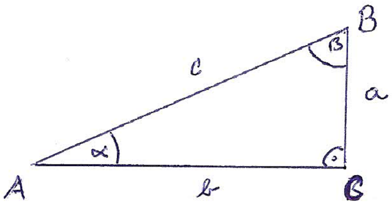
Goniometrické funkce

Jsou to funkce: $\sin x$, $\cos x$, $\operatorname{tg} x$, $\operatorname{cotg} x$.

Pravoúhlý trojúhelník:



$\Rightarrow$ Pro $\alpha \in (0°, 90°) = (0, \frac{\pi}{2})$ platí:

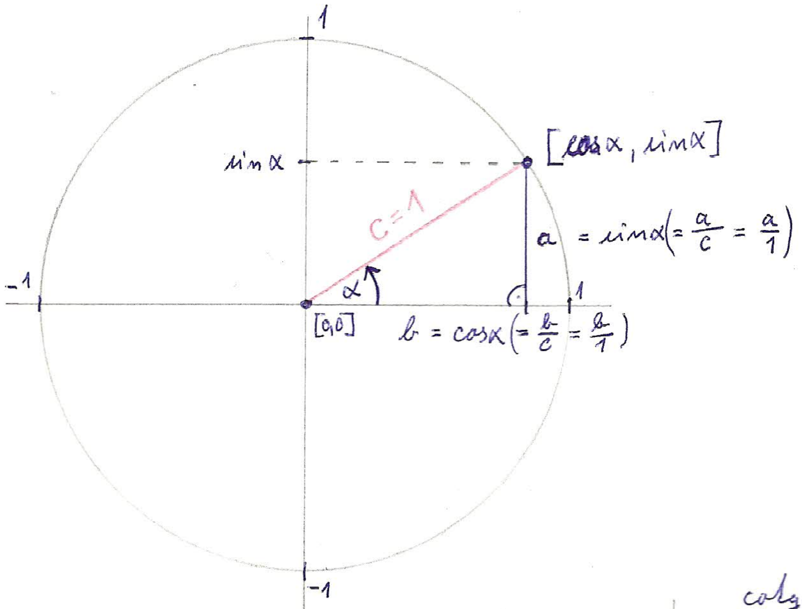
$$\sin \alpha = \frac{a}{c}$$

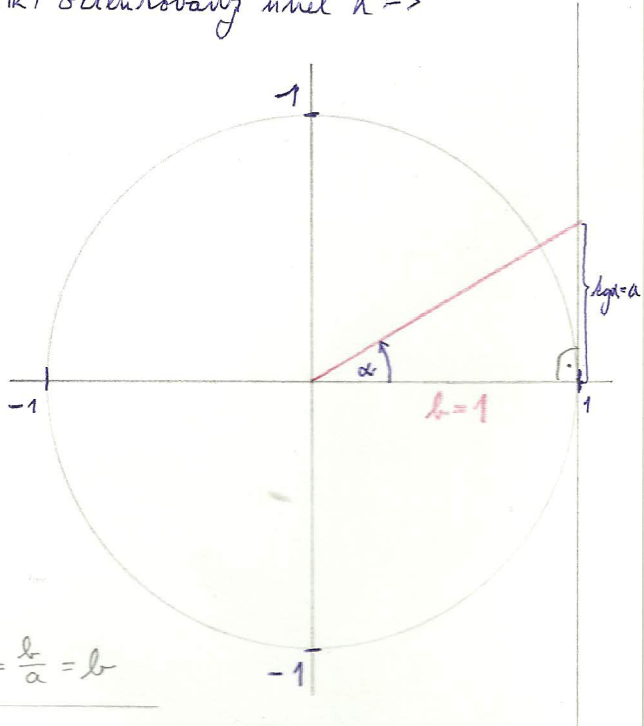
$$\cos \alpha = \frac{b}{c}$$

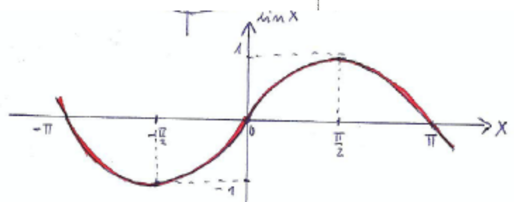
$$\operatorname{tg} \alpha = \frac{a}{b} = \frac{\sin \alpha}{\cos \alpha}$$

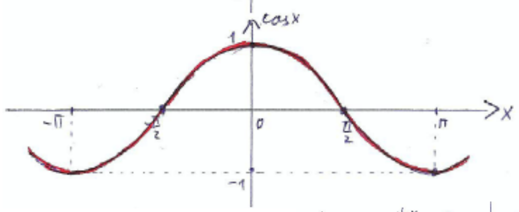
$$\operatorname{cotg} \alpha = \frac{b}{a} = \frac{\cos \alpha}{\sin \alpha}$$

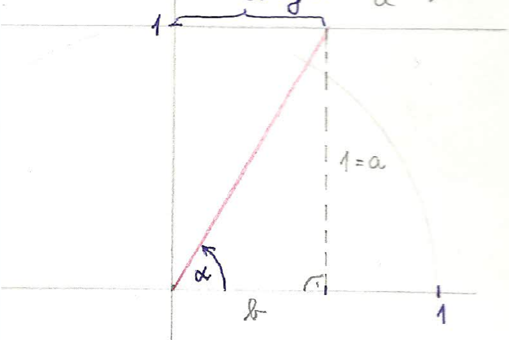
Zobecnění na jednotkové kružnici: Uvažujme libovolný ($\in \mathbb{R}$) orientovaný úhel $\alpha \Rightarrow$

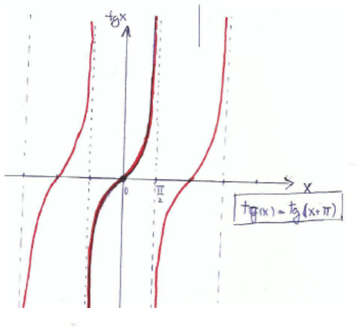


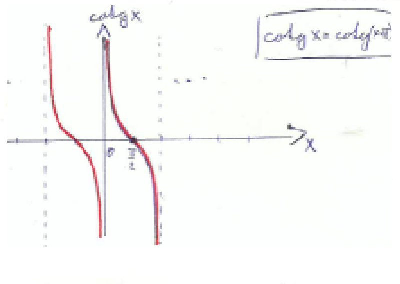








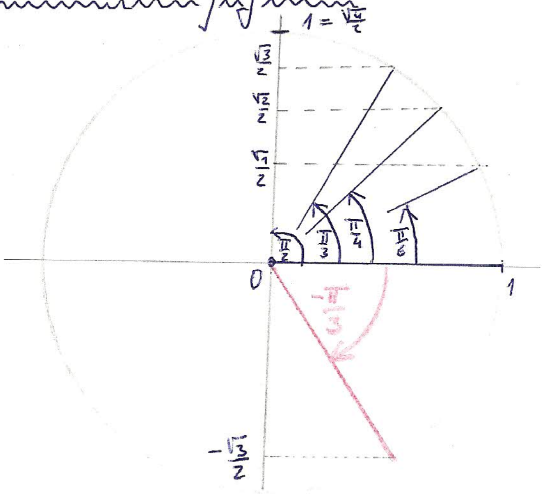




Definiční obory:

$f(x)$	D_f
$\sin x$	$\mathbb{R}$
$\cos x$	$\mathbb{R}$
$\operatorname{tg} x$	$\mathbb{R} - \{\frac{\pi}{2} + k\pi \mid k \in \mathbb{Z}\}$
$\operatorname{cotg} x$	$\mathbb{R} - \{0 + k\pi \mid k \in \mathbb{Z}\}$

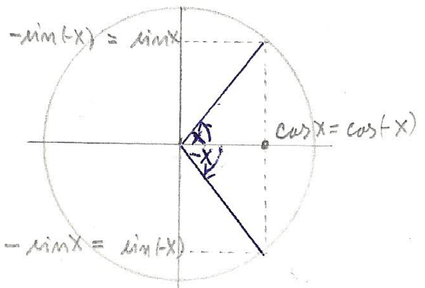
Vybrané hodnoty gon. fcí:



α	$\sin\alpha$	$\cos\alpha$	$\operatorname{tg}\alpha$	$\operatorname{cotg}\alpha$
$\frac{\pi}{2}$	$1 = \frac{\sqrt{4}}{2}$	0	není def.	0
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1
$\frac{\pi}{6}$	$\frac{1}{2} = \frac{\sqrt{1}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$
0	$0 = \frac{\sqrt{0}}{2}$	1	0	není def.

Př: $\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} = -\sin\left(\frac{\pi}{3}\right)$

Vlastnosti gon. fcí:

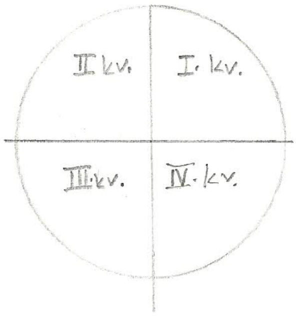


1.) $\forall x \in \mathbb{R}: \quad \sin x = -\sin(-x)$ (lichá ~~sudá~~ funkce)

2.) $\forall x \in \mathbb{R}: \quad \cos x = \cos(-x)$ (sudá funkce)

3.) $\forall x \in \mathbb{R}: \quad \sin^2 x + \cos^2 x = 1$ (Pro $x \in (0, 90°)$ je to Pythagorova věta)

„Kvadrant"



Úhel x patří do:

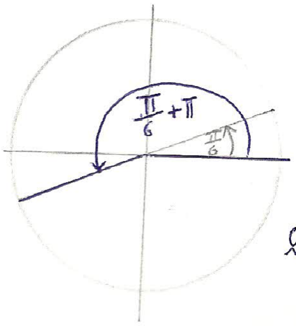
I. kvadrantu $\Leftrightarrow$ $x \in \left(0 + k2\pi\,,\ \frac{\pi}{2} + k\cdot 2\pi\right)$, kde $k \in \mathbb{Z}$

II. kvadrantu $\Leftrightarrow$ $x \in \left(\frac{\pi}{2} + k\cdot 2\pi\,,\ \pi + k\cdot 2\pi\right)$, kde $k \in \mathbb{Z}$

III. kvadrantu $\Leftrightarrow$ $x \in \left(\pi + k2\pi\,,\ \frac{3}{2}\pi + k\cdot 2\pi\right)$, kde $k \in \mathbb{Z}$

IV. kvadrantu $\Leftrightarrow$ $x \in \left(\frac{3}{2}\pi + k2\pi\,,\ 2\pi + k\cdot 2\pi\right)$, kde $k \in \mathbb{Z}$

Př: $x = \frac{31}{6}\pi \quad \Rightarrow \quad x = \frac{1}{6}\pi + \frac{30}{6}\pi = \frac{1}{6}\pi + 5\pi = \frac{1}{6}\pi + \pi + 2\cdot 2\pi \Rightarrow$



$\Rightarrow$ $x \in$ III. kvadrantu

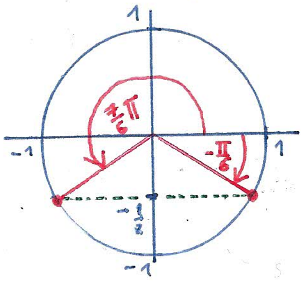
$\frac{1}{6}\pi + \pi = \frac{5}{6}\pi$ ... tzv. základní velikost úhlu x

Obecně: základní velikost úhlu $x \in \mathbb{R}$ je úhel $\alpha \in \langle 0, 2\pi)$:

$$x = \alpha + k2\pi \quad , \text{ kde } k \in \mathbb{Z}$$

Př.: Nalezněte všechna $x \in \mathbb{R}$ splňující:

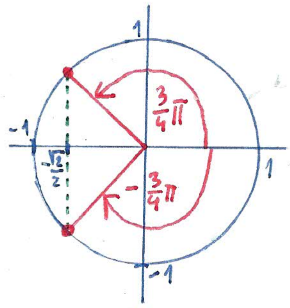
1.) $\sin 3X = -\frac{1}{2}$



$\Rightarrow \quad 3X = -\frac{\pi}{6} + k2\pi \;/:3, \quad$ nebo $\quad 3X = \frac{7}{6}\pi + k\cdot 2\pi \;/:3$

$\underline{\underline{X = -\frac{\pi}{18} + k\frac{2\pi}{3};\ k \in \mathbb{Z}}}$, nebo $\underline{\underline{X = \frac{7}{18}\pi + k\frac{2\pi}{3};\ k \in \mathbb{Z}}}$

2.) $\cos(5X+1) = -\frac{\sqrt{2}}{2}$



$\Rightarrow$

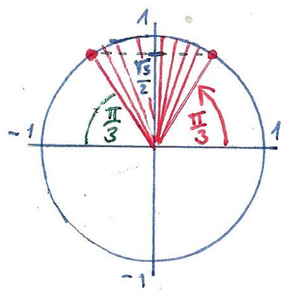
$5X+1 = \frac{3}{4}\pi + k2\pi \qquad$ nebo $\qquad 5X+1 = -\frac{3}{4}\pi + k2\pi$

$5X = \frac{3}{4}\pi - 1 + k2\pi \qquad$ nebo $\qquad 5X = -\frac{3}{4}\pi - 1 + k2\pi$

$\underline{\underline{X = \frac{3}{20}\pi - \frac{1}{5} + k\frac{2\pi}{5},\ k \in \mathbb{Z}}}$ nebo $\underline{\underline{X = -\frac{3}{20}\pi - \frac{1}{5} + k\frac{2\pi}{5},\ k \in \mathbb{Z}}}$

3.) $\sin(2X) \geq \frac{\sqrt{3}}{2} \quad \Rightarrow \quad 2X \in \left\langle \frac{\pi}{3} + k2\pi,\ \frac{2\pi}{3} + k2\pi \right\rangle \quad /:2$

$\underline{\underline{X \in \left\langle \frac{\pi}{6} + k\cdot\pi,\ \frac{2\pi}{6} + k\pi \right\rangle,\ k \in \mathbb{Z}}}$

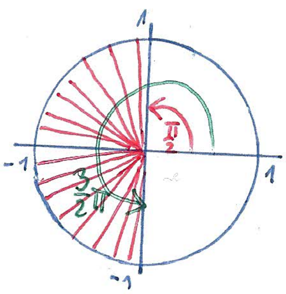


4.) $\cos(1-X) < 0 \quad \Rightarrow$

$\frac{\pi}{2} + k2\pi < 1 - X < \frac{3}{2}\pi + k2\pi$

$\frac{\pi}{2} - 1 + k2\pi < -X < \frac{3}{2}\pi - 1 + k2\pi \quad /\cdot(-1)$

$1 - \frac{\pi}{2} - k2\pi > X > 1 - \frac{3}{2}\pi - k2\pi$

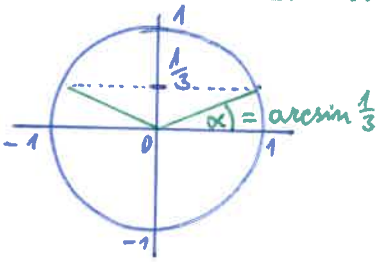


$\underline{\underline{X \in \left(1 - \frac{3}{2}\pi - k2\pi;\ 1 - \frac{\pi}{2} - k2\pi\right);\ k \in \mathbb{Z}}}$

Př. Určete všechna $x \in \mathbb{R}$ splňující

1.) $\sin x = \frac{1}{3}$

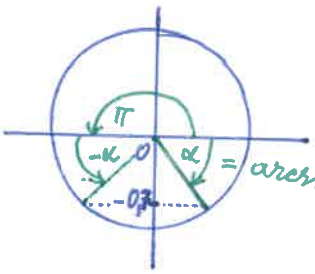
$\arcsin \frac{1}{3} = \alpha \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle : \sin \alpha = \frac{1}{3}$



$$\Rightarrow \quad x = \begin{cases} \arcsin \frac{1}{3} + k \cdot 2\pi \\ \pi - \arcsin \frac{1}{3} + k \cdot 2\pi \end{cases} ; k \in \mathbb{Z}$$

2.) $\sin x = -0{,}7$

$\arcsin(-0{,}7) = \alpha \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle : \sin \alpha = -0{,}7$



α = arcsin(−0,7) − vychází záporné!

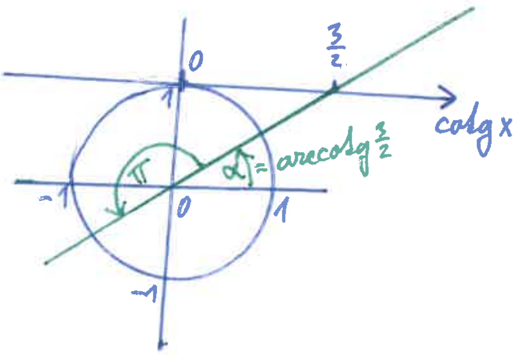
⇓

$-\alpha$ je velikost úhlu α

$$\Rightarrow \quad x = \begin{cases} \arcsin(-0{,}7) + k \cdot 2\pi \\ \pi - \arcsin(-0{,}7) + k \cdot 2\pi \end{cases} ; k \in \mathbb{Z}$$

3.) $\operatorname{cotg} x = \frac{3}{2}$

$\operatorname{arccotg} \frac{3}{2} = \alpha \in (0, \pi) : \operatorname{cotg} \alpha = \frac{3}{2}$



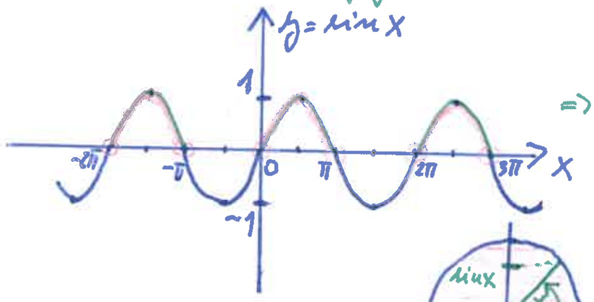
$$\Rightarrow \quad x = \operatorname{arccotg} \frac{3}{2} + k \cdot \pi \quad ; k \in \mathbb{Z}$$

Př. Určete definiční obor funkce dané předpisem.

1.) $f(x) = \ln(\underbrace{\sin x}_{y})$

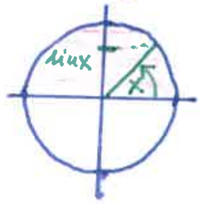
$\sin x$ je definován pro libovolné $x \in \mathbb{R}$, ale $\ln y$ jen pro kladná y.

$\sin x > 0$



$\Rightarrow x \in (0,\pi) \cup (2\pi, 3\pi) \cup (4\pi, 5\pi)$

ale také $(-2\pi, -\pi) \cup (-4\pi, -3\pi)$ ⋮



$$\underline{\underline{D_f = \bigcup_{k \in \mathbb{Z}} (2k\cdot\pi,\ (2k+1)\pi)}}$$

2.) $f(x) = \dfrac{\ln(x+1)}{1-\cos x}$

Logaritmus je definován jen pro kladná čísla. $\wedge$ Ve jmenovateli nesmí být 0.

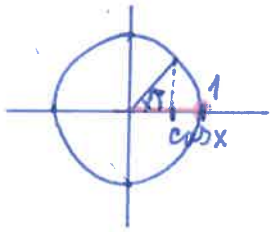
$x+1 > 0$

$x > -1$

$x \in (-1, \infty)$

$1 - \cos x \neq 0$

$\cos x \neq 1$



$\Rightarrow$

$\Rightarrow x \neq k\cdot 2\pi,\ k \in \mathbb{Z}$

$$\Rightarrow \underline{\underline{D_f = (-1, \infty) - \{k2\pi \mid k \in \mathbb{N} \cup \{0\}\}}}$$

Př. Vyřešte

1.) $-2\cos^2 x - 3\sin x + 3 = 0$ $\quad / \quad \cos^2 x = 1 - \sin^2 x$

$-2(1-\sin^2 x) - 3\sin x + 3 = 0$

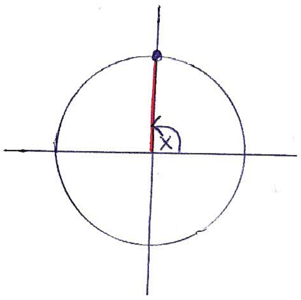
$2\sin^2 x - 3\sin x + 1 = 0$ $\quad / \quad s = \sin x$

$2s^2 - 3s + 1 = 0$

$$s_{1,2} = \frac{3 \pm \sqrt{9-8}}{4} = \frac{3 \pm 1}{4} = \begin{cases} 1 \\ \frac{1}{2} \end{cases}$$

a) $\boxed{s = 1}$

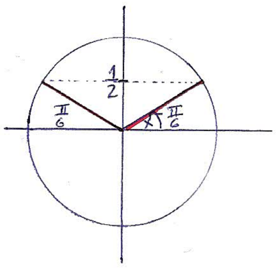
$\sin x = 1$



$\Rightarrow$ $x = \frac{\pi}{2} + k \cdot 2\pi, \ k \in \mathbb{Z}$

b) $\boxed{s = \frac{1}{2}}$

$\sin x = \frac{1}{2}$



$\Rightarrow$ $x = \frac{\pi}{6} + k \cdot 2\pi, \ k \in \mathbb{Z}$

$x = \frac{5}{6}\pi + k \cdot 2\pi, \ k \in \mathbb{Z}$

$\Leftrightarrow$ $x \in \{\frac{\pi}{2} + k \cdot 2\pi \mid k \in \mathbb{Z}\} \cup \{\frac{\pi}{6} + k \cdot 2\pi \mid k \in \mathbb{Z}\} \cup \{\frac{5}{6}\pi + k \cdot 2\pi \mid k \in \mathbb{Z}\}$

3.) $\boxed{\sin^2(3x) - \sin(3x) = 0}$ $\quad / \; t = \sin(3x)$

$$t^2 - t = 0$$

$$t\,(t-1) = 0 \quad \Rightarrow \quad t = \begin{cases} 0 = \sin(3x) \\ \text{nebo} \\ 1 = \sin(3x) \end{cases}$$

$\Rightarrow$ I.) $\sin(3x) = 0$ nebo II.) $\sin(3x) = 1$

$3x = k\pi, \; k \in \mathbb{Z}$ $\qquad$ $3x = \frac{\pi}{2} + k\pi, \; k \in \mathbb{Z}$

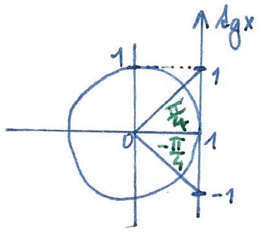
$\underline{\underline{x = k\frac{\pi}{3}; \; k \in \mathbb{Z}, \text{ nebo } \quad x = \frac{\pi}{6} + k\frac{\pi}{3}; \; k \in \mathbb{Z}}}$

4.) $\boxed{\operatorname{tg}^2 x + (1+\sqrt{3})\operatorname{tg} x + \sqrt{3} = 0}$

Má smysl řešit pouze pro $x \in D(\operatorname{tg}) \Rightarrow x \neq \frac{\pi}{2} + k\pi, \; k \in \mathbb{Z}$

$$(\operatorname{tg} x + 1)(\operatorname{tg} x + \sqrt{3}) = 0$$

$$\operatorname{tg} x = \begin{cases} -1 \\ -\sqrt{3} \end{cases}$$



; $\operatorname{tg}\left(-\frac{\pi}{3}\right) = \dfrac{\sin\left(-\frac{\pi}{3}\right)}{\cos\left(-\frac{\pi}{3}\right)} = \dfrac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$

$\Rightarrow \quad x = -\frac{\pi}{4} + k_1\pi, \; k_1 \in \mathbb{Z}$ nebo $x = -\frac{\pi}{3} + k_2\pi, \; k_2 \in \mathbb{Z}$

Musíme ale zkontrolovat, zda taková x náleží do $D(\operatorname{tg})$. Tzn. nestane se náhodou, že:

$x = -\frac{\pi}{4} + k_1\pi = \frac{\pi}{2} + k\pi \; /\cdot\frac{4}{\pi}$, nebo $\qquad x = -\frac{\pi}{3} + k_2\pi = \frac{\pi}{2} + k\pi \; /\cdot\frac{6}{\pi}$?

$\Downarrow \qquad\qquad \Downarrow$

$-1 + 4k_1 = 2 + 4k$ $\qquad$ $-2 + 6k_2 = 3 + 6k$

$4k_1 - 4k = 3$ $\qquad$ $6k_2 - 6k = 5$

násobek 4 $\Rightarrow$ spor! $\qquad$ násobek 6 $\Rightarrow$ spor!

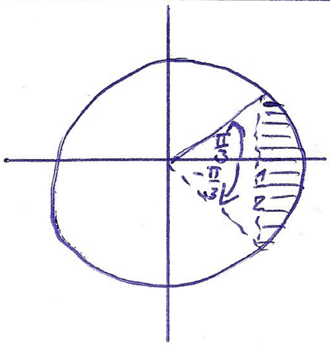
$\Rightarrow$ To se nikdy nestane $\Rightarrow$

$\underline{\underline{x = -\frac{\pi}{4} + k_1\pi; \; k_1 \in \mathbb{Z}, \text{ nebo } \quad x = -\frac{\pi}{3} + k_2\pi; \; k_2 \in \mathbb{Z}}}$

Goniometrické nerovnice

Př: Vyřešte nerovnice

1) $\cos 3x > \frac{1}{2}$



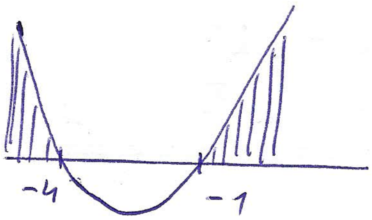
$\Rightarrow \quad 3x \in \left(-\frac{\pi}{3} + k2\pi,\ \frac{\pi}{3} + k2\pi\right)$

$\Rightarrow \quad \underline{\underline{x \in \left(-\frac{\pi}{9} + k\frac{2}{3}\pi;\ \frac{\pi}{9} + k\frac{2}{3}\pi\right),\ k \in \mathbb{Z}}}$

2.) $\sin^2 x + 5\sin x + 4 \geq 0$ subst.: $s = \sin x$

$s^2 + 5s + 4 \geq 0$

$s_{1,2} = \frac{-5 \pm \sqrt{25-16}}{2} = \frac{-5 \pm \sqrt{9}}{2} = \begin{cases} -4 \\ -1 \end{cases}$



$\Leftrightarrow \quad s \in (-\infty, -4\rangle \cup \langle -1, \infty) \quad \Leftrightarrow$

$\Leftrightarrow \quad \sin x \in (-\infty, -4\rangle \cup \langle -1, \infty) \quad \Rightarrow$

$\Rightarrow$ Vzhledem k tomu, že $\sin x \in \langle -1, 1\rangle$, je nerovnost splněna pro libovolné $x \in \mathbb{R}$.

$\sin^2 x + 2{,}5\sin x + 1 = 0$

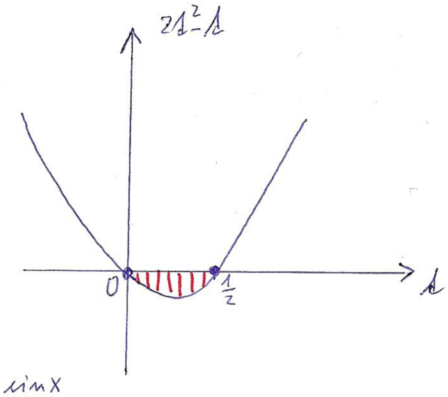
4.) $\sin^2 x - \cos^2 x - \sin x + 1 < 0$

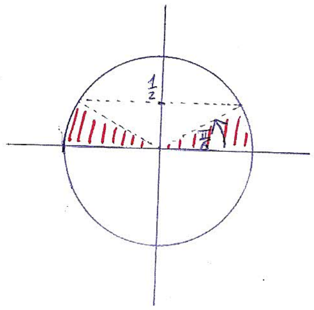
$\sin^2 x - (1-\sin^2 x) - \sin x + 1 < 0$

$2\sin^2 x - \sin x < 0 \quad | \; s = \sin x$

$2s^2 - s < 0$

$s(2s-1) < 0 \iff s \in (0, \frac{1}{2}) \ni \sin x$





$\Rightarrow \quad x \in (0, \frac{\pi}{6}) \cup (0+2\pi, \frac{\pi}{6}+2\pi) \cup (0+2\cdot 2\pi, \frac{\pi}{6}+2\cdot 2\pi) \cup \ldots = \underline{\bigcup_{k\in\mathbb{Z}} (0+k2\pi, \frac{\pi}{6}+k2\pi)}$

nebo

$x \in (\frac{5}{6}\pi, \pi) \cup (\frac{5}{6}\pi+2\pi, \pi+2\pi) \cup (\frac{5}{6}\pi+2\cdot 2\pi, \pi+2\cdot 2\pi) \cup \ldots = \underline{\bigcup_{k\in\mathbb{Z}} (\frac{5}{6}\pi+k2\pi, \pi+k2\pi)}$

$\iff \quad \underline{\underline{x \in \left(\bigcup_{k\in\mathbb{Z}} (0+k2\pi, \frac{\pi}{6}+k2\pi)\right) \cup \left(\bigcup_{k\in\mathbb{Z}} (\frac{5}{6}\pi+k2\pi, \pi+k2\pi)\right)}}$

5.) $\sin x + \cos(2x) \geq 0$

| Určíme $\cos(2x)$: $(\cos x + i\sin x)^2 = \cos(2x) + i\sin(2x) =$
$= \cos^2 x + 2i\cos x \sin x - \sin^2 x$

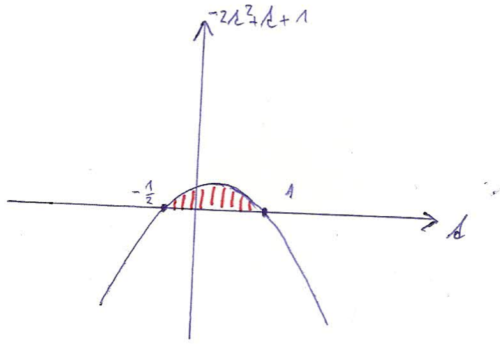
$\sin x + \cos^2 x - \sin^2 x \geq 0$

$\sin x + 1 - \sin^2 x - \sin^2 x \geq 0$

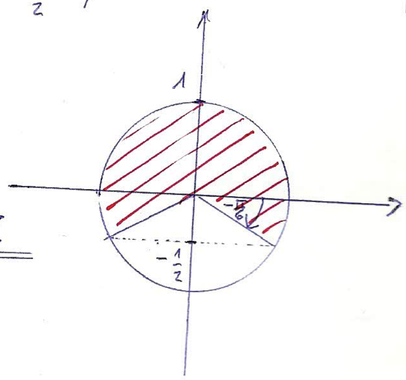
$-2\sin^2 x + \sin x + 1 \geq 0 \quad | \; s = \sin x$

$-2s^2 + s + 1 \geq 0$

$\left(s_{1,2} = \frac{-1 \pm \sqrt{1+8}}{-4} = \frac{-1 \pm 3}{-4} = \begin{cases} 1 \\ -\frac{1}{2} \end{cases} \right)$



$\iff s = \sin x \in \langle -\frac{1}{2}, 1 \rangle$



$\iff \quad \underline{\underline{x \in \langle -\frac{\pi}{6}+k2\pi, \frac{7}{6}\pi+k2\pi \rangle}}, k \in \mathbb{Z}$

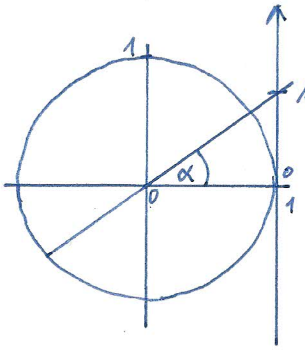
5.) $\boxed{tg(7x) = tg(5x)}$

Má smysl řešit pouze pro x taková, že $7x$ i $5x$ $\in D(tg)$,

tzn. musí být splněno:

$7x \neq \frac{\pi}{2} + k_1\pi$ a $5x \neq \frac{\pi}{2} + k_2\pi$, kde $k_1, k_2 \in \mathbb{Z}$ $\Rightarrow$

$x \neq \frac{\pi}{14} + k_1\frac{\pi}{7}$ a $x \neq \frac{\pi}{10} + k_2\frac{\pi}{5}$



$tg\alpha_1 = tg\alpha_2 \Leftrightarrow (\alpha_1 = \alpha_2 + k\pi, k \in \mathbb{Z} \wedge \alpha_1, \alpha_2 \in D(tg))$

$\Rightarrow (7x) = (5x) + k\pi$

$2x = k\pi$

$\underline{x = k\frac{\pi}{2}}$ je řešením?

Zjišťujeme, zda pro nějaké k najdeme $k_1, k_2 \in \mathbb{Z}$ taková, že:

$k\frac{\pi}{2} = \frac{\pi}{14} + k_1\frac{\pi}{7}$ $/\cdot\frac{14}{\pi}$ nebo $k\frac{\pi}{2} = \frac{\pi}{10} + k_2\frac{\pi}{5}$ $/\cdot\frac{10}{\pi}$

$7k = 1 + 2k_1$ $\qquad$ $5k = 1 + 2k_2$

$2k_1 = 7k - 1$ $\qquad$ $2k_2 = 5k - 1$

$k_1 = \frac{7k-1}{2}$ $\in \mathbb{Z}$ pouze pro k-liché $\qquad$ $k_2 = \frac{5k-1}{2}$ $\in \mathbb{Z}$ pouze pro k-liché

$\Rightarrow$ Pro k-liché platí, že $k\frac{\pi}{2}$ není řešením a pro $k = 2r$, kde $r \in \mathbb{Z}$ je $k\frac{\pi}{2} = \underbrace{2r\frac{\pi}{2}}_{r\cdot\pi}$ řešením. $\Rightarrow$

$$\underline{\underline{x = r\cdot\pi, \text{ kde } r \in \mathbb{Z}}}$$