\infty, \frac{\ln(2 \cdot 3^{-6})}{\ln(2^8 \cdot 3^4)}\right\rangle}}$$

Logaritmická funkce

$\log_a x = y \iff a^y = x$

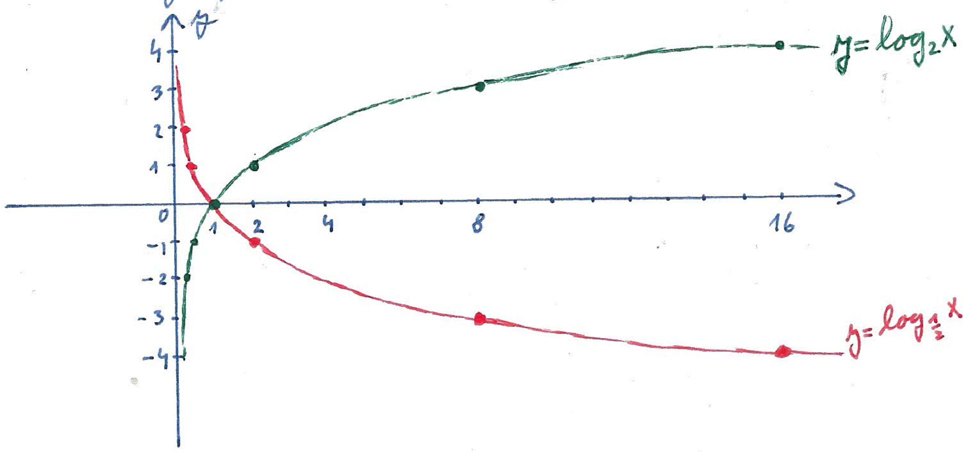
a … základ logaritmu $\in \mathbb{R}^+ \setminus \{1\}$

$D(f) = (0, \infty)$; $H(f) = \mathbb{R}$

– je prostá ; – není: sudá, lichá, periodická, omezená

– $\log_a x = \frac{\ln x}{\ln a}$

Př: Načrtněte graf funkce $f(x) = \log_2(x)$; $g(x) = \log_{\frac{1}{2}} x$



Př: Vyřešte logaritmickou rovnici:

1.) $x = \log_2 16$

$x = \log_2(2^4) = \underline{\underline{4}}$

2.) $x = \log_3 27$

$x = \log_3(3^3) = \underline{\underline{3}}$

3.) $x = \log_{\frac{1}{3}} 3$

$x = \log_{\frac{1}{9}}(9^{\frac{1}{2}}) = \log_{\frac{1}{3}}\left(\frac{1}{3}\right)^{-\frac{1}{2}} = \underline{\underline{-\frac{1}{2}}}$

4.) $x = \log_3 18 + \log_3 \frac{3}{2}$

$x = \log_3 18 \cdot \frac{3}{2} = \log_3 \underset{3^3}{27} = \underline{\underline{3}}$

5.) $x = \log_{10} 500 - \log_{10} 5$

$x = \log_{10} \frac{500}{5} = \log_{10} 10^2 = \underline{\underline{2}}$

6.) $x = \log_{\frac{1}{5}} 5 + \log_5 \frac{1}{125}$

$x = -1 + (-3) = \underline{\underline{-4}}$

Př. Vyřešte logaritmickou rovnici.

1.) $\log_2 \underbrace{(4x+8)}_{\Rightarrow x\in(-2,\infty)} = 2$

$4x+8 = 2^2$

$4x = -4$

$\underline{\underline{x=-1}}$

2.) $\log_{10} \underbrace{x}_{\Rightarrow x\in(0,\infty)} - \log_{10} 5 = 2$

$\log_{10}\frac{x}{5} = 2$

$\frac{x}{5} = 10^2$

$\underline{\underline{x = 500}}$

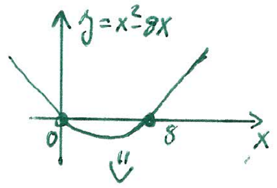
3.) $\log_3 (x^2-8x) = 2$

$x^2-8x = 3^2$

$x^2-8x-9=0$

$(x-9)(x+1)=0$

$x = \left\langle \begin{array}{l} \underline{\underline{9}} \\ \underline{\underline{-1}} \end{array}\right.$



$\Downarrow$

$x\in(-\infty,0)\cup(8,\infty)$

4.) Pozn.: U tohoto typu příkladu:

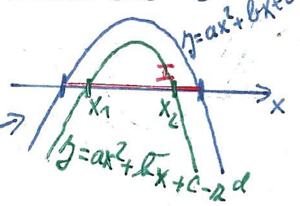
$\log_k (ax^2+bx+c) = d$

musí I.) $x \in I = \left(\frac{-b-\sqrt{b^2-4ac}}{2a}, \frac{-b+\sqrt{b^2-4ac}}{2a}\right)$ pro $a<0$

a $ax^2+bx+c = k^d \Rightarrow ax^2+bx+c-k^d=0$

$\Rightarrow \underline{\underline{x_{1,2} = \frac{-b\pm\sqrt{b^2-4a(c-k^d)}}{2a}}} \in I$

jistě



II.) pro $a>0$ analogicky

Pozn. U typu př. jako v 1.) nalezené x také bude vždy vyhovovat.

Př. Vyřešte nerovnici:

1.) $\log_3(-3x+9) > 2$

I.) x musí splňovat: $-3x+9>0$

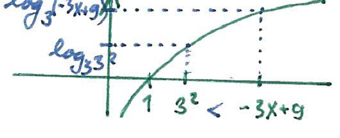
$-3x > -9 \quad /:(-3)$

$\underline{x<3}$

II.) $\log_3(-3x+9) > 2$

$\log_3(-3x+9) > \log_3(3^2)$

($y=\log_3 x$ je rostoucí funkce:



)

$-3x+9 > 3^2 \quad /-9$

$-3x > 0 \quad /:(-3)$

$\underline{x<0}$

$\Rightarrow$ z I.) a II.): $\underline{\underline{x\in(-\infty,0)}}$

2.) $\log_{\frac{1}{2}}(-5x+10) > 3$

I.) x musí splňovat: $-5x+10>0$

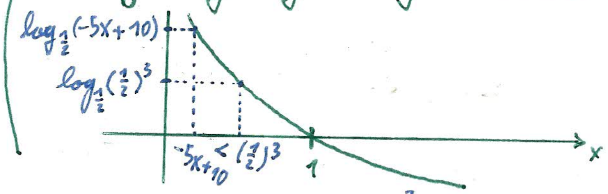
$-5x > -10 \quad /:(-5)$

$\underline{x<2}$

II.) $\log_{\frac{1}{2}}(-5x+10) > 3$

$\log_{\frac{1}{2}}(-5x+10) > \log_{\frac{1}{2}}(\frac{1}{2})^3$

($y=\log x$ je klesající funkce:



)

$-5x+10 < (\frac{1}{2})^3$

$-5x+10 < \frac{1}{8}$

$x > \frac{\frac{1}{8}-10}{-5} = \frac{-\frac{79}{8}}{-5} = \frac{79}{40}$

$\Rightarrow$ z I.) a II.): $\underline{\underline{x\in(\frac{79}{40},2)}}$

3.) $\boxed{\sin^2(3x) - \sin(3x) = 0}$ $\quad / \; t = \sin(3x)$

$$t^2 - t = 0$$

$$t(t-1) = 0 \Rightarrow t = \begin{cases} 0 = \sin(3x) \\ \text{nebo} \\ 1 = \sin(3x) \end{cases}$$

$\Rightarrow$ I.) $\sin(3x) = 0$ nebo II.) $\sin(3x) = 1$

$3x = k\pi \,,\, k \in \mathbb{Z}$ $\qquad$ $3x = \frac{\pi}{2} + k\pi \,,\, k \in \mathbb{Z}$

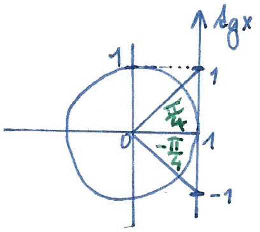
$$\underline{\underline{x = k\frac{\pi}{3} \,;\, k \in \mathbb{Z} \,,\ \text{nebo} \quad x = \frac{\pi}{6} + k\frac{\pi}{3} \,;\, k \in \mathbb{Z}}}$$

4.) $\boxed{tg^2 x + (1+\sqrt{3})\, tg\, x + \sqrt{3} = 0}$

Má smysl řešit pouze pro $x \in D(tg) \Rightarrow x \neq \frac{\pi}{2} + k\pi \,,\, k \in \mathbb{Z}$

$$(tg\,x + 1)(tg\,x + \sqrt{3}) = 0$$

$$tg\,x = \begin{cases} -1 \\ -\sqrt{3} \end{cases}$$



; $\quad tg\left(-\frac{\pi}{3}\right) = \frac{\sin\left(-\frac{\pi}{3}\right)}{\cos\left(-\frac{\pi}{3}\right)} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$

$\Rightarrow \quad x = -\frac{\pi}{4} + k_1\pi \,,\, k_1 \in \mathbb{Z}$ nebo $x = -\frac{\pi}{3} + k_2\pi \,,\, k_2 \in \mathbb{Z}$

Musíme ale zkontrolovat, zda taková x náleží do $D(tg)$. Tzn. nestane se náhodou, že:

$x = -\frac{\pi}{4} + k_1\pi = \frac{\pi}{2} + k\pi \;/ \cdot \frac{4}{\pi}$, nebo $\qquad x = -\frac{\pi}{3} + k_2\pi = \frac{\pi}{2} + k\pi \;/ \cdot \frac{6}{\pi}$

$\Downarrow \qquad\qquad \Downarrow$

$-1 + 4k_1 = 2 + 4k$ $\qquad$ $-2 + 6k_2 = 3 + 6k$

$4k_1 - 4k = 3$ $\qquad$ $6k_2 - 6k = 5$

násobek 4 $\Rightarrow$ spor! $\qquad$ násobek 6 $\Rightarrow$ spor!

$\Rightarrow$ To se nikdy nestane $\Rightarrow$

$$\underline{\underline{x = -\frac{\pi}{4} + k_1\pi \,;\, k_1 \in \mathbb{Z} \,,\ \text{nebo} \quad x = -\frac{\pi}{3} + k_2\pi \,;\, k_2 \in \mathbb{Z}}}$$

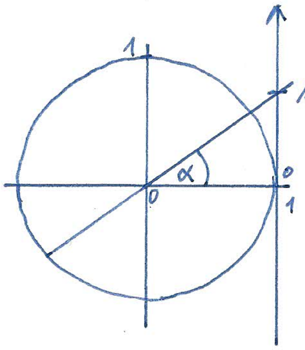
5.) $\boxed{tg(7x) = tg(5x)}$

Má smysl řešit pouze pro x taková, že $7x$ i $5x \in D(tg)$,

tzn. musí být splněno:

$7x \neq \frac{\pi}{2} + k_1\pi$ a $5x \neq \frac{\pi}{2} + k_2\pi$, kde $k_1, k_2 \in \mathbb{Z}$ =>

$x \neq \frac{\pi}{14} + k_1\frac{\pi}{7}$ a $x \neq \frac{\pi}{10} + k_2\frac{\pi}{5}$



$tg\alpha_1 = tg\alpha_2 \Leftrightarrow (\alpha_1 = \alpha_2 + k\pi, k \in \mathbb{Z} \wedge \alpha_1, \alpha_2 \in D(tg))$

$\Rightarrow (7x) = (5x) + k\pi$

$2x = k\pi$

$\underline{x = k\frac{\pi}{2}}$ je řešením?

Zjišťujeme, zda pro nějaké k najdeme $k_1, k_2 \in \mathbb{Z}$ takové, že:

$k\frac{\pi}{2} = \frac{\pi}{14} + k_1\frac{\pi}{7} \quad /\cdot\frac{14}{\pi}$ nebo $k\frac{\pi}{2} = \frac{\pi}{10} + k_2\frac{\pi}{5} \quad /\cdot\frac{10}{\pi}$

$7k = 1 + 2k_1$ $\qquad$ $5k = 1 + 2k_2$

$2k_1 = 7k - 1$ $\qquad$ $2k_2 = 5k - 1$

$k_1 = \frac{7k-1}{2} \in \mathbb{Z}$ pouze pro k-liché $\qquad$ $k_2 = \frac{5k-1}{2} \in \mathbb{Z}$ pouze pro k-liché

=> Pro k-liché platí, že $k\frac{\pi}{2}$ není řešením a pro $k = 2r$, kde $r \in \mathbb{Z}$

je $k\frac{\pi}{2} = \underbrace{2r\frac{\pi}{2}}_{r\cdot\pi}$ řešením. =>

$$\underline{\underline{x = r\cdot\pi, \text{ kde } r \in \mathbb{Z}}}$$