Př. Dělte polynom polynomem :

1.) $(x^2+2x+5):(x+1) = x+1$

$$-(x^2+x)$$
$$0+x+5$$
$$-(x+1)$$
$$4$$

$$\Rightarrow \frac{x^2+2x+5}{x+1} = \frac{(x+1)(x+1)+5}{x+1} = \underline{\underline{x+1+\frac{5}{x+1}}}$$

Zk.: $(x+1)(x+1)+4 = x^2+2x+1+4 = x^2+2x+5$ ✓

2.) $(4x^5-2x^3+x+1):(x^3-1) = 4x^2-2$

$$-(4x^5-4x^2)$$
$$0\quad -2x^3+4x^2+x+1$$
$$-(-2x^3+2)$$
$$0+4x^2+x-1$$

$$\Rightarrow \frac{4x^5-2x^3+x+1}{x^3-1} = \frac{(x^3-1)(4x^2-2)+4x^2+x-1}{x^3-1} =$$
$$= \underline{\underline{4x^2-2+\frac{4x^2+x-1}{x^3-1}}}$$

Zk.: $(x^3-1)(4x^2-2)+4x^2+x-1 = 4x^5-2x^3-4x^2+2+4x^2+x-1 =$
$= 4x^5-2x^3+x+1$ ✓

3.) $(3x^8-2x^5+x^4-5):(x^5+x+1) = 3x^3-2$

$$-(3x^8+3x^4+3x^3)$$
$$0-2x^5-2x^4-3x^3-5$$
$$-(-2x^5-2x-2)$$
$$-2x^4-3x^3+2x-3$$

$$\frac{3x^8-2x^5+x^4-5}{x^5+x+1} = \underline{\underline{3x^3-2+\frac{-2x^4-3x^3+2x-3}{x^5+x+1}}}$$

Zk.: $(x^5+x+1)(3x^3-2)-2x^4-3x^3+2x-3 =$
$= 3x^8-2x^5+3x^4-2x+3x^3-2-2x^4-3x^3+2x-3 =$
$= 3x^8-2x^5+x^4-5$ ✓

Integrace racionálních funkcí

Každou polynomickou funkci $q(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, kde $a_n, \dots, a_0 \in \mathbb{R}$ lze napsat ve tvaru:

$$\boxed{q(x) = a_n (x-\alpha_1)^{m_1} \cdots (x-\alpha_k)^{m_k} (x^2+\beta_1 x+\gamma_1)^{m_1} \cdots (x^2+\beta_l x+\gamma_l)^{m_l},} \quad (*)$$

kde α_i jsou navzájem různé kořeny (reálné) polynomu $q(x)$; $\beta_i, \gamma_j \in \mathbb{R}$; polynomy $x^2+\beta_i x+\gamma_i$ nemají reálné kořeny; $m_i, m_j \in \mathbb{N} \cup \{0\}$.

Př.: $q(x) = x^6 - x^2 = x^2(x^4-1) = x^2(x^2-1)(x^2+1) = (x-0)^2(x-1)^1(x+1)^1 \underbrace{(x^2+1)}$

$D = 0^2 - 4 \cdot 1 \cdot 1 < 0 \Rightarrow$ nemá reálné kořeny

Věta (Rozklad na parciální zlomky): Nechť $p(x)$ a $q(x)$ jsou polynomické funkce, kde stupeň $p(x)$ je menší, než stupeň $q(x)$. Jestliže $q(x)$ má tvar $(*)$, pak existují $a_{ij}, b_{rs}, c_{rs} \in \mathbb{R}$:

v rozkladu q: $(x-\alpha_1)^{m_1}$ ⇓ $(x-\alpha_k)^{m_k}$ ⇓

$$\frac{p(x)}{q(x)} = \left(\frac{a_{11}}{(x-\alpha_1)^1} + \frac{a_{12}}{(x-\alpha_1)^2} + \dots + \frac{a_{1m_1}}{(x-\alpha_1)^{m_1}}\right) + \dots + \left(\frac{a_{k1}}{(x-\alpha_k)^1} + \frac{a_{k2}}{(x-\alpha_k)^2} + \dots + \frac{a_{km_k}}{(x-\alpha_k)^{m_k}}\right) +$$

$(x^2+\beta_1 x+\gamma_1)^{m_1}$ ⇓

$$+\left(\frac{b_{11}x + c_{11}}{(x^2+\beta_1 x+\gamma_1)^1} + \frac{b_{12}x + c_{12}}{(x^2+\beta_1 x+\gamma_1)^2} + \dots + \frac{b_{1m_1}x + c_{1m_1}}{(x^2+\beta_1 x+\gamma_1)^{m_1}}\right) +$$

$$\vdots$$

$$+ \frac{b_{l1}x + c_{l1}}{(x^2+\beta_l x+c_l)^1} + \frac{b_{l2}x + c_{l2}}{(x^2+\beta_l x+c_l)^2} + \dots - \frac{b_{lm_l}x + c_{lm_l}}{(x^2+\beta_l x+c_l)^{m_l}}$$

Př: Určete integrál $\int \frac{4x-3}{x^2+6x+8}\,dx$

Rozklad na parciální zlomky:

$$\frac{4x-3}{x^2+6x+8} = \frac{4x-3}{(x+2)(x+4)} = \frac{a}{x+2} + \frac{b}{x+4} = \frac{a(x+4)+b(x+2)}{(x+2)(x+4)}$$

Určíme a, b: $4x-3 = a(x+4)+b(x+2)$ => dosaďme za x:

$\boxed{x=-2}$ => $4(-2)-3 = a(-2+4)+b(-2+2)$

$-11 = 2a$

$a = \frac{-11}{2}$

$\boxed{x=-4}$ => $4(-4)-3 = a(-4+4)+b(-4+2)$

$-19 = -2b$

$b = \frac{19}{2}$

$$\Rightarrow \frac{4x-3}{x^2+6x+8} = \frac{-\frac{11}{2}}{x+2} + \frac{\frac{19}{2}}{x+4}$$

$\Downarrow$

$$\int \frac{4x-3}{x^2+6x+8}\,dx = \int \frac{-\frac{11}{2}}{x+2} + \frac{\frac{19}{2}}{x+4}\,dx = -\frac{11}{2}\int \frac{1}{x+2}\,dx + \frac{19}{2}\int \frac{1}{x+4}\,dx =$$

$t = x+2$, $dt = dx$; $y = x+4$, $dy = dx$

$$= -\frac{11}{2}\int \frac{1}{t}\,dt + \frac{19}{2}\int \frac{1}{y}\,dy = -\frac{11}{2}\ln|t| + \frac{19}{2}\ln|y| =$$

$$= \underline{\underline{-\frac{11}{2}\ln|x+2| + \frac{19}{2}\ln|x+4|}}$$

Př: Určete integrál

$$\int \frac{5x}{x^2+x-6}\,dx \Rightarrow$$

$$\frac{5x}{x^2+x-6} = \frac{5x}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2} = \frac{A(x-2)+B(x+3)}{(x+3)(x-2)}$$

$$\Rightarrow \quad 5x = A(x-2) + B(x+3)$$

$\boxed{x=2}$ $\quad 10 = 5B \quad \Rightarrow \quad B = 2$

$\boxed{x=-3}$ $\quad -15 = -5A \quad \Rightarrow \quad A = 3$

$$\Rightarrow \int \frac{5x}{x^2+x-6}\,dx = \int \frac{3}{x+3} + \frac{2}{x-2}\,dx = \underline{\underline{3\ln|x+3| + 2\ln|x-2|}}$$

Př. Určete $\int \frac{5x-12}{x^2-5x+6}\,dx$

$$\frac{5x-12}{x^2-5x+6} = \frac{5x-12}{(x-2)(x-3)} = \frac{a}{x-2} + \frac{b}{x-3} = \frac{a(x-3)+b(x-2)}{(x-2)(x-3)}$$

$\underline{x=3} \Rightarrow \quad 5\cdot 3-12 = a\cdot 0 + b\cdot 1$

$\underline{3 = b}$

$\boxed{x=2} \Rightarrow \quad 5\cdot 2-12 = a(-1) + b\cdot 0$

$-2 = -a$

$\underline{a = 2}$

$$\Rightarrow \frac{5x-12}{x^2-5x+6} = \frac{3}{x-2} + \frac{2}{x-3}$$

$$\Rightarrow \int \frac{5x-12}{x^2-5x+6}\,dx = \int\left(\frac{3}{x-2} + \frac{2}{x-3}\right)dx$$

$$\int \frac{3}{x-2}\,dx = \left|\begin{matrix} t = x-2 \\ dt = dx \end{matrix}\right| = \int \frac{3}{t}\,dt = 3\ln|t| = 3\ln|x-2|$$

$$\int \frac{2}{x-3}\,dx = \left|\begin{matrix} t = x-3 \\ dt = dx \end{matrix}\right| = \int \frac{2}{t}\,dt = 2\ln|t| = 2\ln|x-3|$$

$$\Rightarrow \int \frac{5x-12}{x^2-5x+6}\,dx = \underline{\underline{2\ln|x-2| + 3\ln|x-3|}}$$

Př: Určete integrál

$$\int \frac{8x^2+4x-6}{x^3+x^2-2x}\,dx$$

$$\frac{8x^2+4x-6}{x^3+x^2-2x} = \frac{8x^2+4x-6}{x(x^2+x-2)} = \frac{8x^2+4x-6}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2} = \frac{A(x-1)(x+2)+B(x+2)x+Cx(x-1)}{x(x-1)(x+2)}$$

$$\Rightarrow \quad 8x^2+4x-6 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$$

$\lvert x=0 \rvert$	$-6 = -2A$	$\Rightarrow A=3$
$\lvert x=1 \rvert$	$6 = 3B$	$\Rightarrow B=2$
$\lvert x=-2 \rvert$	$18 = 6C$	$\Rightarrow C=3$

$$\Rightarrow \int \frac{8x^2+4x-6}{x^3+x^2-2x}\,dx = \int \frac{3}{x} + \frac{2}{x-1} + \frac{3}{x+2}\,dx$$

$$\int \frac{3}{x}\,dx = 3\ln|x|$$

$$\int \frac{2}{x-1}\,dx = \left| \begin{matrix} t = x-1 \\ dt = dx \end{matrix} \right| = \int \frac{2}{t}\,dt = 2\ln|t| = 2\ln|x-1|$$

$$\int \frac{3}{x+2}\,dx = \left| \begin{matrix} t = x+2 \\ dt = dx \end{matrix} \right| = \int \frac{3}{t}\,dt = 3\ln|t| = 3\ln|x+2|$$

$$\Rightarrow \int \frac{8x^2+4x-6}{x^3+x^2-2x}\,dx = \underline{\underline{3\ln|x| + 2\ln|x-1| + 3\ln|x+2|}}$$

Př. Určete integrál $\int \frac{7x^2+3x+5}{x^3+x}\,dx$.

Rozklad na parciální zlomky:

$$\frac{7x^2+3x+5}{x^3+x} = \frac{7x^2+3x+5}{x\underbrace{(x^2+1)}_{\text{již nelze rozložit } (D<0)}} = \frac{a}{x} + \frac{b+cx}{x^2+1} = \frac{a(x^2+1)+bx+cx^2}{x(x^2+1)}$$

$\boxed{x=0}$ $\Rightarrow$ $7\cdot 0^2+3\cdot 0+5 = a(0^2+1)+b\cdot 0+c\cdot 0^2$

$\underline{5 = a}$

$\boxed{x=1}$ $7\cdot 1^2+3\cdot 1+5 = a(1^2+1)+b\cdot 1+c\cdot 1^2$

$15 = 2a+b+c$ $\quad / \ a=5$

$15 = 10+b+c$

$\underline{b+c = 5}$

$\boxed{x=-1}$ $7(-1)^2+3(-1)+5 = a((-1)^2+1)+b(-1)+c(-1)^2$

$9 = 2a-b+c$ $\quad / \ a=5$

$\underline{-1 = -b+c}$

$$\begin{array}{r} b+c=5 \\ -b+c=-1 \\ \hline 2c=4 \\ \underline{c=2} \end{array}$$

} sečteme

dosadíme do

$b+2=5$

$\underline{b=3}$

$$\Rightarrow \int \frac{7x^2+3x+5}{x^3+x}\,dx = \int \frac{5}{x} + \frac{3+2x}{x^2+1}\,dx = 5\int\frac{1}{x}\,dx + 3\int\frac{1}{x^2+1}\,dx + \int\frac{2x}{x^2+1}\,dx =$$

dt

$\Big/ \ t = x^2+1$, $dt = 2x\,dx$

$$= 5\cdot\ln|x| + 3\,\mathrm{arctg}\,x + \int\frac{1}{t}\,dt = 5\ln|x| + 3\,\mathrm{arctg}\,x + \ln|t| =$$

$$\underline{\underline{= 5\ln|x| + 3\,\mathrm{arctg}\,x + \ln|x^2+1|}}$$

<u>Př:</u> Určete $\int \frac{x^2+x+1}{x^4-1}\,dx$

$$\frac{x^2+x+1}{x^4-1} = \frac{x^2+x+1}{(x^2-1)(x^2+1)} = \frac{x^2+x+1}{(x-1)(x+1)(x^2+1)} = \frac{a}{x-1} + \frac{b}{x+1} + \frac{cx+d}{x^2+1} \quad \Rightarrow$$

$\Rightarrow a=?,\ b=?,\ c=?,\ d=? \qquad \Rightarrow \quad \forall x \in \mathbb{R}-\{1,-1\}:$

$$\frac{x^2+x+1}{x^4-1} = \frac{a(x+1)(x^2+1) + b(x-1)(x^2+1) + cx(x-1)(x+1) + d(x-1)(x+1)}{(x-1)(x+1)(x^2+1)}$$

$\Rightarrow \forall x \in \mathbb{R}$ (ze spojitosti polynomů) $:\quad x^2+x+1 = a(x+1)(x^2+1) + b(x-1)(x^2+1) + cx(x-1)(x+1) + d(x-1)(x+1)$

zvolíme: $\boxed{x=1} \Rightarrow 3 = a\cdot 2\cdot 2 + 0 + 0 + 0 \Rightarrow a = \frac{3}{4}$

$\boxed{x=-1} \Rightarrow 1 = 0 - 4b + 0 + 0 \Rightarrow b = -\frac{1}{4}$

$\boxed{x=0} \Rightarrow 1 = \frac{3}{4} + \frac{1}{4} + 0 - d \Rightarrow d = 0$

$\boxed{x=2} \Rightarrow 7 = \frac{45}{4} - \frac{5}{4} + 6c + 0 \Rightarrow c = -\frac{1}{2}$

$\Rightarrow$

$$\int \frac{x^2+x+1}{x^4-1}\,dx = \int \frac{\frac{3}{4}}{x-1} + \frac{-\frac{1}{4}}{x+1} + \frac{-\frac{1}{2}x}{x^2+1}\,dx =$$

$$\left| \begin{array}{l} \int \frac{3}{4}\frac{1}{x-1}\,dx = \frac{3}{4}\ln|x-1| \\ \int -\frac{1}{4}\frac{1}{x+1}\,dx = -\frac{1}{4}\ln|x+1| \\ -\frac{1}{2}\int \frac{x}{x^2+1}\,dx = \left| \begin{array}{l} t = x^2+1 \\ dt = 2x\,dx \end{array} \right| = -\frac{1}{2}\int \frac{x}{t}\frac{dt}{2x} = \\ \qquad dx = \frac{dt}{2x} \\ = -\frac{1}{4}\ln|t| = -\frac{1}{4}\ln|x^2+1| \end{array} \right|$$

$$= \frac{3}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{4}\ln|x^2+1|$$

Př: Určete $\int \frac{x^2-x+2}{x^4+3x^3+2x^2}\,dx$

$$\frac{x^2-x+2}{x^4+3x^3+2x^2} = \frac{x^2-x+2}{x^2(x^2+3x+2)} = \frac{x^2-x+2}{x^2(x+1)(x+2)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x+1} + \frac{d}{x+2} \Rightarrow$$

$$\frac{x^2-x+2}{x^4+3x^3+2x^2} = \frac{a\,x(x+1)(x+2) + b\,(x+1)(x+2) + c\,x^2(x+2) + d\,x^2(x+1)}{x^2(x+1)(x+2)}$$

$\Rightarrow \forall x \in \mathbb{R}: \quad x^2-x+2 = a\,x(x+1)(x+2) + b(x+1)(x+2) + c\,x^2(x+2) + d\,x^2(x+1)$

$$\left.\begin{array}{llll}
\boxed{x=0} \Rightarrow & 2 = 0+2b+0+0 & \Rightarrow & b=1 \\
\boxed{x=-1} \Rightarrow & 4 = 0+0+c+0 & \Rightarrow & c=4 \\
\boxed{x=-2} \Rightarrow & 8 = 0+0+0-4d & \Rightarrow & d=-2 \\
\boxed{x=1} \Rightarrow & 2 = 6a+6+12-4 & \Rightarrow & a=-2
\end{array}\right\} \Rightarrow$$

$$\int \frac{x^2-x+2}{x^4+3x^3+2x^2}\,dx = \int \frac{-2}{x} + \frac{1}{x^2} + \frac{4}{x+1} + \frac{2}{x+2}\,dx =$$

$$= \underline{\underline{-2\ln x - x^{-1} + 4\ln|x+1| + 2\ln|x+2|}}$$

Př:

Př: Určete $\int \frac{x^4+4x^3-4x^2-11x+14}{x^2+6x+8}\,dx$

Pozor! Není splněna podmínka st $(p(x)) <$ st $(q(x))$! $\Rightarrow$ nejprve podělíme:

$$\left.\begin{array}{l} (x^4+4x^3-4x^2-11x+14) : (x^2+6x+8) = x^2-2x \\ -(x^4+6x^3+8x^2) \\ \quad -2x^3-12x^2-11x+14 \\ \quad -(-2x^3-12x^2-16x) \\ \quad\quad \text{zb: } \underline{5x+14} \end{array}\right\} \Rightarrow$$

$$\int \frac{x^4+4x^3-4x^2-11x+14}{x^2+6x+8}\,dx = \int x^2-2x+\frac{5x+14}{x^2+6x+8}\,dx$$

$\alpha)$ $\int x^2\,dx = \underline{\underline{\frac{x^3}{3}}}$ $\qquad$ $\beta)$ $\int -2x\,dx = \underline{\underline{-x^2}}$

$\gamma)$ $\int \frac{5x+14}{x^2+6x+8}\,dx$ $\Rightarrow$ parciální zlomky

$$\frac{5x+14}{x^2+6x+8} = \frac{5x+14}{(x+2)(x+4)} = \frac{a}{x+2} + \frac{b}{x+4}$$

$$\frac{5x+14}{x^2+6x+8} = \frac{a(x+4)+b(x+2)}{(x+2)(x+4)} \Rightarrow \quad 5x+14 = a(x+4)+b(x+2)$$

$\lfloor x=-4 \rfloor \Rightarrow \quad -6 = 0-2b \quad \Rightarrow b=3$

$\lfloor x=-2 \rfloor \Rightarrow \quad 4 = 2a+0 \quad \Rightarrow a=2$

$$\Rightarrow \int \frac{5x+14}{x^2+6x-8}\,dx = \int \frac{2}{x+2}+\frac{3}{x+4}\,dx = \underline{\underline{2\ln|x+2| + 3\ln|x+4|}}$$

$$\Rightarrow \int \frac{x^4+4x^3-4x^2-11x+14}{x^2+6x+8}\,dx = \underline{\underline{\frac{x^3}{3} - x^2 + 2\ln|x+2| + 3\ln|x+4|}}$$

Př: Určete integrál

$$I = \int \frac{x^3+2x^2-x+3}{x^3-6x^2+5x}\, dx$$

- zde není splněno, že stupeň $p(x)$ > stupeň $q(x)$ $\Rightarrow$
$\Rightarrow$ nejprve polynomy podělíme:

$$(x^3+2x^2-x+3) : (x^3-6x^2+5x) = 1 + \underbrace{\frac{8x^2-6x+3}{x^3-6x^2+5x}}$$
$$-(x^3-6x^2+5x)$$
$$8x^2-6x+3$$

rozložíme na parciální zlomky

$$\Rightarrow x^3-6x^2+5x = x(x^2-6x+5) = x(x-5)(x-1)$$ $\Rightarrow$ kořeny: $x_1=0$, $x_2=5$, $x_3=1$

$$\Rightarrow \frac{8x^2-6x+3}{x^3-6x^2+5x} = \frac{a}{x} + \frac{b}{x-5} + \frac{c}{x-1}$$

$$\frac{8x^2-6x+3}{x^3-6x^2+5x} = \frac{a(x-5)(x-1)+b(x-1)x+c(x-5)x}{x(x-5)(x-1)}$$

$$\Rightarrow 8x^2-6x+3 = a(x-5)(x-1)+b(x-1)x+c(x-5)x$$ $\Rightarrow$ dosadíme kořeny $\Rightarrow$

$$\left.\begin{aligned} &\underline{x_1=0} \Rightarrow 3 = a(-5)(-1) \\ &\underline{x_2=5} \Rightarrow 8\cdot 25-30+3 = b\cdot 20 \\ &\underline{x_3=1} \Rightarrow 8-6+3 = c(-4)\cdot 1 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} a &= \frac{3}{5} \\ b &= \frac{173}{20} \\ c &= -\frac{5}{4} \end{aligned}\right\} \Rightarrow \frac{8x^2-6x+3}{x^3-6x^2+5x} = \frac{3}{5x} + \frac{173}{20(x-5)} - \frac{5}{4(x-1)}$$

$$\Rightarrow \int \frac{x^3+2x^2-x+3}{x^3-6x^2+5x}\, dx = \int 1 + \frac{8x^2-6x+3}{x^3-6x^2+5x}\, dx =$$

$$= \int 1 + \frac{3}{5x} + \frac{173}{20(x-5)} - \frac{5}{4(x-1)}\, dx = \underline{\underline{x + \frac{3}{5}\ln|x| + \frac{173}{2}\ln|x-5| - \frac{5}{4}\ln|x-}}$$