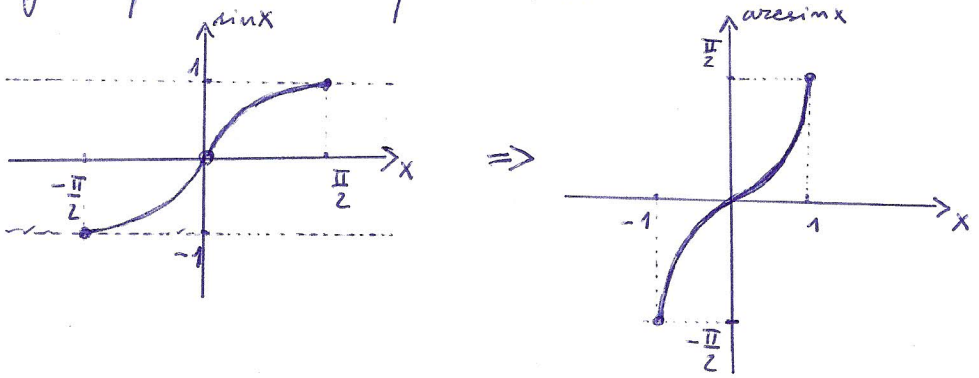
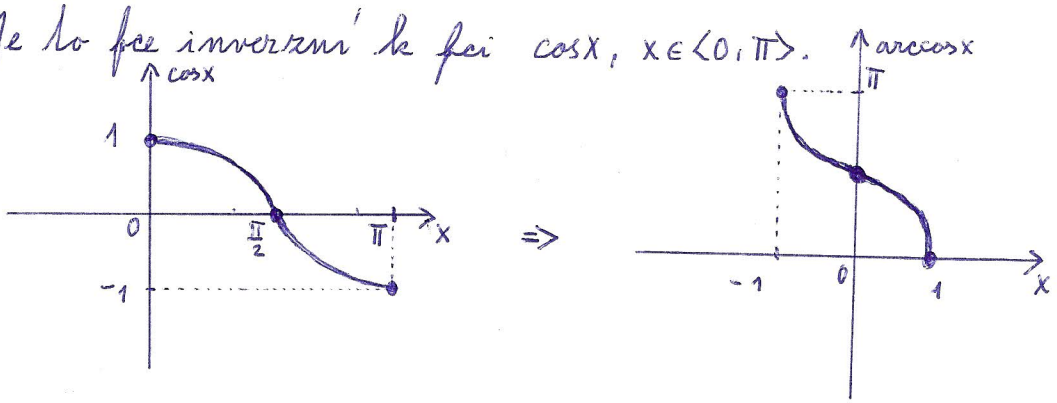
Cyklometrické funkce

Jsou to funkce inverzní ke goniometrickým funkcím.

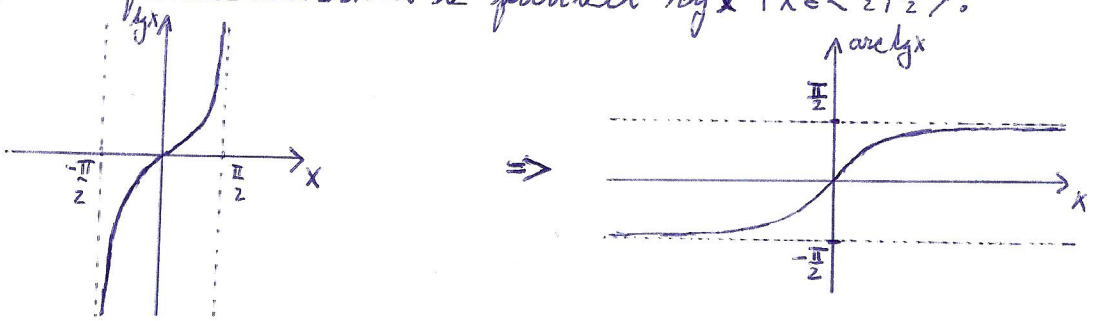
1.) arcsin x : Je to fce inverzní k fci $\sin x$, $x \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$.



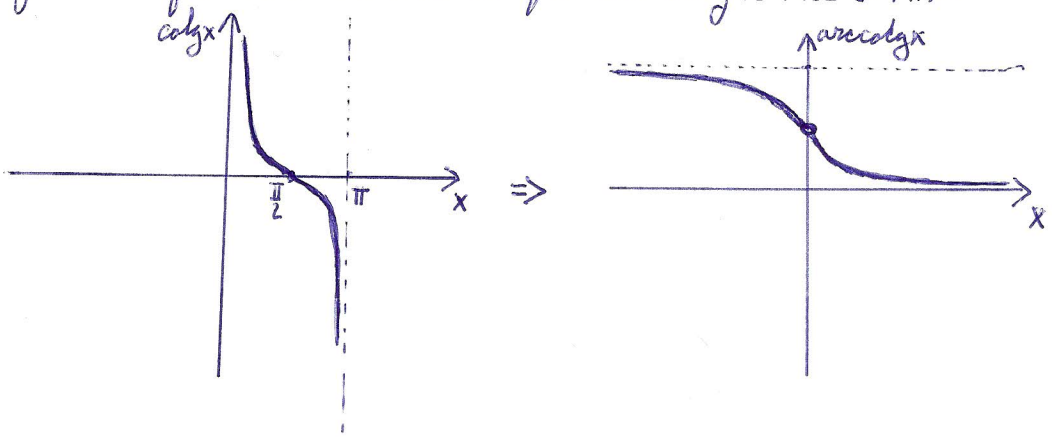
2.) arccos x : Je to fce inverzní k fci $\cos x$, $x \in \langle 0, \pi \rangle$.



3.) arctg x : Je to funkce inverzní k funkci $\operatorname{tg} x$, $x \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$.



4.) arccotg x : Je to funkce inverzní k funkci $\operatorname{cotg} x$, $x \in \langle 0, \pi \rangle$.



Př: Načrtněte grafy funkcí:

a) $f(x) = \sin(\arcsin x)$

b) $f(x) = \arcsin(\sin x)$

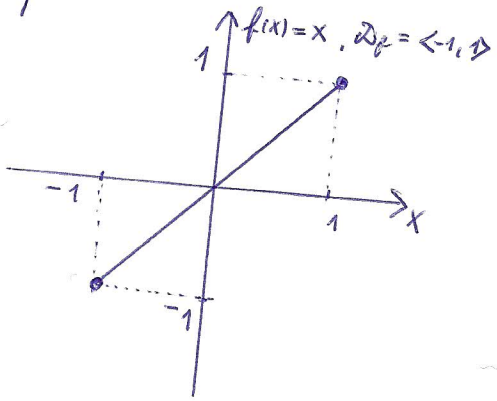
c) $f(x) = \sin(-\arccos x)$

d) $f(x) = \arccos(\sin x)$

a) $f(x) = \sin(\arcsin x)$

$\Rightarrow D_f = \langle -1, 1 \rangle$. Navíc: podle definice fce arcsin platí:

$\forall x \in \langle -1, 1 \rangle$: $f(x) = \sin(\underbrace{\arcsin x}_{=y \Leftrightarrow \sin y = x}) = x$



b) $f(x) = \arcsin(\sin x)$

$\Rightarrow \alpha)$ $D_f \in \mathbb{R}$, neboť $\forall x \in \mathbb{R}$: $\sin x \in \langle -1, 1 \rangle$.

$\beta)$ Funkce f je periodická s periodou 2π.

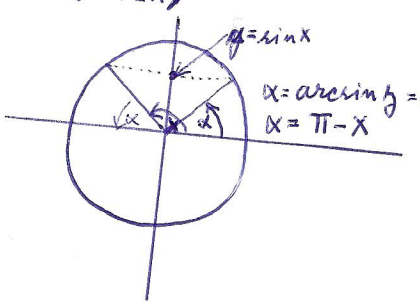
$\Rightarrow$ I.) $x \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$

Podle definice fce arcsin:

$f(x) = \arcsin(\underbrace{\sin x}_{=y}) = x$

$\Leftrightarrow \arcsin y = x$

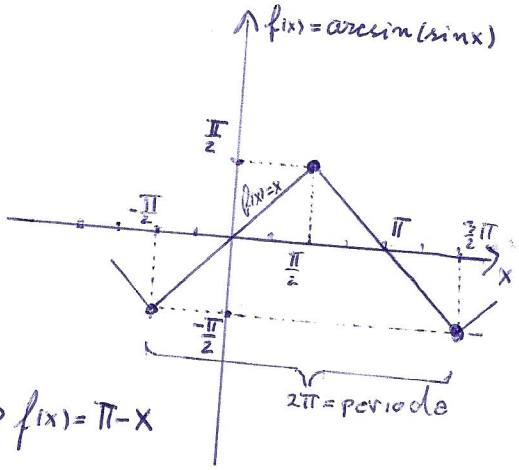
$\Rightarrow$ II.) $x \in (\frac{\pi}{2}, \frac{3}{2}\pi)$



$\alpha = \arcsin y = \arcsin(\sin x) = f(x)$

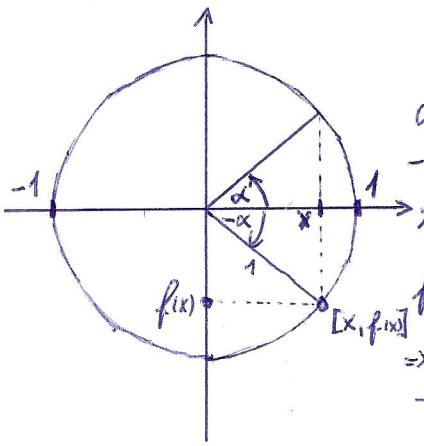
$\alpha = \pi - x$

$\Rightarrow f(x) = \pi - x$



c) $f(x) = \sin(-\arccos x)$

$\Rightarrow D_f = \langle -1, 1 \rangle$

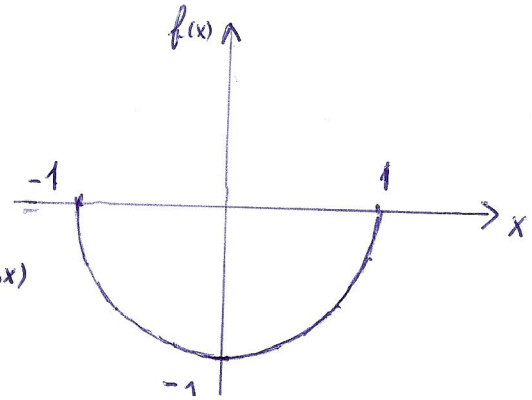


$\alpha = \arccos x$

$-\alpha = -\arccos x$

$f(x) = \sin(-\alpha) = \sin(-\arccos x)$

$\Rightarrow f(x) = -\sqrt{1-x^2}$

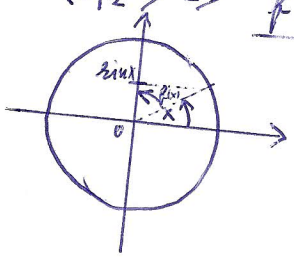


d) $f(x) = \arccos(\sin x)$

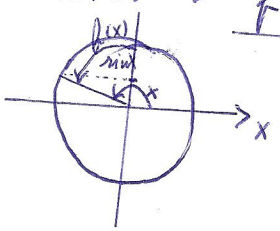
$\Rightarrow D_f = \mathbb{R}$: neboť $\forall x \in \mathbb{R}: \sin x \in \langle -1, 1 \rangle$

Navíc f je periodická funkce s periodou 2π.

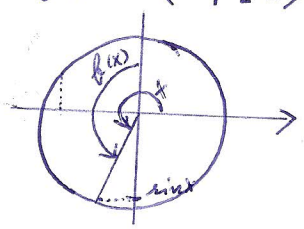
α) $x \in \langle 0, \frac{\pi}{2} \rangle \Rightarrow f(x) = \frac{\pi}{2} - x$



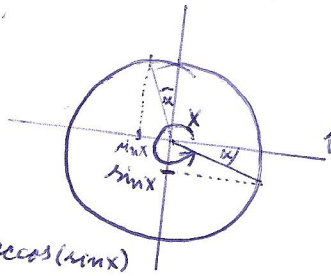
β) $x \in (\frac{\pi}{2}, \pi \rangle \Rightarrow f(x) = x - \frac{\pi}{2}$



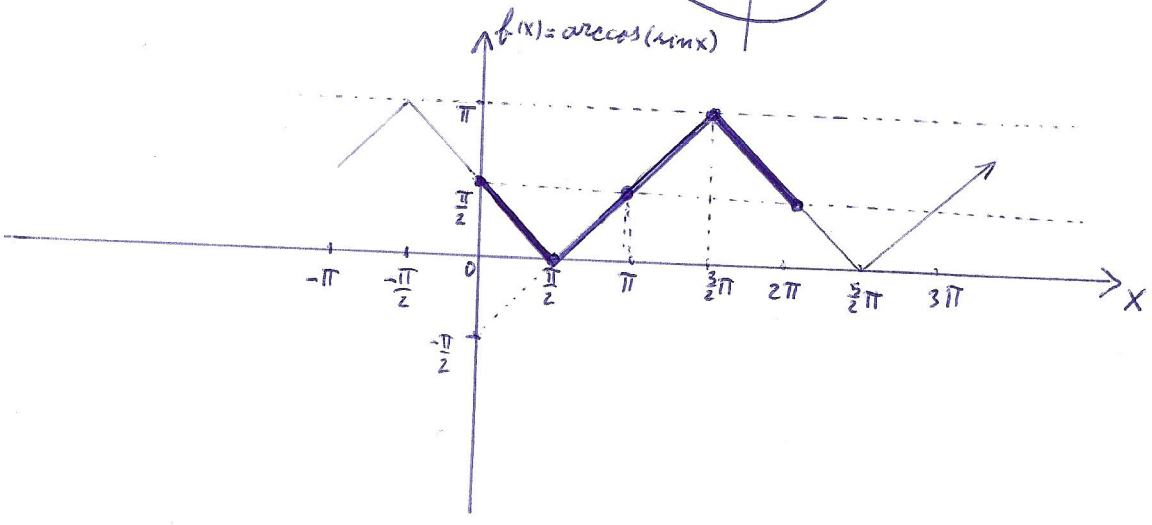
γ) $x \in \langle \pi, \frac{3}{2}\pi \rangle \Rightarrow f(x) = x - \frac{\pi}{2}$



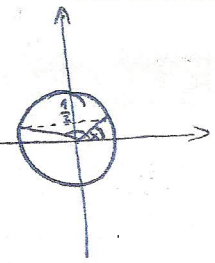
δ) $x \in \langle \frac{3}{2}\pi, 2\pi \rangle \Rightarrow f(x) = \frac{5}{2}\pi - x$



$f(x) = \frac{\pi}{2} + \alpha = \frac{\pi}{2} + (2\pi - x)$

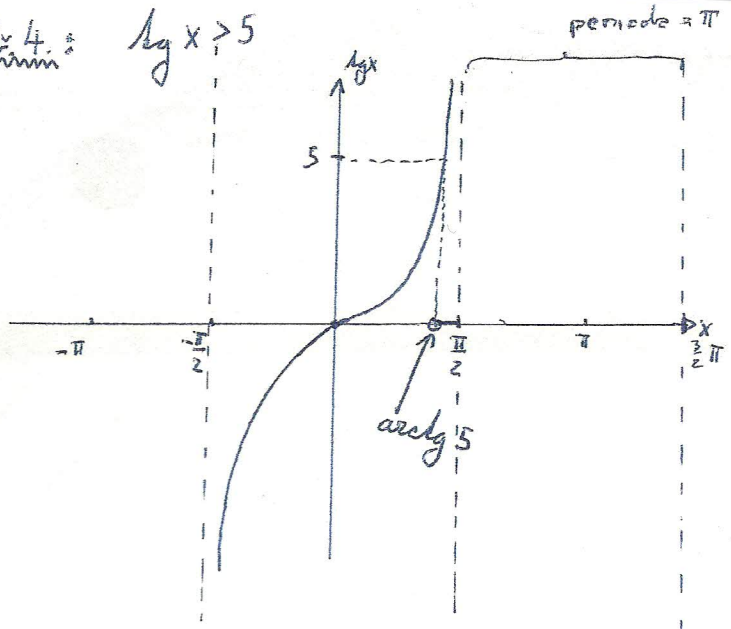


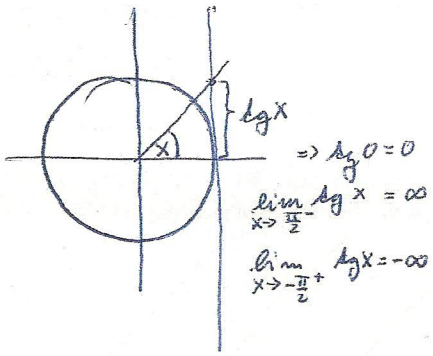
Př. 1.: $\sin x = \frac{1}{3}$, určete x.



$$\Rightarrow \quad x = \begin{cases} \arcsin\frac{1}{3} + k\cdot 2\pi, \ k\in\mathbb{Z} \\ \pi - \arcsin\frac{1}{3} + k\cdot 2\pi, \ k\in\mathbb{Z} \end{cases}$$

Př. 4.: $\operatorname{tg} x > 5$





$$x \in (\operatorname{arctg} 5, \tfrac{\pi}{2}) \cup (\operatorname{arctg} 5 + \pi \ ; \ \tfrac{\pi}{2}+\pi) \cup (\operatorname{arctg} 5 + 2\pi, \tfrac{\pi}{2}+2\pi) \cup \dots$$

$$\Rightarrow \quad x \in \bigcup_{k\in\mathbb{Z}} (\operatorname{arctg} 5 + k\pi \ ; \ \tfrac{\pi}{2} + k\cdot\pi)$$

Exponenciální funkce

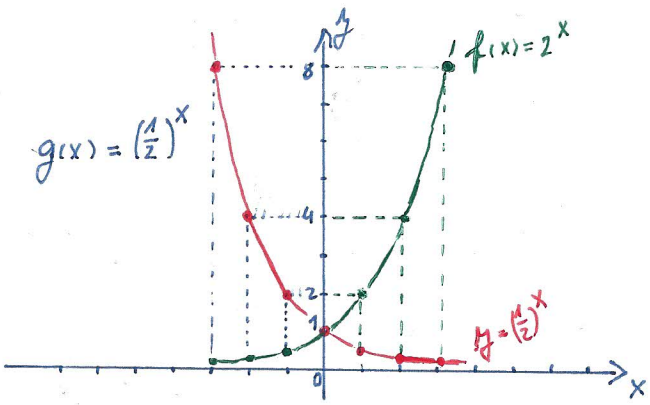
$$f(x) = a^x = e^{x \cdot \ln a}$$

(x ← exponent; a ← základ $\in \mathbb{R}^+ - \{1\}$)

$D(f) = \mathbb{R}$; $H(f) = (0, \infty)$;

- je prostá
- je omezená zdola
- není sudá ani lichá
- není periodická

Př.: Načrtněte graf funkce $f(x) = 2^x$ a $g(x) = \left(\frac{1}{2}\right)^x$



Př.: Vyřešte exponenciální rovnici:

1.) $5^{3x+2} = 25^{x+1}$

$5^{3x+2} = (5^2)^{x+1}$

$3x+2 = 2x+2$

$x = 0$

2.) $8^x = 16^{2-x}$

$(2^3)^x = (2^4)^{2x}$

$3x = 8-4x$

$x = \frac{8}{7}$

3.) $3^{x-2} = \left(\frac{1}{3}\right)^{-2x}$

$3^{x-2} = \left(3^{-1}\right)^{-2x}$

$x-2 = 2x$

$x = -2$

4.) $2^{3x-4} = \left(\frac{1}{8}\right)^{x+1}$

$2^{3x-4} = \left(2^{-3}\right)^{x+1}$

$3x-4 = -3x-3$

$x = \frac{1}{6}$

5.) $2^x \cdot 5^x = 0{,}1\,(10^{x-1})^5$

$10^x = 10^{-1} \cdot 10^{5x-5}$

$10^x = 10^{5x-6}$

$x = 5x-6$

$x = \frac{6}{4} = \frac{3}{2}$

6.) $4^x \cdot 5^{x+1} = 5 \cdot 20^{2-x}$

$20^x \cdot 5 = 5 \cdot 20^{2-x}$

$x = 2-x$

$x = 1$

Př. Vyřešte exponenciální rovnici:

1.) $3^{x+1} = 2^{2x+3}$

$3 \cdot 3^x = 8 \cdot 2^{2x}$

$3 \cdot 3^x = 8 \cdot 4^x$

$\frac{3^x}{4^x} \quad \frac{8}{3}$

$\left(\frac{3}{4}\right)^x = \frac{8}{3} \quad / \ln$

$\ln\left(\frac{3}{4}\right)^x = \ln\frac{8}{3}$

$x \cdot \ln\left(\frac{3}{4}\right) = \ln\frac{8}{3}$

$x = \frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{8}{3}\right)}$

2.) $5^{2x-3} = 8^{3x+7} \quad / \ln$

$\ln 5^{2x-3} = \ln 8^{3x+7}$

$(2x-3)\ln 5 = (3x+7)\ln 8$

$(2\ln 5)x - 3\ln 5 = (3 \cdot \ln 8)x + 7 \cdot \ln 8$

$x(2\ln 5 - 3\ln 8) = 7 \cdot \ln 8 + 3\ln 5$

$x(\ln 5^2 - \ln 8^3) = \ln 8^7 + \ln 5^3$

$x \ln \frac{5^2}{8^3} = \ln 8^7 \cdot 5^3$

$x = \frac{\ln 8^7 \cdot 5^3}{\frac{5^2}{8^3}}$

Př. Vyřešte exponenciální nerovnici:

1.) $3^{2x} > 5^{x-1} \quad / \ln$

$\ln 3^{2x} > \ln 5^{x-1}$

$2x \cdot \ln 3 > (x-1)\ln 5$

$x \cdot \ln 3^2 > x \cdot \ln 5 - \ln 5$

$x(\ln 9 - \ln 5) > -\ln 5$

$x > \frac{-\ln 5}{\ln\frac{9}{5}}$

$x \in \left(-\frac{\ln 5}{\ln\frac{9}{5}}, \infty\right)$

2.) $2^{8x-1} \leq \left(\frac{1}{3}\right)^{4x+6}$

$2^{8x-1} \leq (3^{-1})^{4x+6}$

$2^{8x-1} \leq 3^{-4x-6} \quad / \ln$

$\ln 2^{8x-1} \leq \ln 3^{-4x-6}$

$(8x-1)\ln 2 \leq (-4x-6)\ln 3$

$8x\ln 2 - \ln 2 \leq -4x\ln 3 - 6\ln 3$

$x\ln 2^8 - \ln 2 \leq x\ln 3^{-4} - \ln 3^6$

$x(\ln 2^8 - \ln 3^{-4}) \leq \ln 2 - \ln 3^6$

$x \ln \frac{2^8}{3^{-4}} \leq \ln \frac{2}{3^6}$

$x \ln(2^8 \cdot 3^4) \leq \ln(2 \cdot 3^{-6})$

$x \leq \frac{\ln(2 \cdot 3^{-6})}{\ln(2^8 \cdot 3^4)}$

$x \in \left(-\infty, \frac{\ln(2 \cdot 3^{-6})}{\ln(2^8 \cdot 3^4)}\right\rangle$

Logaritmická funkce

$\log_a x = y \iff a^y = x$

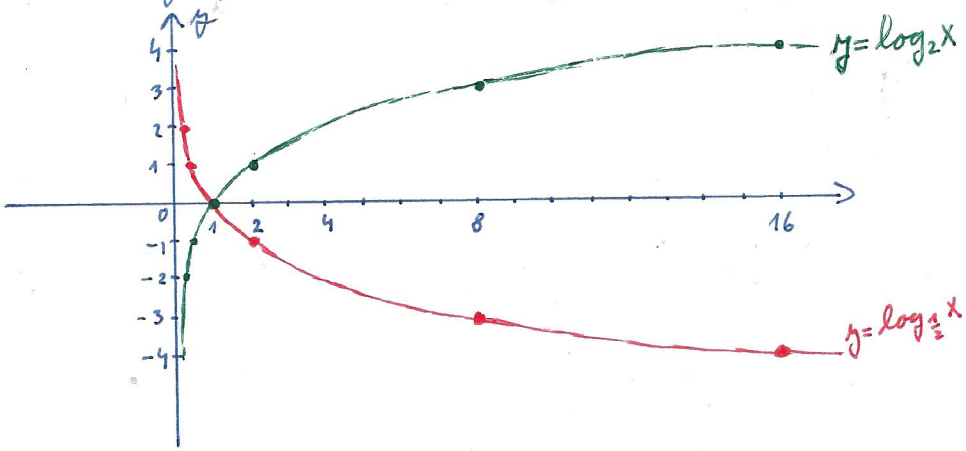
a ← základ logaritmu $\in \mathbb{R}^+ - \{1\}$

$D(f) = (0, \infty)$; $H(f) = \mathbb{R}$

- je prostá ; - není: sudá, lichá, periodická, omezená

- $\log_a x = \frac{\ln x}{\ln a}$

Př.: Načrtněte graf funkce $f(x) = \log_2(x)$; $g(x) = \log_{\frac{1}{2}} x$



Př.: Vyřešte logaritmickou rovnici:

1.) $x = \log_2 16$

$x = \log_2(2^4) = \underline{4}$

2.) $x = \log_3 27$

$x = \log_3(3^3) = \underline{3}$

3.) $x = \log_{\frac{1}{3}} 3$

$x = \log_{\frac{1}{3}}(9^{\frac{1}{2}}) = \log_{\frac{1}{3}}(\frac{1}{3})^{-\frac{1}{2}} = \underline{-\frac{1}{2}}$

4.) $x = \log_3 18 + \log_3 \frac{3}{2}$

$x = \log_3 18 \cdot \frac{3}{2} = \log_3 27 = \log_3 3^3 = \underline{3}$

5.) $x = \log_{10} 500 - \log_{10} 5$

$x = \log_{10} \frac{500}{5} = \log_{10} 10^2 = \underline{2}$

6.) $x = \log_{\frac{1}{5}} 5 + \log_5 \frac{1}{125}$

$x = -1 + (-3) = \underline{-4}$

Př. Vyřešte logaritmickou rovnici.

1.) $\log_2(4x+8) = 2$ $\quad \Rightarrow x \in (-2,\infty)$

$4x+8 = 2^2$

$4x = -4$

$x = -1$

2.) $\log_{10} x - \log_{10} 5 = 2$ $\quad \Rightarrow x \in (0,\infty)$

$\log_{10} \frac{x}{5} = 2$

$\frac{x}{5} = 10^2$

$x = 500$

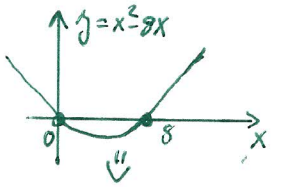
3.) $\log_3(x^2-8x) = 2$

$x^2-8x = 3^2$

$x^2-8x-9 = 0$

$(x-9)(x+1) = 0$

$x = \begin{cases} 9 \\ -1 \end{cases}$



$x \in (-\infty,0) \cup (8,\infty)$

4.) Pozn.: U tohoto typu příkladu:

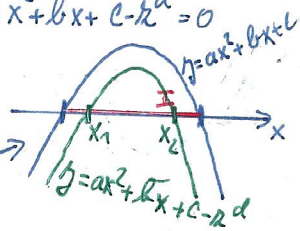
$\log_k(ax^2+bx+c) = d$

musí I.) $x \in I = \left(\frac{-b-\sqrt{b^2-4ac}}{2a}, \frac{-b+\sqrt{b^2-4ac}}{2a}\right)$ pro $a<0$

a $ax^2+bx+c = k^d \Rightarrow ax^2+bx+c-k^d = 0$

$\Rightarrow x_{1,2} = \frac{-b \pm \sqrt{b^2-4a(c-k^d)}}{2a} \in I$ jistě

II.) pro $a>0$ analogicky



Pozn. U typu př. jako v 1.) nalezené x také bude vždy vyhovovat.

Př. Vyřešte nerovnici:

1.) $\log_3(-3x+9) > 2$

I.) x musí splňovat: $-3x+9 > 0$

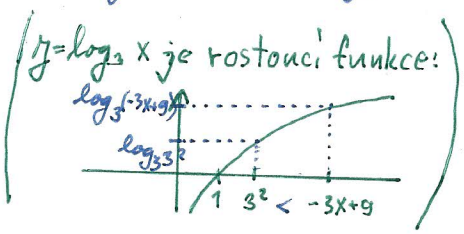
$-3x > -9 \quad /:(-3)$

$x < 3$

II.) $\log_3(-3x+9) > 2$

$\log_3(-3x+9) > \log_3(3^2)$

$y = \log_3 x$ je rostoucí funkce:



$-3x+9 > 3^2 \quad /-9$

$-3x > 0 \quad /:(-3)$

$x < 0$

$\Rightarrow \wedge$ I.) a II.): $x \in (-\infty, 0)$

2.) $\log_{\frac{1}{2}}(-5x+10) > 3$

I.) x musí splňovat: $-5x+10 > 0$

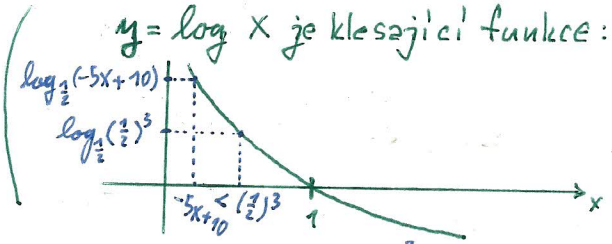
$-5x > -10 \quad /:(-5)$

$x < 2$

II.) $\log_{\frac{1}{2}}(-5x+10) > 3$

$\log_{\frac{1}{2}}(-5x+10) > \log_{\frac{1}{2}}(\frac{1}{2})^3$

$y = \log x$ je klesající funkce:



$-5x+10 < (\frac{1}{2})^3$

$-5x+10 < \frac{1}{8}$

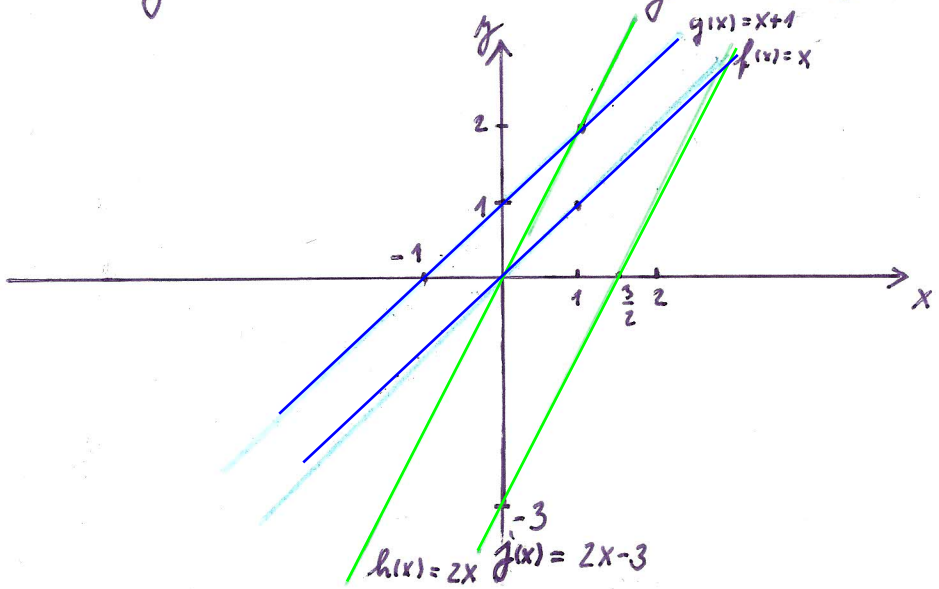
$x > \frac{\frac{1}{8}-10}{-5} = \frac{-\frac{79}{8}}{-5} = \frac{79}{40}$

$\Rightarrow \wedge$ I.) a II.): $x \in (\frac{79}{40}, 2)$

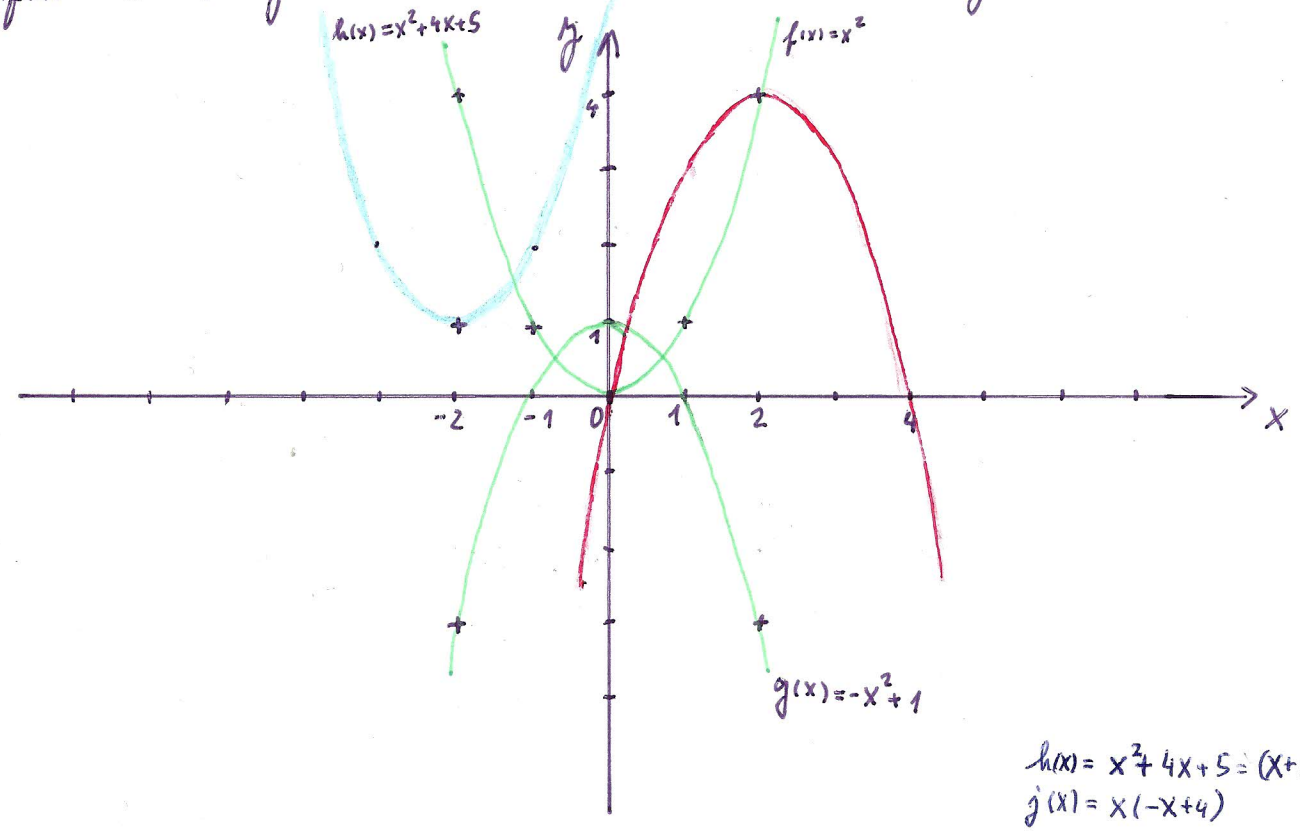
Elementární funkce a transformace grafu funkce

Př: Načrtněte grafy funkcí

1.) $f(x) = x$, $g(x) = x + 1$, $h(x) = 2x$, $j(x) = 2x - 3$



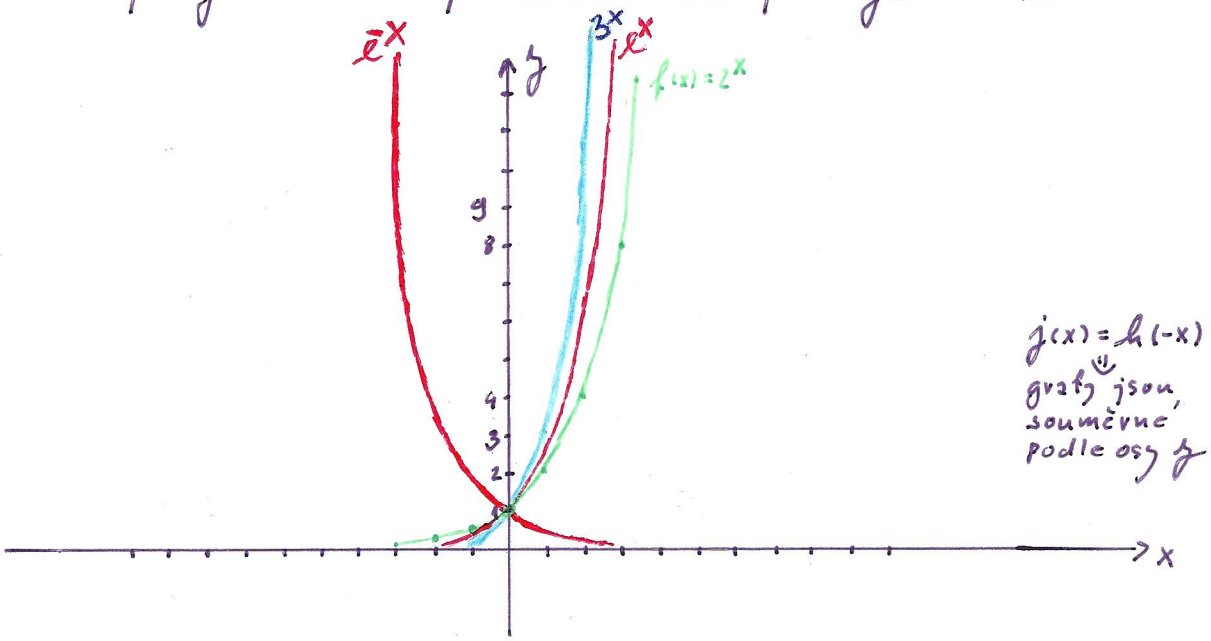
2.) $f(x) = x^2$, $g(x) = -x^2 + 1$, $h(x) = x^2 + 4x + 5$, $j(x) = -x^2 + 4x$



$h(x) = x^2 + 4x + 5 = (x+2)^2 + 1$

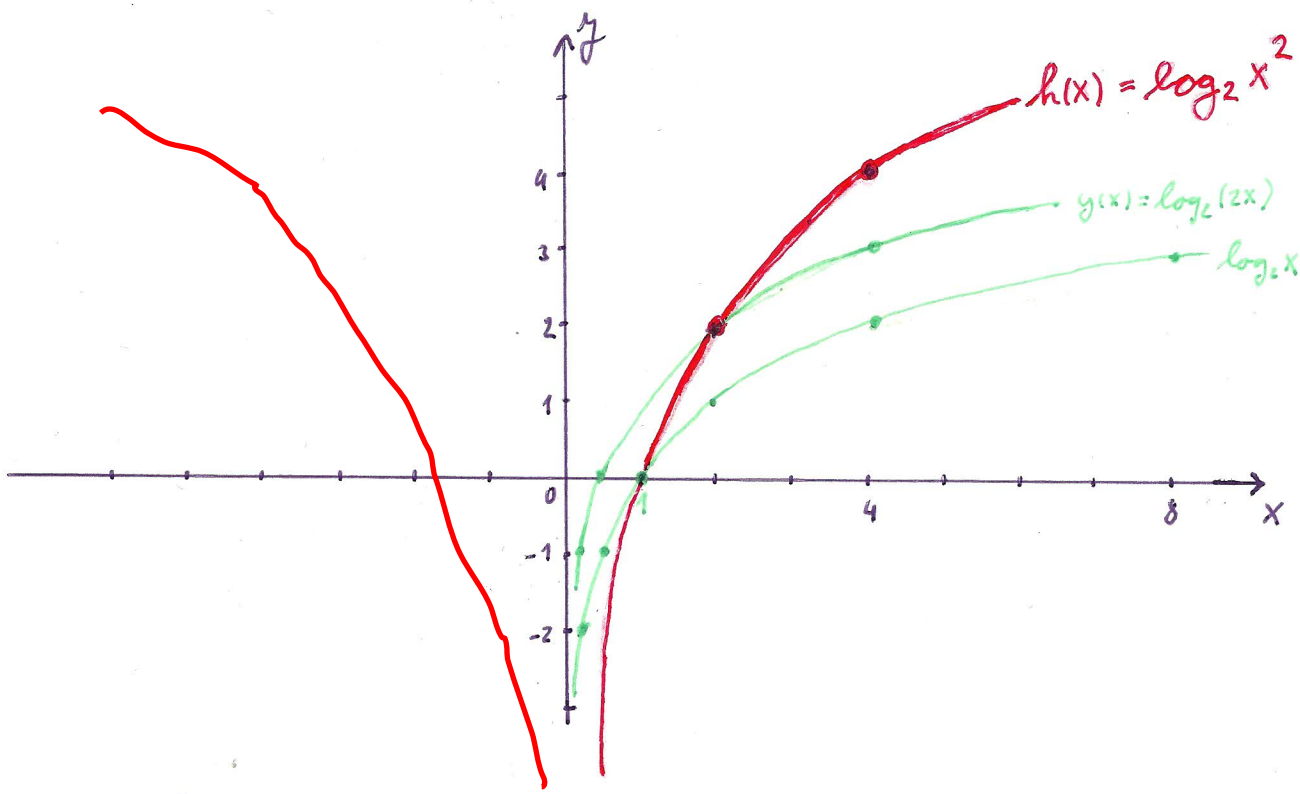
$j(x) = x(-x+4)$

3.) $f(x) = 2^x$, $g(x) = 3^x$, $h(x) = e^x$, $j(x) = e^{-x}$



$j(x) = h(-x)$
grafy jsou souměrné podle osy y

4.) $f(x) = \log_2 x$, $g(x) = \log_2(2 \cdot x)$, $h(x) = \log_2 x^2$, $j(x) = \ln(-x^2)$

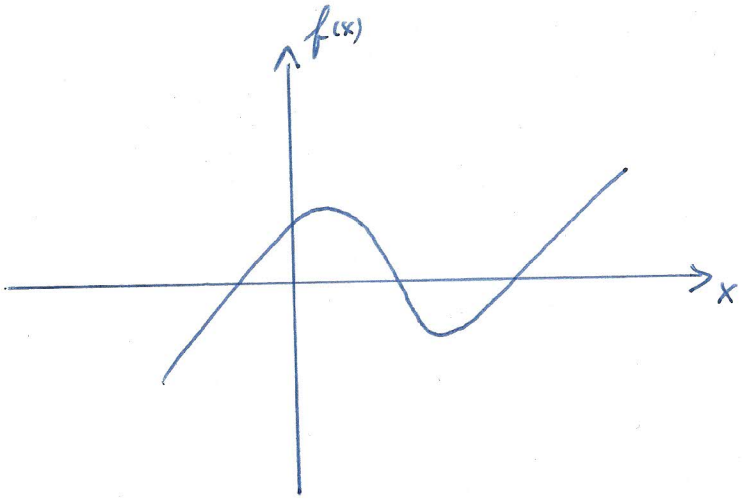


$g(x) = \log(2x) = \underbrace{\log_2 2}_{+1} + \log_2 x = \log_2 x + 1$

$h(x) = \log_2 x^2 = 2 \cdot \log_2 x$

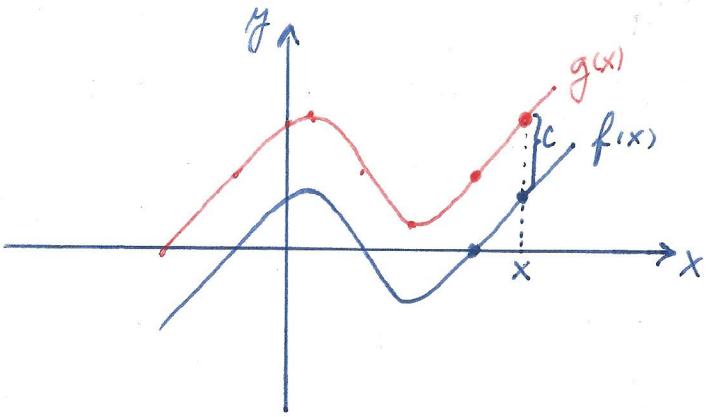
$j(x) = \ln(-x^2)$ ale $\forall x \in \mathbb{R}: -x^2 \leq 0 \Rightarrow D_j = \emptyset \Rightarrow$ nemá žádný graf

Transformace grafu funkce



1.) $g(x) = f(x) + c$

Graf fce g je stejný jako graf fce f, ale je posunuto o c ve směru nahoru / dolů



2.) $g(x) = c \cdot f(x)$