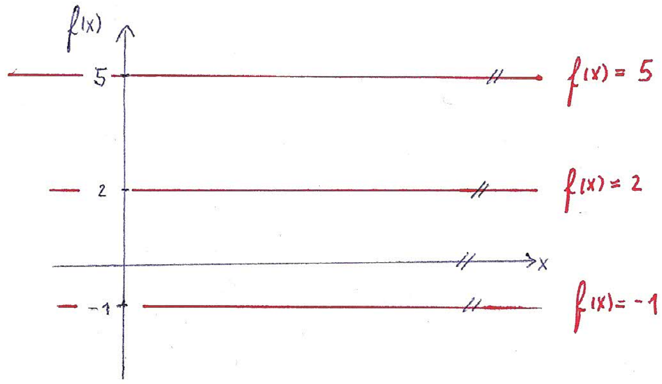
Základní elementární funkce

1.) Konstantní :

$$\boxed{f(x) = c, \quad \text{kde} \quad c \in \mathbb{R}}$$

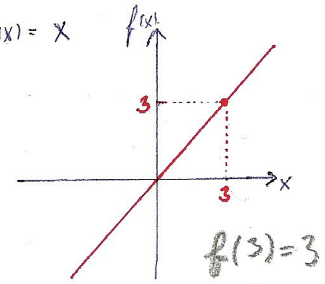
Př:



2.) Mocninné funkce s přirozeným exponentem $n \in \mathbb{N}$:

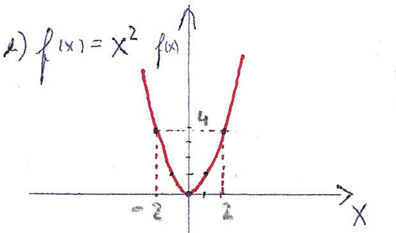
$$\boxed{f(x) = x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{n\text{-krát}}} \qquad \boxed{D_f = \mathbb{R}}$$

a) $f(x) = x$



$f(3) = 3$

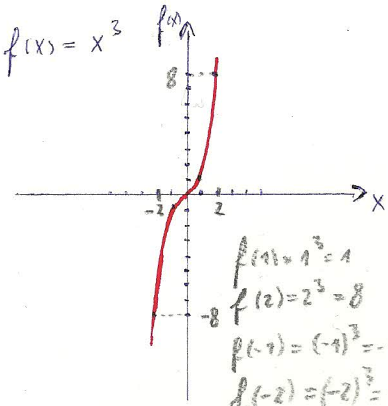
b) $f(x) = x^2$



$f(2) = 2^2 = 4$

$f(-2) = (-2)^2 = 4$

c) $f(x) = x^3$



$f(1) = 1^3 = 1$

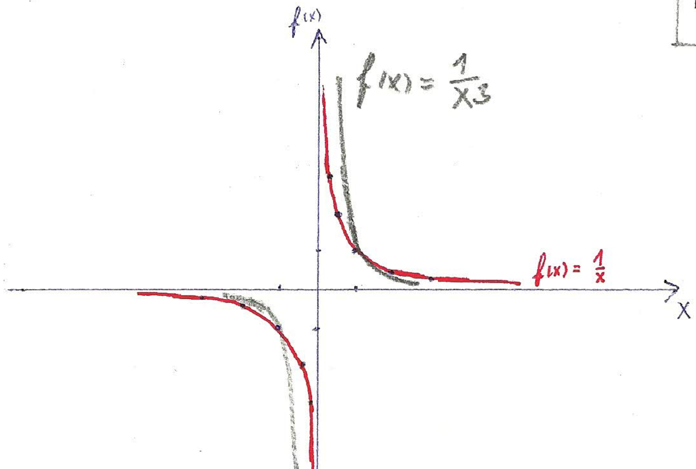
$f(2) = 2^3 = 8$

$f(-1) = (-1)^3 = -1$

$f(-2) = (-2)^3 = -8$

3.) Mocninné funkce s exponentem $-n$, kde $n \in \mathbb{N}$:

$$\boxed{f(x) = x^{-n} = \frac{1}{x^n} = \frac{1}{\underbrace{x \cdot x \cdot \ldots \cdot x}_{n\text{-krát}}}} \qquad \boxed{D_f = \mathbb{R} - \{0\}}$$



$f(x) = \frac{1}{x} \Rightarrow f(1) = \frac{1}{1} = 1$

$f(-1) = \frac{1}{-1} = -1$

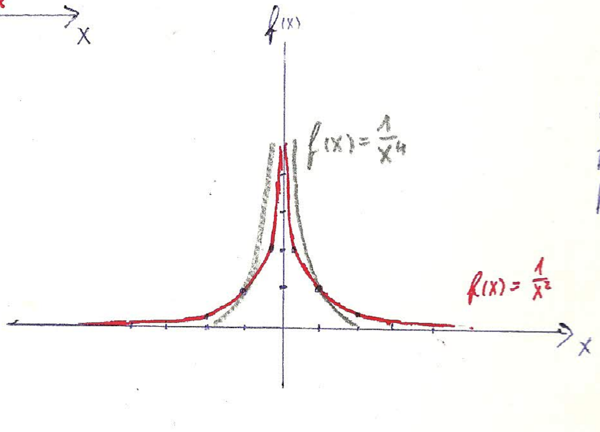
$f(2) = \frac{1}{2}$

$f(3) = \frac{1}{3}$

$f(\frac{1}{2}) = 2$

$f(\frac{1}{3}) = 3$

⋮



$f(x) = \frac{1}{x^2} \Rightarrow$

$f(1) = \frac{1}{1^2} = 1$

$f(-1) = \frac{1}{(-1)^2} = 1$

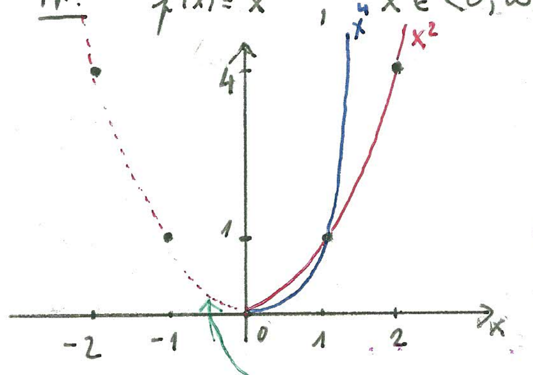
$f(2) = \frac{1}{4}$

$f(\frac{1}{2}) = \frac{1}{(\frac{1}{2})^2} = 4$

4.) n-tá odmocnina, kde $n \in \mathbb{N}$ je sudé:

n-tá odmocnina, kde $n \in \mathbb{N}$, je sudé, je definována jako funkce inverzní k funkci $f(x) = x^n$, kde $D_f = \langle 0, \infty)$.

Př: $f(x) = x^2$, $x \in \langle 0, \infty)$

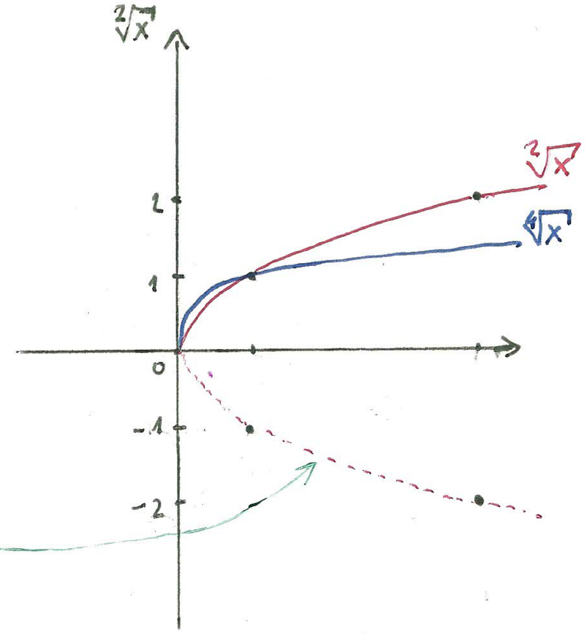


$\Rightarrow$ Funkce inverzní: druhá odmocnina:

Př: $3^2 = 9 \iff \sqrt{9} = 3$

obecně pro $x \in \langle 0, \infty)$:

$x^2 = y \iff \sqrt{y} = x$

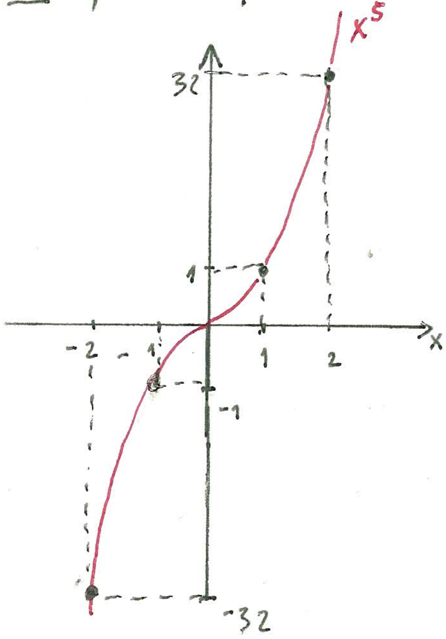


kdybychom připustili $x < 0 \Rightarrow$ $\sqrt{x}$ by nebyla funkce!

5.) n-tá odmocnina, kde $n \in \mathbb{N} - \{1\}$ je liché:

n-tá odmocnina, kde $n \in \mathbb{N} - \{1\}$ je liché, je definována jako funkce inverzní k funkci $f(x) = x^n$, kde $D_f = \mathbb{R}$

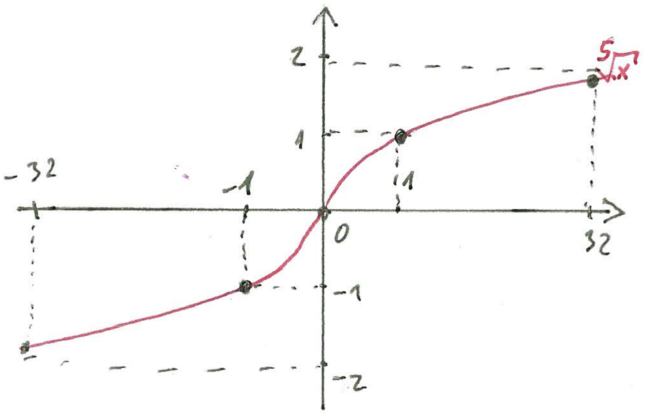
Př: $f(x) = x^5$, $x \in \mathbb{R}$



$\Rightarrow$ Funkce inverzní: pátá odmocnina

Př: $2^5 = 32 \iff \sqrt[5]{32} = 2$

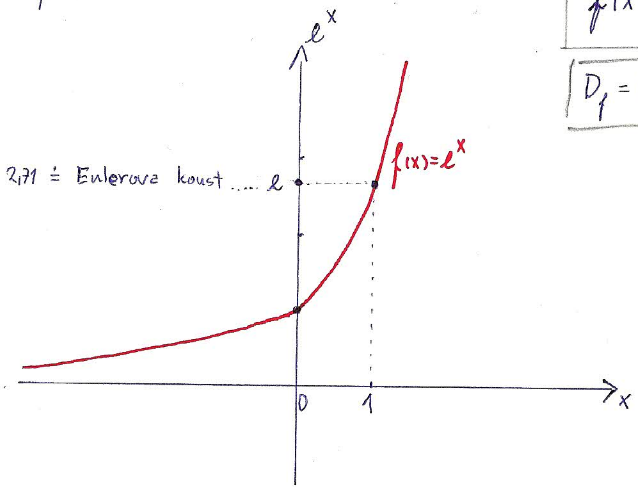
$(-2)^5 = -32 \iff \sqrt[5]{-32} = -2$



6.) Exponenciální funkce:

$$\boxed{f(x) = e^x = \exp(x) := 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!}}$$

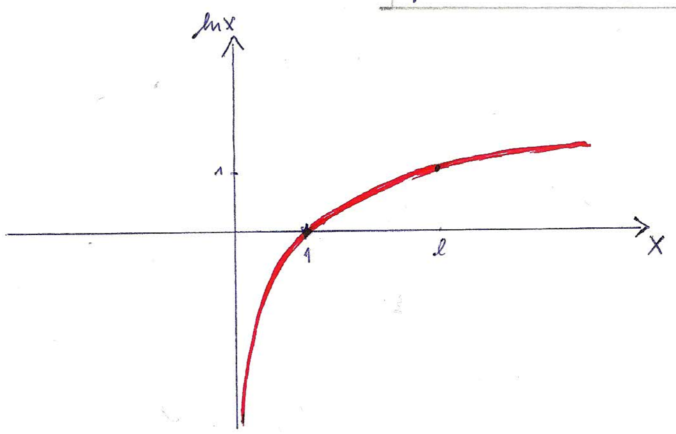
$$\boxed{D_f = \mathbb{R}}$$



7.) Logaritmická funkce:

$$\boxed{f(x) = \ln x = y \iff e^y = x}$$

$$\boxed{D_f = (0, \infty)}$$



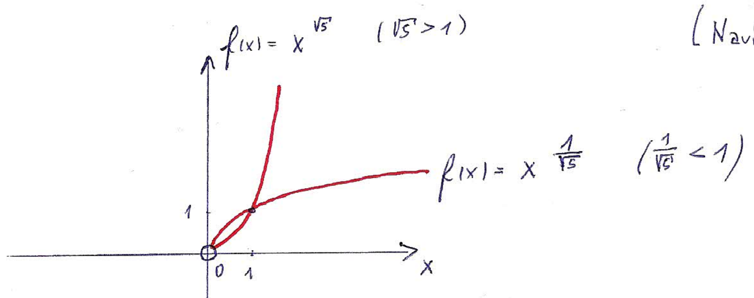
$e^0 = 1 \Rightarrow \ln 1 = 0$

$e^1 = e \doteq 2{,}71 \Rightarrow \ln e = 1$

8.) Mocninná funkce s reálným exponentem $r \in \mathbb{R} - \mathbb{Z}$

$$\boxed{f(x) = x^r := e^{r \cdot \ln x} \quad \text{pro } x \in (0, \infty)}$$

(Navíc definujeme: $x^0 = 1$ pro každé $x \in \mathbb{R}$)



$$f(1) = 1^r = e^{r \cdot \ln 1} = e^{r \cdot 0} = e^0 = 1$$

Poznámka: Lze dokázat, že platí: $\forall p, q \in \mathbb{Z},\ q \geq 2$ (aby byla def. odmocnina) $\forall x \in \mathbb{R}^+ : x^{\frac{p}{q}} = e^{\frac{p}{q} \cdot \ln x} = \sqrt[q]{x^p}$

Proč pro q sudá nedefinujeme $x^{\frac{p}{q}} := \sqrt[q]{x^p}$ i pro $x<0$? Nebylo by to korektní!

Např.: $-1 = (-1)^1 = (-1)^{\frac{2}{2}} = \sqrt[2]{(-1)^2} = \sqrt[2]{1} = 1 \Rightarrow$ z takové definice by plynulo $1 = -1$!

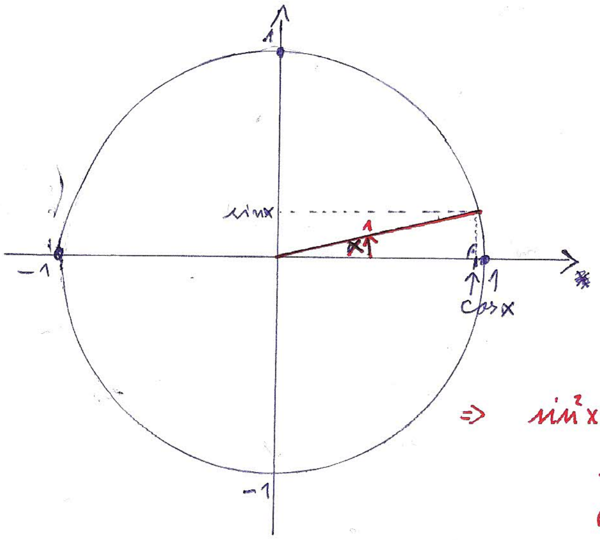
9.) Goniometrické funkce (sinus, kosinus, tangens, kotangens)

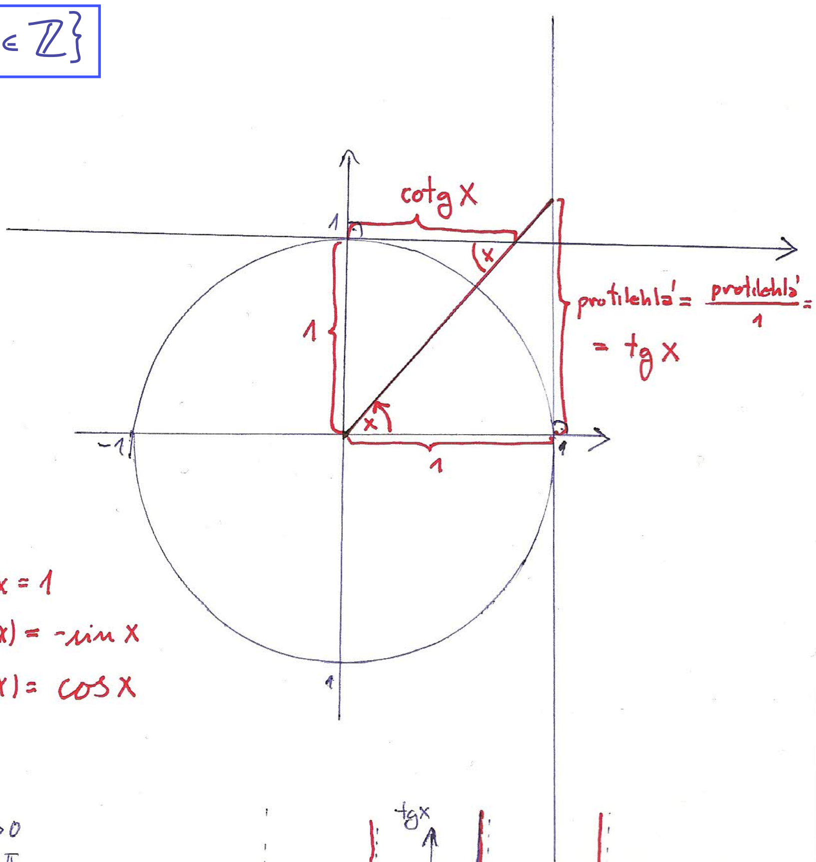
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \qquad \boxed{D(f) = \mathbb{R}}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \qquad \boxed{D(f) = \mathbb{R}}$$

$$\operatorname{tg} x = \frac{\sin x}{\cos x} \qquad \boxed{D(f) = \mathbb{R} \setminus \{\tfrac{\pi}{2} + k\pi \mid k \in \mathbb{Z}\}}$$

$$\operatorname{cotg} x = \frac{\cos x}{\sin x} \qquad \boxed{D(f) = \mathbb{R} \setminus \{k\pi \mid k \in \mathbb{Z}\}}$$

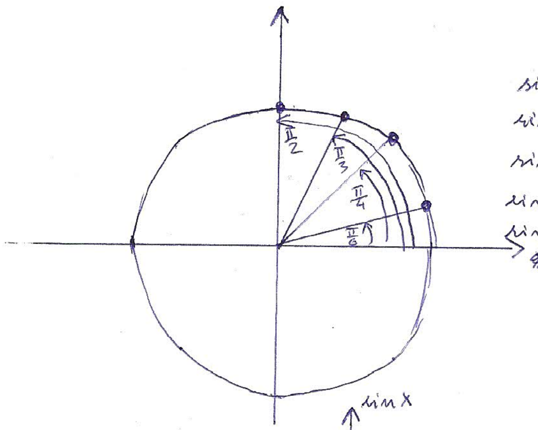




$\Rightarrow \sin^2 x + \cos^2 x = 1$

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$



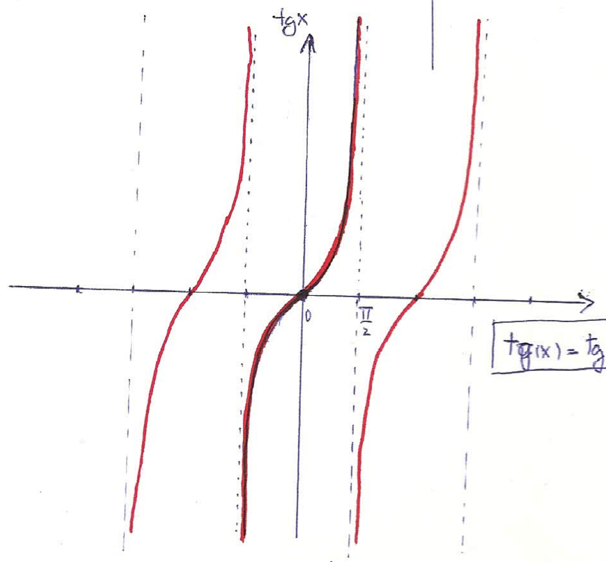
$\sin\frac{\pi}{2} = 1 = \frac{\sqrt{4}}{2} = \cos 0$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$

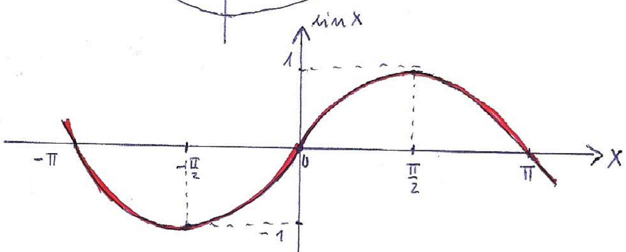
$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} = \cos\frac{\pi}{4}$

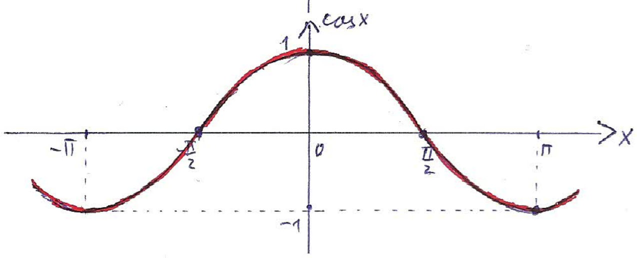
$\sin\frac{\pi}{6} = \frac{1}{2} = \cos\frac{\pi}{3}$

$\sin 0 = 0 = \cos\frac{\pi}{2}$

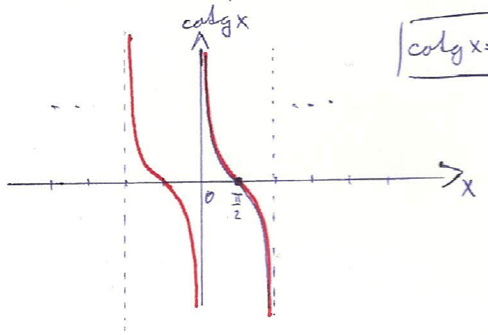


$\boxed{\operatorname{tg}(x) = \operatorname{tg}(x+\pi)}$



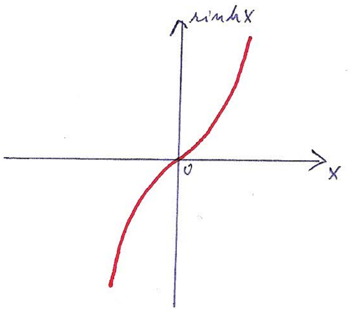


$\boxed{\operatorname{cotg} x = \operatorname{cotg}(x+\pi)}$

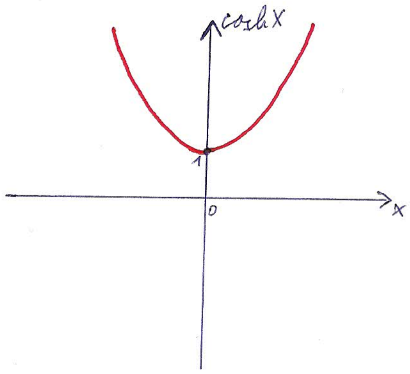


10.) Hyperbolické funkce (sinh, cosh, tgh, cotgh):

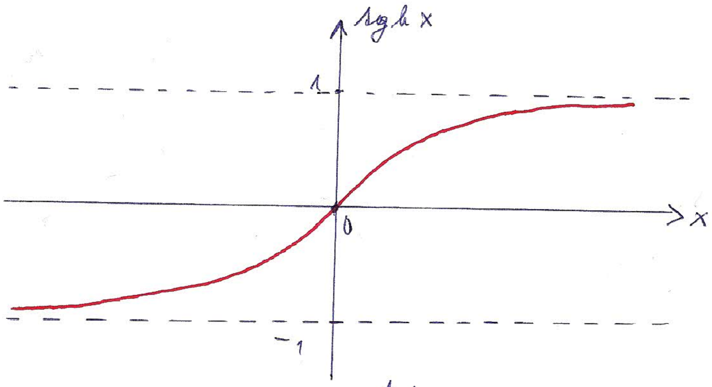
$$\sinh x := \frac{e^x - e^{-x}}{2}$$



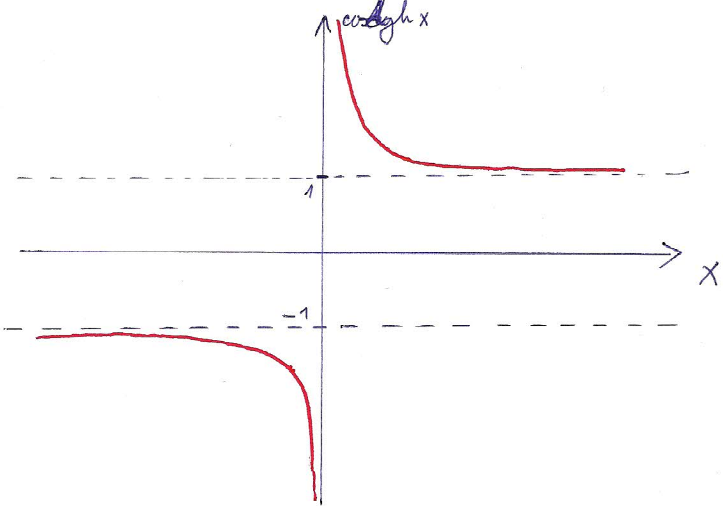
$$\cosh x := \frac{e^x + e^{-x}}{2}$$



$$\operatorname{tgh} x := \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$



$$\operatorname{cotgh} x := \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$



Operace s funkcemi

Def: Nechť f a g jsou funkce. Definujme funkce $f+g$, $f-g$, $f\cdot g$, $\frac{f}{g}$ a $g\circ f$ předpisy:

1.) $(f+g)(x) := f(x) + g(x)$

2.) $(f-g)(x) := f(x) - g(x)$

3.) $(f\cdot g)(x) := f(x)\cdot g(x)$

4.) $\left(\frac{f}{g}\right)(x) := \frac{f(x)}{g(x)}$

5.) $(g\circ f)(x) := g(f(x))$

Př: f je dáno předpisem $f(x) = 2x$, g je dáno předpisem $g(x) = \sin x$

$\Rightarrow$ $(f\pm g)(x) = 2x \pm \sin x$

$(f\cdot g)(x) = (2x)\cdot(\sin x)$

$\left(\frac{f}{g}\right)(x) = \frac{2x}{\sin x}$

$(g\circ f)(x) = g(f(x)) = g(2x) = \sin(2x)$

$(f\circ g)(x) = f(g(x)) = f(\sin x) = 2\cdot \sin x$

Def (Elementární funkce): Elementárními funkcemi nazveme ty funkce, které lze vytvořit ze základních elementárních funkcí pomocí konečného počtu operací $+, -, \cdot, \frac{\cdot}{\cdot}$ (lomeno) a skládání funkcí ($\circ$).

Př: $f(x) = x^2$, $g(x) = 1$, $h(x) = \frac{1}{x}$ (jsou základní elementární fce) $\Rightarrow$

$\Rightarrow$ $\left(h\circ(f+g)\right)(x) = \frac{1}{x^2+1}$ je elementární funkce